Lecture 12

* HW2 posted, due 3/5 (Sunday)

Recap (Lec 11)

End-to-end sol. for

Laplace's eq. + Dirichlet b.c.

impose "u" on the boundary

sol. ~~is~~ exists and is unique *

near the end of Lec 11:

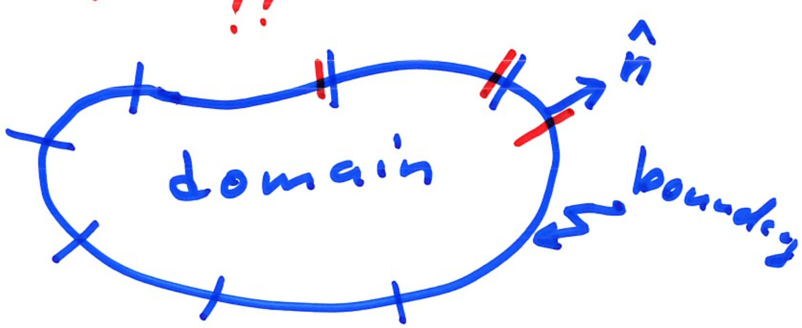
Laplace's eq. + Neumann b.c.

Cannot have unique sol. !!

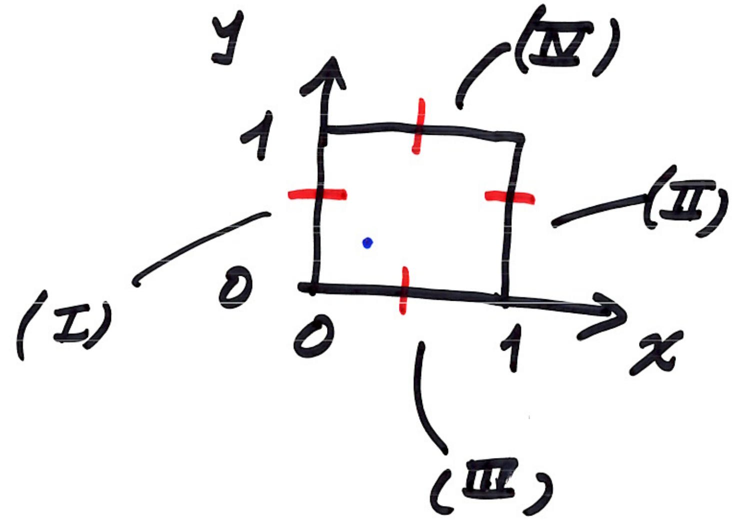
impose

impose derivative of u on the boundary

in the direction normal to



End of Lec 11



☆
$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \text{— } (*) \\ \text{(I)} \ u_x(0, y) = F_1(y) \\ \text{(II)} \ u_x(1, y) = F_2(y) \\ \text{(III)} \ u_y(x, 0) = F_3(x) \\ \text{(IV)} \ u_y(x, 1) = F_4(x) \end{cases}$$

if

Say, we find a solution of ☆ $u(x, y)$

Then $\hat{u}(x, y) = u(x, y) + C$ is also a solution of ☆

any const!

☆ has either infinitely many sols.

(or no solution) ← ?

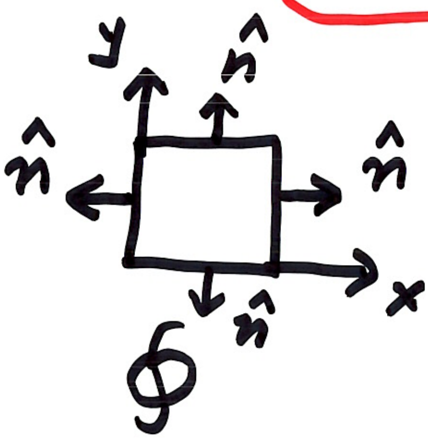
$$\int_0^1\int_0^1 (*)\, dx\, dy$$ area integral of $(*)$

$$\underbrace{\int_0^1\int_0^1 \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\} dx\, dy}_{\parallel} = \int_0^1\int_0^1 0\, dx\, dy = 0$$

$$\int_0^1 \left[\int_0^1 \frac{\partial^2 u}{\partial x^2} dx \right] dy + \int_0^1 \left[\int_0^1 \frac{\partial^2 u}{\partial y^2} dy \right] dx$$

$$= \int_0^1 [u_x(1,y) - u_x(0,y)]\, dy + \int_0^1 [u_y(x,1) - u_y(x,0)]\, dx$$

$$= \int_0^1 [F_2(y) - F_1(y)]\, dy + \int_0^1 [F_4(x) - F_3(x)]\, dx \quad (\star\star)$$



Solvability condition:

$$\boxed{\star\star = 0} \Rightarrow$$ ~~solution exists~~ infinite many sol.

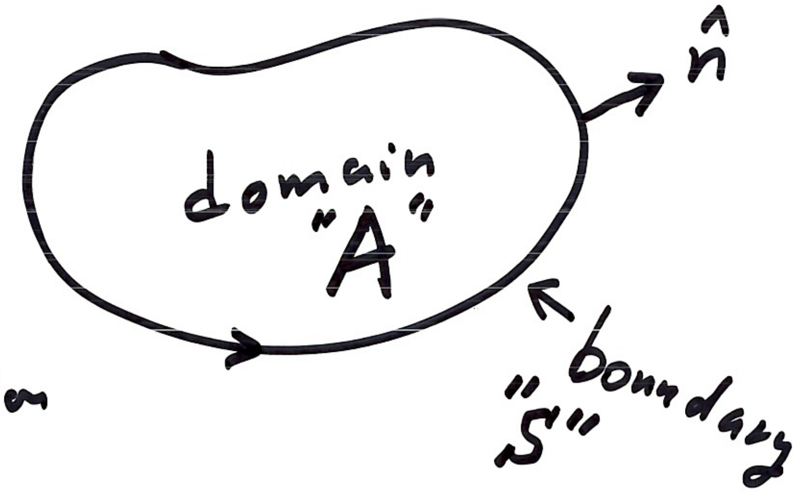
if $\star\star \neq 0 \Rightarrow$ NO SOLUTION!

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$(***) \quad \boxed{\nabla \cdot \nabla u = 0}$$

Laplace's eq.

(cf. derivation of 2-D Heat eq.)



Neumann b.c.

impose $\underline{\nabla u \cdot \hat{n}} = f$ on the boundary

heat flux $\propto$ going in/out of the domain

f = given.

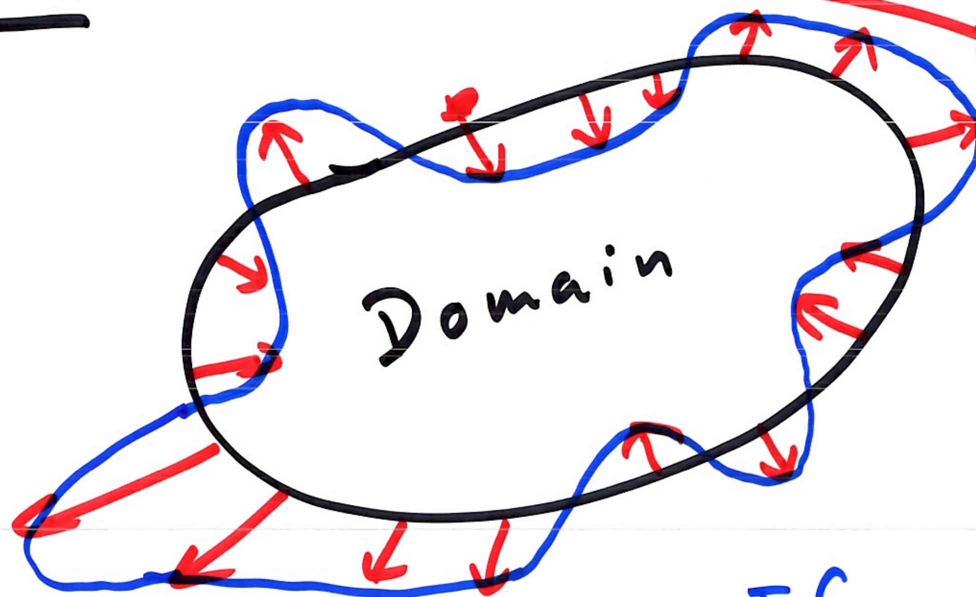
$$\iint_A (***)\, dA$$

$$\iint_A \nabla \cdot \nabla u \, dA = 0$$

$$= \oint_S (\nabla u) \cdot \hat{n} \, dl$$

$$= \oint_S f \, dl$$

Laplace's eq. governs the steady sol. of 2-D Heat eq. → impose heat flux at boundary



only when in = out

$\Rightarrow$ steady sol.

If total heat flux crossing the boundary $\neq 0$

$\Rightarrow$ total energy $\nearrow +\infty$ or $\searrow -\infty$

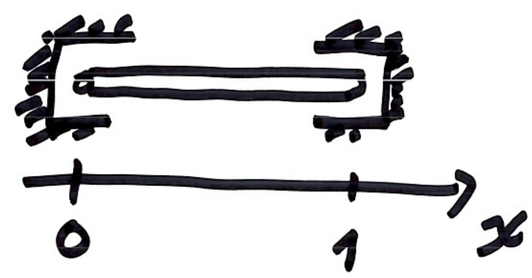
$\Rightarrow$ NO STEADY SOL!

$\Rightarrow$ Laplace's eq. has NO SOL!

Recall (<u>Lec 7</u> ?)

1-D Heat eq. (impose heat flux in x-dir)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \quad (*) \\ \left.\begin{aligned} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{aligned}\right\} \\ u(x,0) = p(x) \end{cases}$$



heat flux = 0 at $x = 0, 1$

total energy

$$E(t) \equiv \int_0^1 u(x,t)\,dx$$

$$\frac{dE}{dt} \equiv \frac{d}{dt}\int_0^1 u\,dx \overset{\text{Calc 101}}{=} \int_0^1 \frac{\partial u}{\partial t}\,dx \overset{(*)}{=} \int_0^1 \frac{\partial^2 u}{\partial x^2}\,dx$$

$$= u_x(1,t) - u_x(0,t) \overset{\text{b.c.}}{=} 0$$

$$E(t) = \text{Const} = E(0) = E(\infty)$$

$$\text{“}\int_0^1 p(x)\,dx = \text{''} = \text{“}a_0\text{''}$$

What if

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u_x(0,t) = A \\ u_x(1,t) = B \\ u(x,0) = p(x) \end{cases}$$

steady sol. $u_s(x)$

$$\frac{d^2 u_s}{dx^2} = 0$$

$u_s'(0) = A \quad u_s'(1) = B$

1-D Laplace's eq.

$$\Rightarrow \quad \frac{dE}{dt} = \dots = u_x(1,t) - u_x(0,t) = B - A$$

$$E(t) = E(0) + (B-A)\,t$$

* If $B > A \Rightarrow E(t) \to \infty$ as $t \to \infty$

$B < A \Rightarrow E(t) \to -\infty$ "

No steady sol. !!

Only $B = A \Rightarrow$ steady sol.

$\boxed{B - A = 0}$

Solvability cond. in 1-D

Laplace's eq + Dirichlet b.c.

$\longrightarrow$ sol. exists & is unique

Laplace's eq + Neumann b.c.

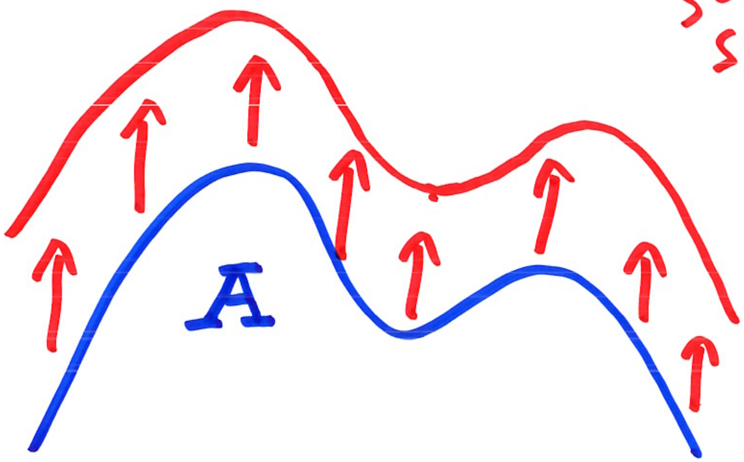
$\longrightarrow$ ~~so~~ infinite many sol. (all differ by <u>a</u> <u>const.</u>)

if solvability condition ✓

NO SOLUTION

if solvability condition ✗

$u(x,y)$



same slope
same curvature

lift up by $\underline{10}$

1 sol.

Mixed b.c.'s ?

(e.g. HW2-Q3)

→ find the full sol.

you will see !

HW2
Q2-b

Even in the case of
pure ~~Di~~ Neumann or Dirichlet b.c.
you will see as well
by brute force solution
(by separation of var. etc.!)

If multiple sol. exist, then some expansion coefficients will remain undetermined.