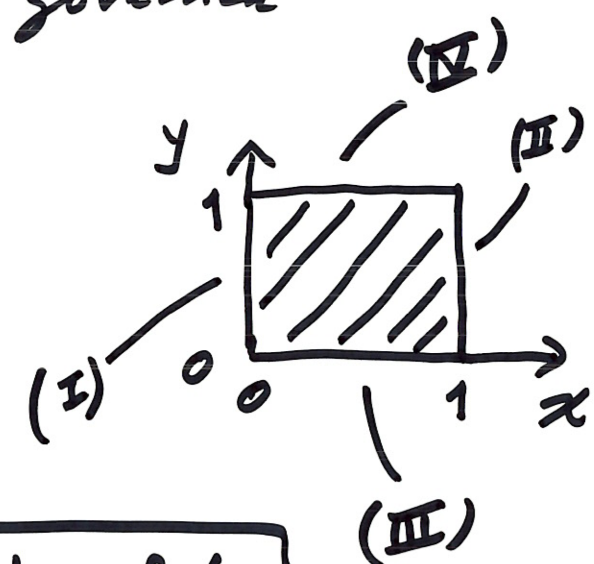
Lecture 10

Recap (Lec 9) 2D Heat eq.

→ steady sol. of which is governed by **Laplace's eq.**



Full

(★) **2-D Heat** eq. $u(x,y,t)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

(I) ~~u(0, t) = P~~ $u(0,y,t) = P$

(II) ~~u(1, t) = Q~~ $u(1,y,t) = Q$

(III) ~~u(~~ $u(x,0,t) = R$

(IV) $u(x,1,t) = S$

(V) $u(x,y,0) = F(x,y)$

≠

steady sol. of (★) $u(x,y)$

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{— Laplace's eq.}$$

(I) $u(0,y) = P$

(II) $u(1,y) = Q$

(III) $u(x,0) = R$

(IV) $u(x,1) = S$

Dirichlet b.c.

(★★)

Qualitative behavior...

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Downarrow$$

(Lec 9):

$$\underbrace{\frac{\partial^2 u}{\partial x^2}} = - \underbrace{\frac{\partial^2 u}{\partial y^2}}$$

$u(x,y)$ needs not be a "plane"

e.g. "saddle" surface

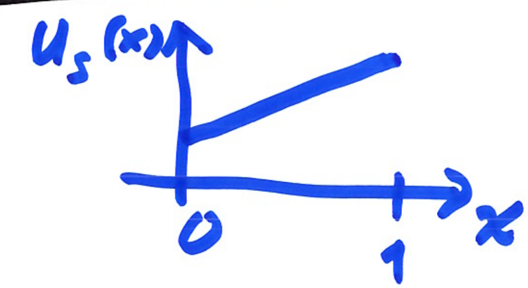
1-D Heat eq

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

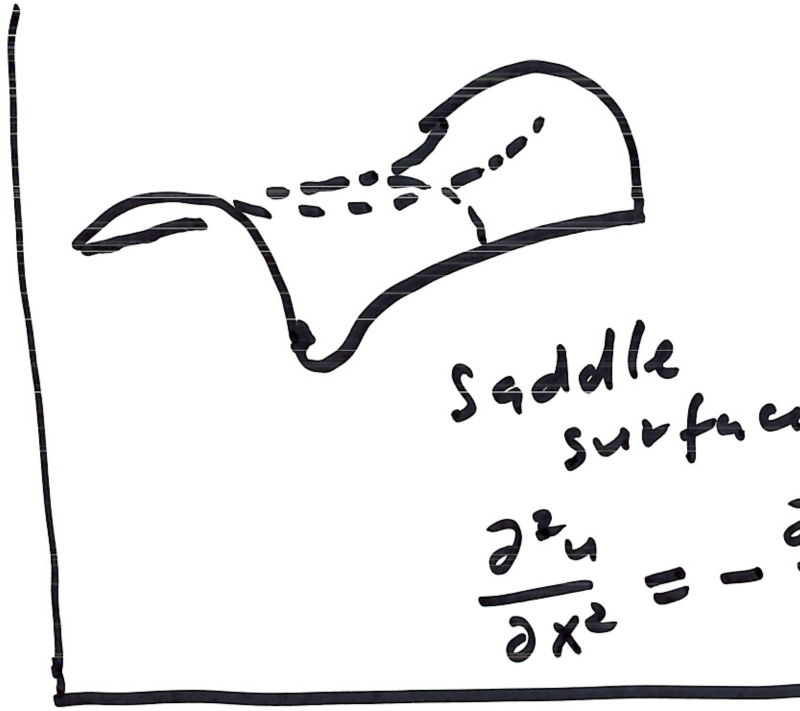
$\downarrow$ steady sol.

$$\frac{d^2 u_s}{d x^2} = 0$$

is a line!

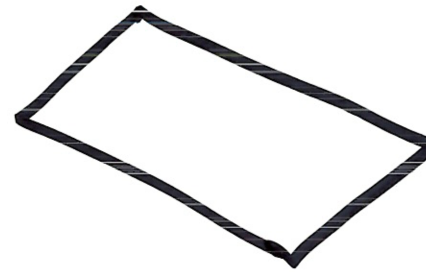


$u(x,y)$



Saddle surface

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \checkmark$$



plane ✓

$$\frac{\partial^2 u}{\partial x^2}=0, \ \frac{\partial^2 u}{\partial y^2}=0$$

X $u(x,y)$

?

Local max (in all direction)

Calculus 101:

$$\frac{\partial^2 u}{\partial x^2}<0 \quad \frac{\partial^2 u}{\partial y^2}<0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}<0 \neq 0$$

Violates Laplace's eq.

✓ Also, cannot have local min

*

Max/Min only occurs at the boundary!

Ex:

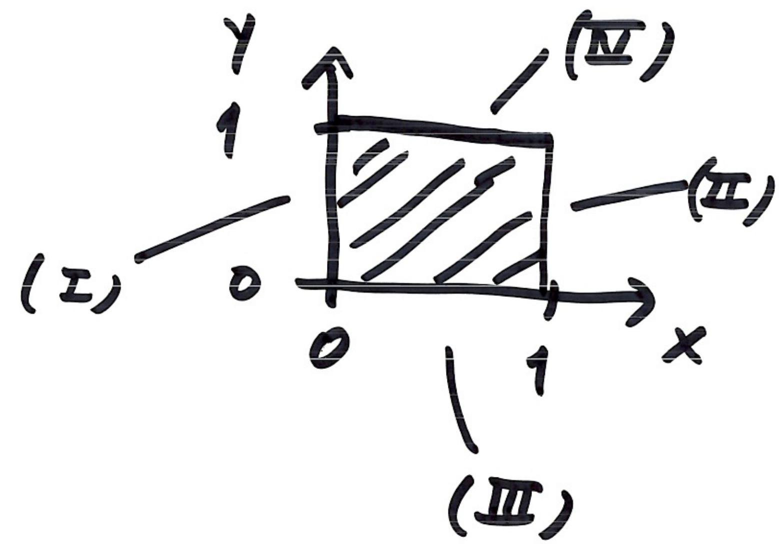
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(I) $u(0, y) = 2$

(II) $u(1, y) = 2$

(III) $u(x, 0) = 2$

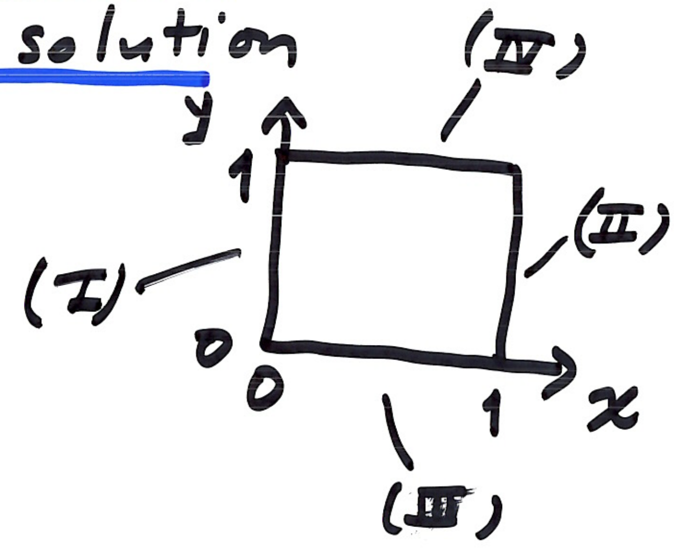
(IV) $u(x, 1) = 2$



full sol: $u(x, y) = 2$ for all x, y

Laplace's eq + Dirichlet b.c.

$\Longrightarrow$ unique solution



(☆)

$$\cancel{\frac{\partial}{\partial t}} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (*)$$

(I) $u(0, y) = P(y)$

(II) $u(1, y) = Q(y)$

(III) $u(x, 0) = R(x)$

(IV) $u(x, 1) = S(x)$

Suppose that system (✱) has 2 sols.

u_1 and u_2

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \quad (*1)$$

(I) $u_1(0, y) = P(y)$

(II) $u_1(1, y) = Q(y)$

(III) $u_1(x, 0) = R(x)$

(IV) $u_1(x, 1) = S(x)$

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \quad (*2)$$

$u_2(0, y) = P(y)$

$u_2(1, y) = Q(y)$

$u_2(x, 0) = R(x)$

$u_2(x, 1) = S(x)$

Define $u_* \equiv u_2 - u_1$

$(*2) - (*1)$: $\dfrac{\partial^2 u_*}{\partial x^2} + \dfrac{\partial^2 u_*}{\partial y^2} = 0$ ✓

✓ b.c. (I): $u_*(0,y) = u_2(0,y) - u_1(0,y) = P(y) - P(y) = 0$

✓ (II) $u_*(1,y) = \dots\dots = 0$

✓ (III) $u_*(x,0) = \dots = 0$

✓ (IV) $u_*(x,1) = \dots = 0$

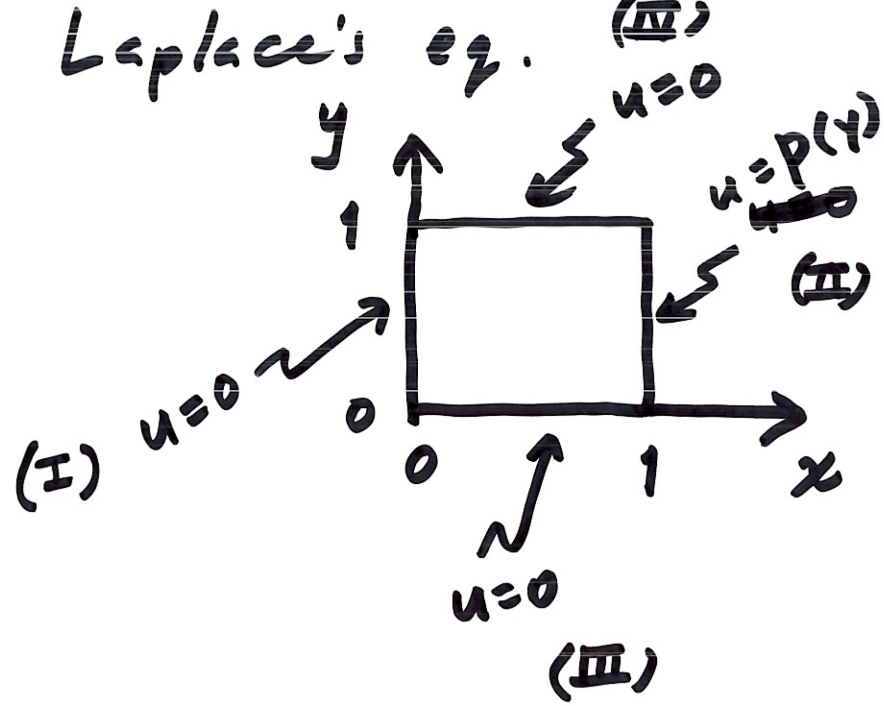
$\Rightarrow$ $u_*(x,y) = 0$ everywhere

$\Rightarrow$ $u_1 \equiv u_2$

<u>Solution is unique!</u>

(If b.c. is of <u>Dirichlet type</u>)

End-to-end sol. of Laplace's eq. w/ Dirichlet b.c. (simplest case)



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—} (*)$$

(I) $u(0, y) = 0$ ✓

(II) $u(1, y) = p(y)$

(III) $u(x, 0) = 0$

(IV) $u(x, 1) = 0$

↳ homogeneous subsystem

<u>Sep of var.</u>

$u \sim G(x)H(y)$

(*)
$$\frac{\partial^2 GH}{\partial x^2} + \frac{\partial^2 GH}{\partial y^2} = 0$$

$$HG'' + G\ddot{H} = 0$$

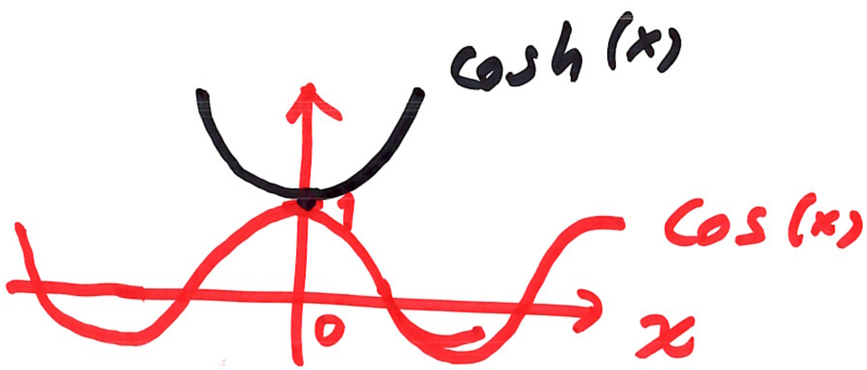
$$-\frac{G''}{G} = \frac{\ddot{H}}{H} = c$$

$( )' \equiv \frac{d( )}{dx}$

$\dot{( )} \equiv \frac{d( )}{dy}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$



sin(x)

cos(x) $\quad u'' = -u$

sinh(x)

cosh(x) $\quad u'' = u$

$$\ddot{H} = cH$$

$$G'' = -cG$$

sep. of var on b.c. (III), (IV)

$$H(0) = 0, \quad H(1) = 0$$

$$\left\{ \begin{array}{l} \boxed{\ddot{H} = cH \quad H(0)=0,\ H(1)=0} \\ G'' = -cG \end{array} \right.$$

Nothing new! (Lec 4, 5)

$c_n = -(n\pi)^2$ $n = 1, 2, 3$

$H_n(y) = \sin(n\pi y)$

* if b.b. is $\left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\}$ in one dir. the b.b. in the other dir has to be $\left\{ \begin{array}{l} \sinh \\ \cosh \end{array} \right\}$

$$G_n'' = -c_n G_n = -(-(n\pi)^2) G_n = +(n\pi)^2 G_n$$

$$\Rightarrow \quad G_n(x) = A_n \cosh(n\pi x) + B_n \sinh(n\pi x)$$

b.c. (I) is also separable, so let's use it!

$$u(0,y) = 0 \quad \Longrightarrow \quad G(0) = 0$$

$$0 = G_n(0) = A_n \underbrace{\overset{\cosh}{\cancel{\cos}}(0)}_{1} + B_n \underbrace{\sinh(0)}_{0}$$

$$\Rightarrow \quad A_n = 0 \quad \text{for all } n.$$

$$\Rightarrow \quad G_n(x) = \sinh(n\pi x)$$

don't care what B_n is $\rightarrow$ set to 1

$$u_n(x,y) \sim G_n(x)\,H_n(y)$$

	G	H	u
$n=1$	$\sinh(\pi x)$	$\sin(\pi y)$	$u_1(x,y) = \sinh(\pi x)\sin(\pi y)$
$n=2$	$\sinh(2\pi x)$	$\sin(2\pi y)$	$u_2(x,y) = \cancel{\sin(2\pi x)}$ $\sinh(2\pi x)\sin(2\pi y)$
	$\vdots$	$\vdots$	$\vdots$

$$\frac{\partial^2 u_1}{\partial x^2} = -\frac{\partial^2 u_1}{\partial y^2}$$

$$\frac{\partial^2 u_2}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \text{etc. . .}$$

Full sol.

$$u(x,y) = \sum_{n=1}^{\infty} a_n u_n(x,y)$$

$$= \sum_{n=1}^{\infty} a_n \sinh(n\pi x)\sin(n\pi y) \quad \text{—— } (**)$$

Want to use b.c. (II) to determine $\{a_n\}$

match

$u(1,y) = P(y) \Leftarrow$ given

set x to 1 in $(**)$

=

$$\underbrace{P(y) = u(1,y)}_{\text{b.c. (II)}} = \underbrace{\sum_{n=1}^{\infty} a_n \sinh(n\pi)\sin(n\pi y)}_{\text{Setting } x \text{ to } 1 \text{ in } (**)}$$

$$\sum_{n=1}^{\infty} \underbrace{\boxed{a_n \sinh(n\pi)}}_{\equiv A_n} \sin(n\pi y) = P(y)$$

(Lec 4,5)

$$A_n = 2\int_0^1 P(y)\sin(n\pi y)\,dy$$

$$\Rightarrow \quad \underline{\underline{a_n = \frac{2}{\sinh(n\pi)}\int_0^1 P(y)\sin(n\pi y)\,dy}}$$

#