Lecture 13

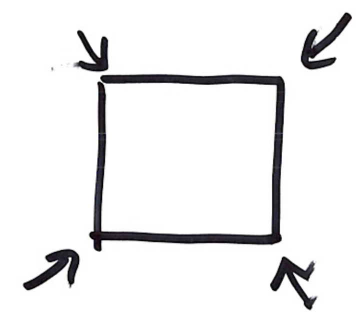
2/22

Tips on HW2

* "Splitting" is never needed for all probs in HW2

All can be solved with standard technique of **sep of var.** ~~*~~

except **Q4** actually simple



* **Solvability condition** → check if sol exist

Laplace's eq + Neumann

solutions

Q2

Tip on HW2-Q3

sep of var $\quad u \sim G(x)H(y)$

$\dashrightarrow$ Need to solve

$$\ddot{H} + p\dot{H} + qH = 0$$

(p, q some constant)

already discussed before

$$H(y) \sim e^{\alpha y}$$

$$\Rightarrow \dot{H} \sim \alpha e^{\alpha y}, \ \ddot{H} \sim \alpha^2 e^{\alpha y}$$

$$\alpha^2 + p\alpha + q = 0$$

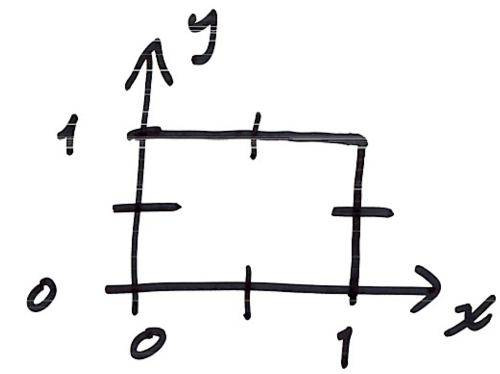
2 roots $\alpha_1 \quad \alpha_2$

$$H(y) = Ae^{\alpha_1 y} + Be^{\alpha_2 y}$$

Explicit examples Laplace's eq. + Neumann b.c.

E1

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ u_x(0, y) = 0 \\ u_x(1, y) = 0 \\ u_y(x, 0) = 0 \\ u_y(x, 1) = 0 \end{cases}$$



$u(x,y) \equiv 0$ is a sol. ✓

$u(x,y) = 0 + \underline{\underline{C}} = C$

infinite many

test solvability

$$\int_0^1 (0-0)\,dx + \int_0^1 (0-0)\,dy = \underline{\underline{0}} \quad ✓$$

E2

$$\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{— } (*) \ \checkmark$$

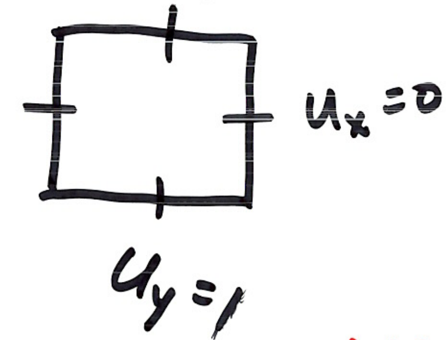
$$u_x(0,y) = 0 \quad \text{— } (\mathrm{I}) \ \checkmark$$

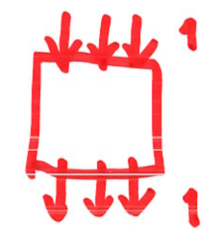
$$u_x(1,y) = 0 \quad \text{— } (\mathrm{II}) \ \checkmark$$

homog. sub system.

$$u_y(x,0) = 1 \quad \text{— } (\mathrm{III})$$

$$u_y(x,1) = 1 \quad \text{— } (\mathrm{IV})$$





Solvability ?

$$\int_0^1 (0-0)\,dx + \int_0^1 (1-1)\,dx = 0 \quad \checkmark$$

anticipate infinite many sols !

Full sol. :

sep of var on $\underline{(*)}$, $\underline{(\mathrm{I})}$, $\underline{(\mathrm{II})}$ $\quad u \sim G(x)H(y)$

$(\ )' \equiv \frac{d(\ )}{dx}$

$(\dot{\ }) \equiv \frac{d(\ )}{dy}$

$$\begin{cases} \boxed{G'' = cG \qquad G'(0) = 0 \qquad G'(1) = 0} & (**) \\ \ddot{H} = -cH \end{cases}$$

Lec 6, 7
HW1 Q1a

(**): $c = 0\ ,\ -(n\pi)^2\ ,\ n = 1, 2, 3, \dots$ also $G_0(x) = 1$, $G_n(x) = \cos(n\pi x)$

$\Rightarrow$ for $c = 0$ $\ddot{H}_0 = 0 \Rightarrow H_0(y) = A_0 y + B_0$

for $c > 0$ $\ddot{H}_n = -c_n H_n = +(n\pi)^2 H_n$

$$H_n(y) = A_n \cosh(n\pi y) + B_n \sinh(n\pi y)$$

Observing b.c. III, IV, only $c = 0$ matters

$\Rightarrow$ • only b.b. that will survive in full sol. $u_0(x,y) = G_0(x) H_0(y) = 1 \cdot (A_0 y + B_0)$

Full sol: $u(x,y) = a_0\, u_0(x,y)$
$= A_0 y + B_0$

$a_0 A_0 \to A_0$ redundant
$a_0 B_0 \to B_0$

If you insist on

$u(x,y) = \cancel{a_0 \ldots} \underline{A_0 y + B_0} + \cancel{\sum_{n=1}^{\infty} \left[A_n \cosh(n\pi y) + B_n \sinh(n\pi y) \right]} \cos(n\pi x)$

then, all $A_n = B_n = 0$ anyway!!

Full sol: $u(x,y) = A_0 y + B_0$

~~b.c. (III)~~ $\Rightarrow u_y(x,y) = A_0$

b.c. (III) $\Rightarrow u_y(x,0) = 1 \Rightarrow A_0 = 1$

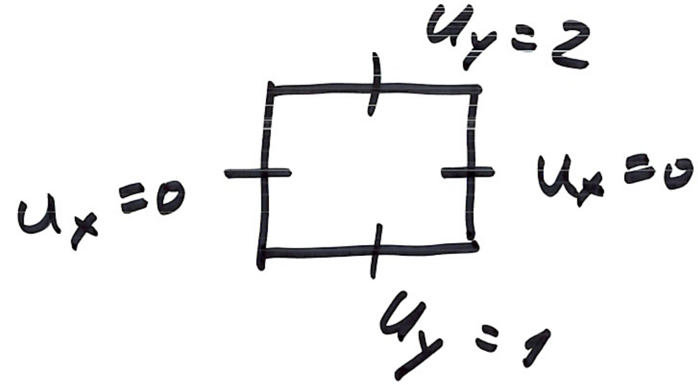
b.c. (IV) $\Rightarrow u_y(x,1) = 1 \Rightarrow A_0 = 1$

B_0 is not determined.

Full sol: $\boxed{u(x,y) = y + B_0}$ any const!

infinite many sols!

E3 $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$



$u_x(0, y) = 0$

$u_x(1, y) = 0$

$u_y(x, 0) = 1$ — III

$u_y(x, 1) = 2$ — (IV)

test solvability

$$\int_0^1 (0 - 0)\,dx + \int_0^1 (2 - 1)\,dy = 1 \neq 0$$

NO SOLUTION!

Full sol: Repeat the procedure for E2 until the last step

Full sol $u(x, y) = A_0 y + B_0$

$\Rightarrow u_y(x, y) = A_0$

b.c. (III) $\Rightarrow A_0 = 1$

b.c. (IV) $\Rightarrow A_0 = 2$

Contradiction!

$\Rightarrow$ NO SOLUTION!

(1-D) Wave equation

Will derive in Lec14 ←

Hyperbolic PDE

1-D Heat eq.

Parabolic PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Laplace's eq.

Elliptic PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

General classification of 2nd-order linear PDE: $u(x, y)$

$$\underline{A}\, u_{xx} + \underline{B}\, u_{xy} + \underline{C}\, u_{yy} + D u_x + E u_y + F u = G$$

$\frac{\partial^2 u}{\partial x^2}$ $\quad$ $\frac{\partial^2 u}{\partial x \partial y}$

$A \rightarrow A(x,y)$

$B \rightarrow B(x,y)$

in general

$\underline{B^2 - 4AC}$
- > 0 hyperbolic
- $= 0$ parabolic
- < 0 elliptic

Laplace's eq.

$A = 1 \quad B = 0 \quad C = 1 \quad (D = E = F = G = 0)$

$B^2 - 4AC = -4 < 0$

1-D Heat "t" → "y"

$A = 1 \quad B = C = 0$

$B^2 - 4AC = 0$