Lecture 11

2/15

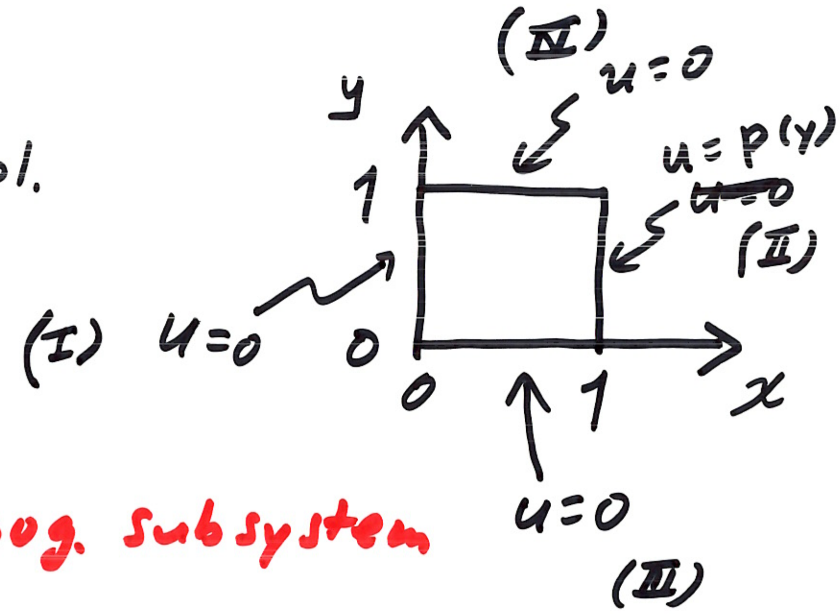
Hw1 Q2, Q3: "How do I know my solution is correct?"

* plug the solution back to PDE and all b.c.'s → should match all !

~~*~~

Laplace's eq.

Recap: (Lec 10) End-to-end sol.



old b.c (I)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ✓$$

(I) $u(0,y)=0$

✓ (II) $u(1,y)=P(y)$

(III) $u(x,0)=0$ ✓

(IV) $u(x,1)=0$ ✓

→ homog. subsystem

→ sep of var works.

Nothing new

Same approach works even if b.c. ≠ (I) → $u(0,y)=Q(y)$

New b.c. (I)

New example

Can still do sep. of var. on PDE, b.c. (III), (IV)

(cf. Lec 10 — same procedure)

$u \sim G(x)H(y)$

$$\boxed{\ddot{H} = cH \quad H(0)=0 \quad H(1)=0}$$

$(\dot{\ }) = \frac{d(\ )}{dy}$

$(\ )' = \frac{d(\ )}{dx}$

solve

Lec 4,5

$G'' = -cG$ $\cancel{G'' = cG}$

$c_n = -(n\pi)^2, \; n = 1, 2, 3, \ldots$

$H_n(y) = \sin(n\pi y)$

$$\Rightarrow \quad G_n'' = -c_n G_n = +(n\pi)^2 G_n$$

$$G_n(x) = A_n \cosh(n\pi x) + B_n \overset{\sinh}{\cancel{\sin}}(n\pi x)$$

* In the example in Lec 10, we eliminate A_n by invoking b.c. (I) $[u(0,y)=0]$, which we can no longer do here!

$$u_1(x,y) = G_1(x)\,H_1(y)$$
$$= A_1 \cosh(\overset{1}{\cancel{n}}\pi x)\sin(\overset{1}{\cancel{n}}\pi y) + B_1 \sinh(\overset{1}{\cancel{n}}\pi x)\sin(\overset{1}{\cancel{n}}\pi y)$$

$$u_2(x,y) = G_2(x)\,H_2(y)$$
$$= \dots$$

used PDE,
b.c. III, IV

Full sol.

still have
b.c. I, II

$$u(x,y) = \sum_{n=1}^{\infty} a_n u_n(x,y)$$

$$= \sum_{n=1}^{\infty} \underline{a_n A_n} \cosh(n\pi x)\sin(n\pi y) + \underline{a_n B_n} \sinh(n\pi x)\sin(n\pi y)$$

redundant
let's call it A_n

redundant
let's call it B_n

New notation

$$\rightarrow = \sum_{n=1}^{\infty} A_n \cosh(n\pi x)\sin(n\pi y) + B_n \sinh(n\pi x)\sin(n\pi y)$$

(*)

Solve $\{A_n\}$ and $\{B_n\}$ using b.c. (I), (II).

known

b.c. (I) $u(0, y) = Q(y)$

b.c. (II) $u(1, y) = P(y)$

known

* set x to 0 in (*) to match

① — $$\sum_{n=1}^{\infty} \underbrace{[A_n \cdot 1 + B_n \cdot 0]}_{\alpha_n} \sin(n\pi y) = Q(y)$$

* set x to 1 in (*) to match

② — $$\sum_{n=1}^{\infty} [\underbrace{A_n \cosh(n\pi) + B_n \sinh(n\pi)}_{\beta_n}] \sin(n\pi y) = \cancel{P(y)}\, P(y)$$

$$\alpha_n = 2\int_0^1 Q(y) \sin(n\pi y)\, dy \quad ✓$$

$$\beta_n = 2\int_0^1 P(y) \sin(n\pi y)\, dy \quad ✓$$

Solve a 2×2 linear algebra problem:

$$\begin{cases} 1 \cdot A_n + 0 \cdot B_n = \alpha_n \\ \cosh(n\pi) A_n + \boxed{\sinh(n\pi) B_n = \beta_n} \end{cases}$$

Lec 10

✓

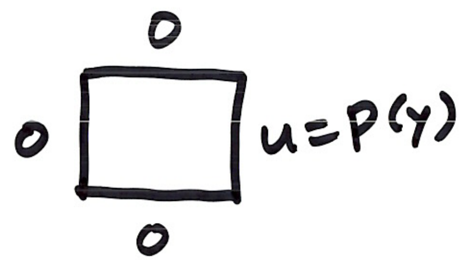
$\Rightarrow$ obtain A_n, B_n for all n

\#

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{w/ Dirichlet b.c.}$$

Lec 10

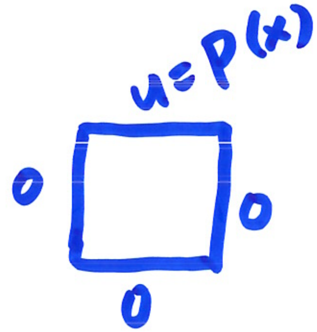
Case (I)



$\Rightarrow$ Sol. ~~u(x,y)= f(~~

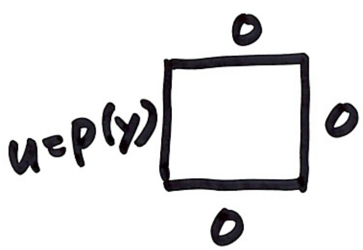
$u(x,y) = \underline{f(x,y)}$

Case (II)



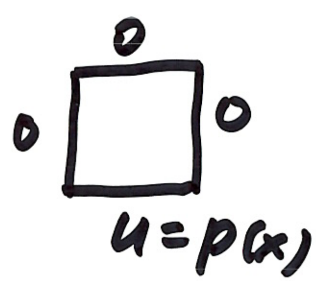
Sol. $u(x,y) = f(y,x)$ (I) swap $x \leftrightarrow y$

Case (III)



$\Rightarrow u(x,y) = f(1-x, y)$ (I) $x \leftarrow 1-x$

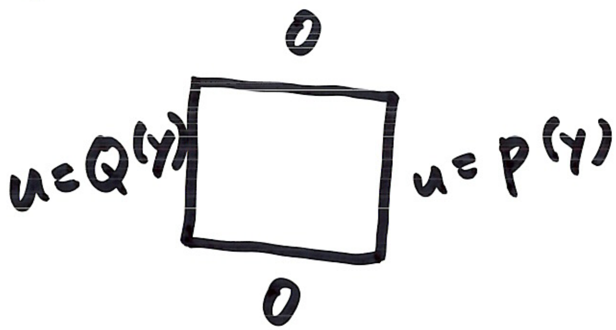
Case (IV)



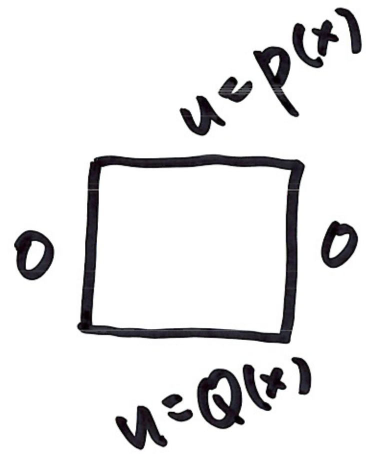
$\Rightarrow u(x,y) = f(1-y, x)$ (II) $y \leftarrow 1-y$

Lec 11

Case (I)



Case (II)



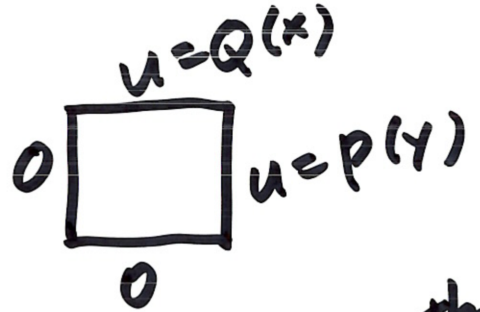
Sol.

$$u(x,y) = F(x,y)$$

swap $x \longleftrightarrow y$

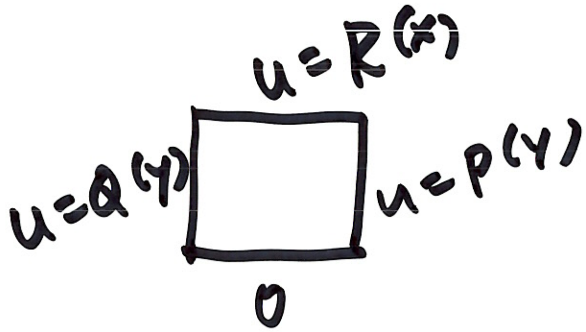
$$u(x,y) = F(y,x).$$

BUT, what about ...

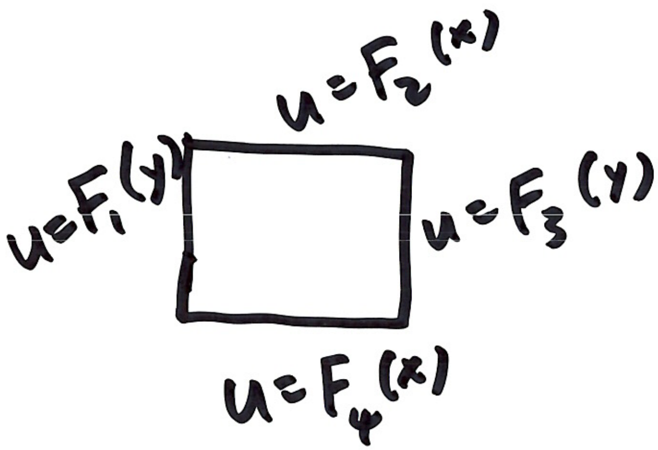


sep of var <u>fails</u>

3 other variation



"



"

Superposition (of the "responses to boundary forcing")

Full system:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

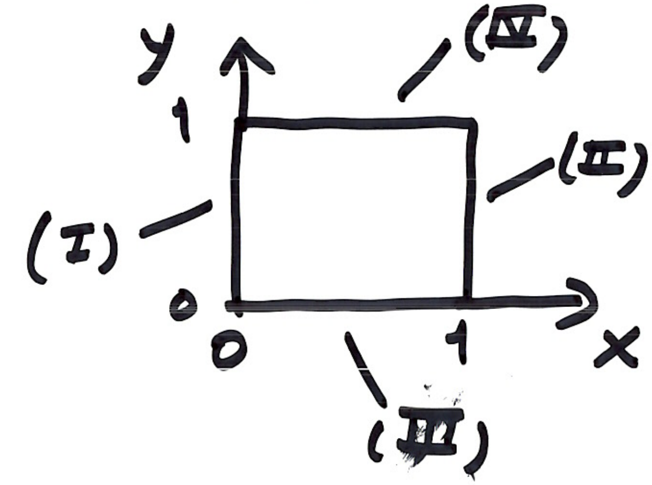
(I) $u(0,y) = F_1(y)$

(II) $u(1,y) = F_2(y)$

(III) $u(x,0) = F_3(x)$

(IV) $u(x,1) = F_4(x)$

— (★)



Split it into 4 independent system for u_1, u_2, u_3, u_4, each in response to only 1 non-zero b.c.

or split it into 2 systems, u_1, u_2 each in response to 2 b.c.'s in the same direction.

etc. etc.

Sys (I)

$$\frac{\partial^2 u_I}{\partial x^2} + \frac{\partial^2 u_I}{\partial y^2} = 0$$

(I) $u_I(0, y) = F_1(y)$

(II) $u_I(1, y) = 0$

(III) $u_I(x, 0) = 0$

(IV) $u_I(x, 1) = 0$

Sys (II)

$$\frac{\partial^2 u_{II}}{\partial x^2} + \frac{\partial^2 u_{II}}{\partial y^2} = 0$$

(I) $u_{II} = 0$

(II) $u_{II}(1, y) = F_2(y)$

(III) $u_{II} = 0$

(IV) $u_{II} = 0$

Sys (III)

Sys (IV)

$u \equiv u_I + u_{II} + u_{III} + u_{IV}$

$\Rightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ✓

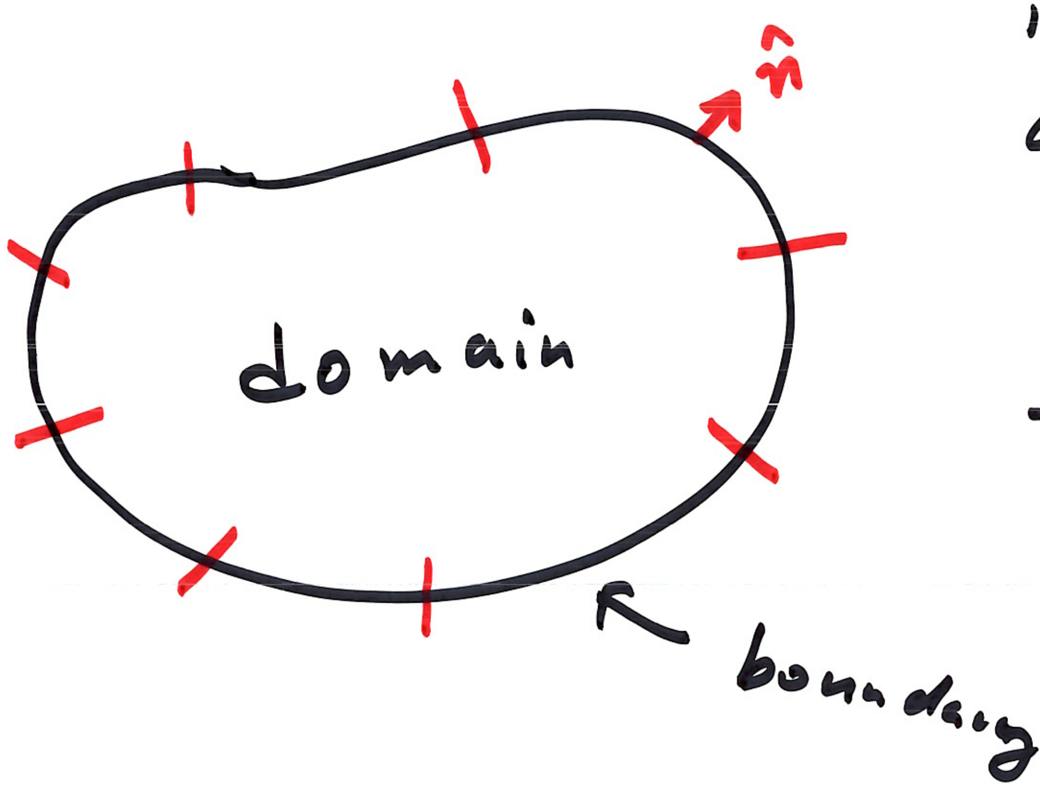
$\Rightarrow$ u also satisfies all 4 original b.c.s

u is full sol. of original sys.

We have shown, by direct construction, that the solution of Laplace's eq + Dirichlet b.c. exists and is unique

#.

Laplace's eq + "<u>Neumann b.c.</u>"



imposing the value of the derivative of u <u>in the direction normal to the boundary</u> to the boundary.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(I) $u_x(0, y) = P(y)$

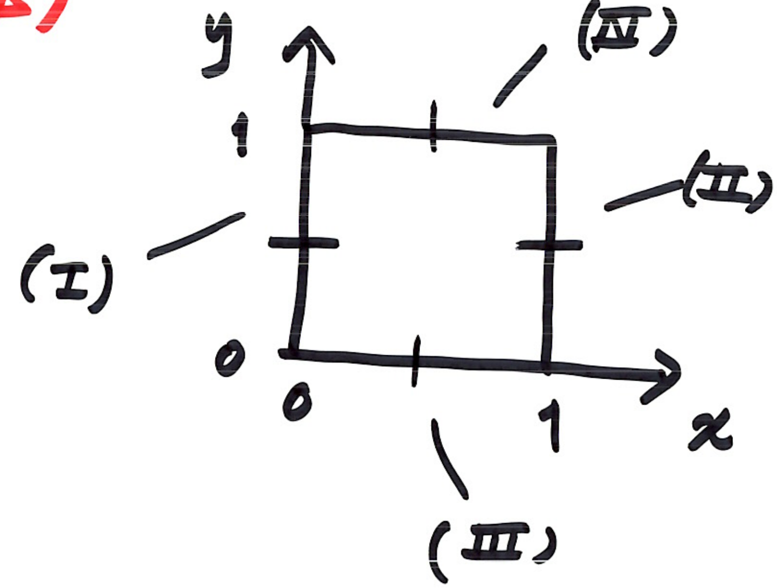
(II) $u_x(1, y) = Q(y)$

(III) $u_y(x, 0) = R(x)$

(IV) $u_y(x, 1) = S(x)$

(☆☆)

→ Neumann b.c.



Claim: (☆☆) cannot have a unique solution

Proof

If u is a solution of (☆☆), then $u + C$ is also a solution.

C : const

Laplace's eq. + Neumann b.c.

either has <u>infinite may solutions</u>

or <u>no solution at all</u>

next (Lec 12)

Laplace's eq + Dirichlet b.c.

always has <u>unique sol.</u>