Lecture 14

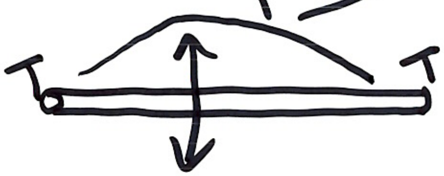
2/27

1-D Wave eq. "Hyperbolic"

vibration of a "1-D string"

thin

pluck

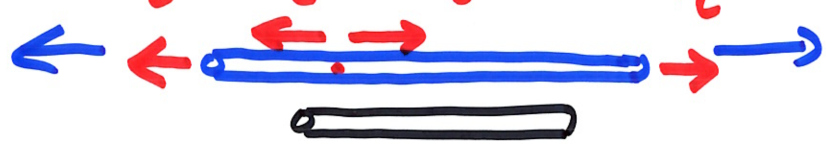


Lecture 1-14 scope of midterm

"tension"

$\neq$

τ τ τ τ



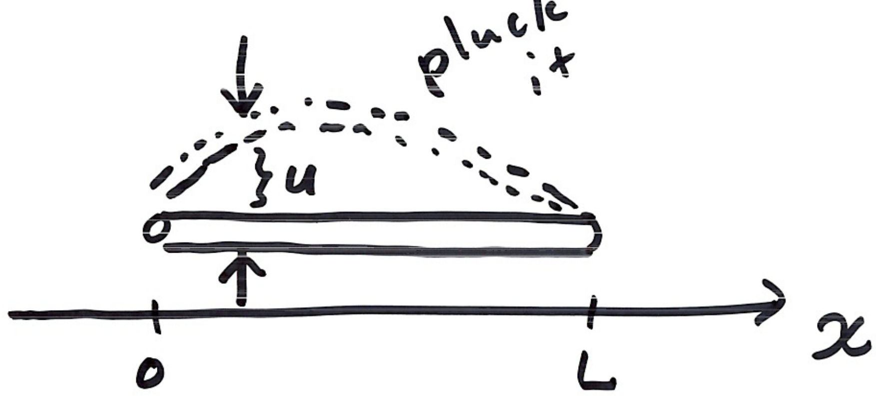
neutral configuration

string

force per unit area

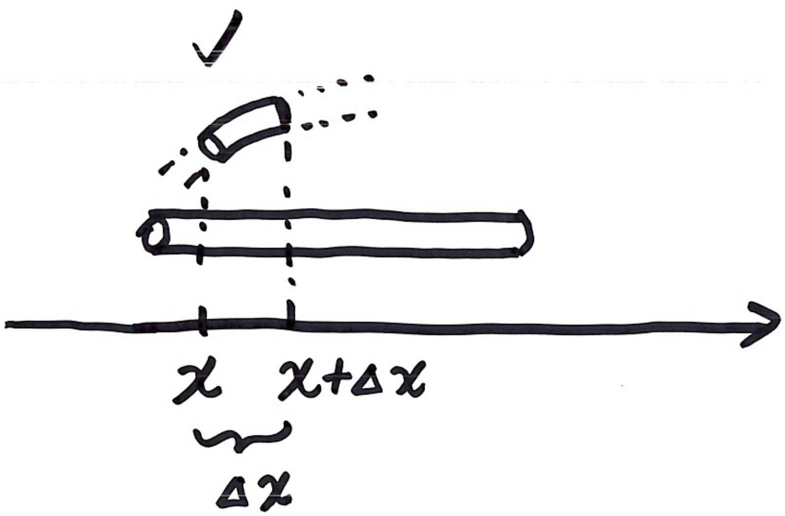
A : x-sec area

$\tau \cdot \underline{\underline{A}} =$ force

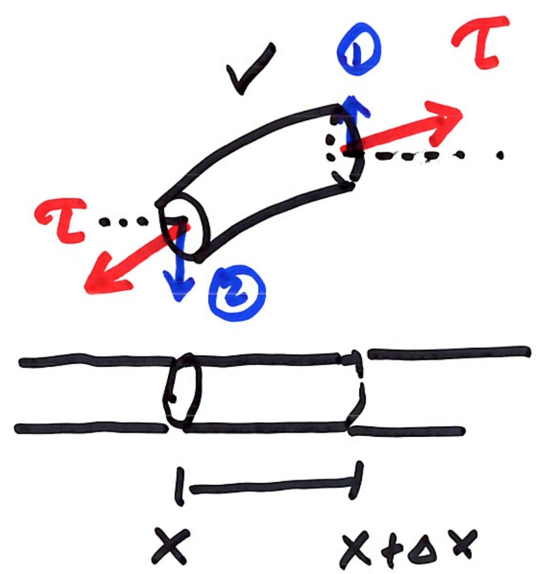


u: displacement

$u(x, t)$

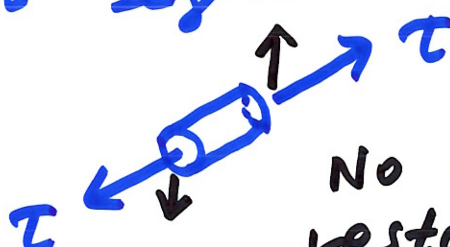


balance of force on the small segment

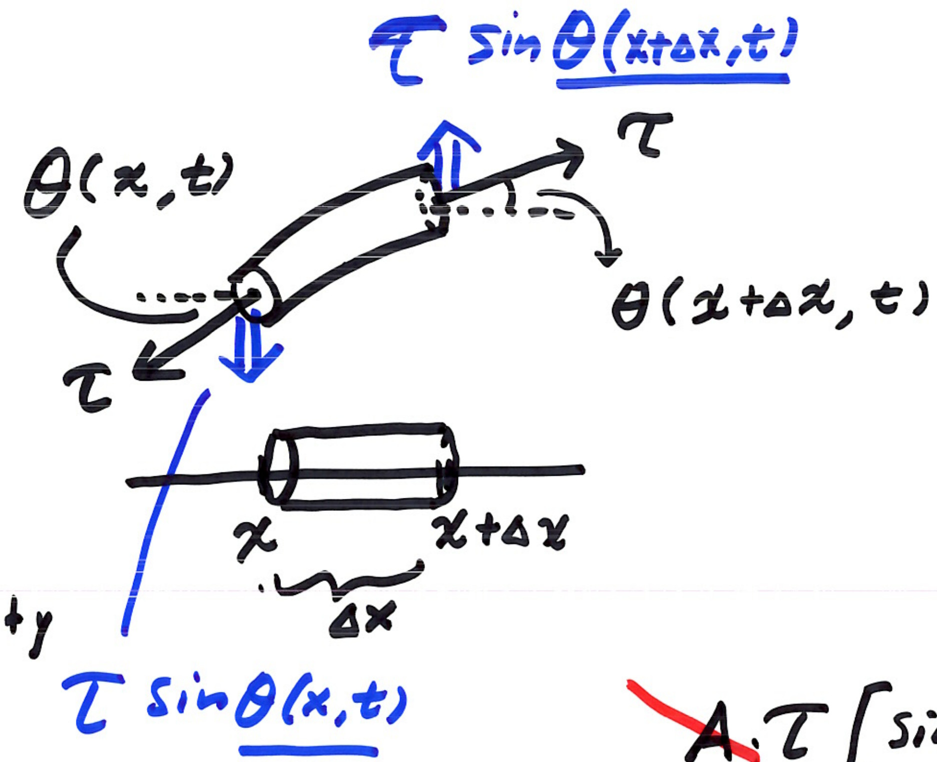


|②| > |①|

"straight" segment

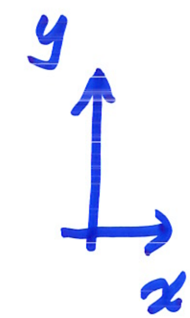


No net restoring force !!



ρ density

u is displacement in y dir



(in y dir)
Newton's law

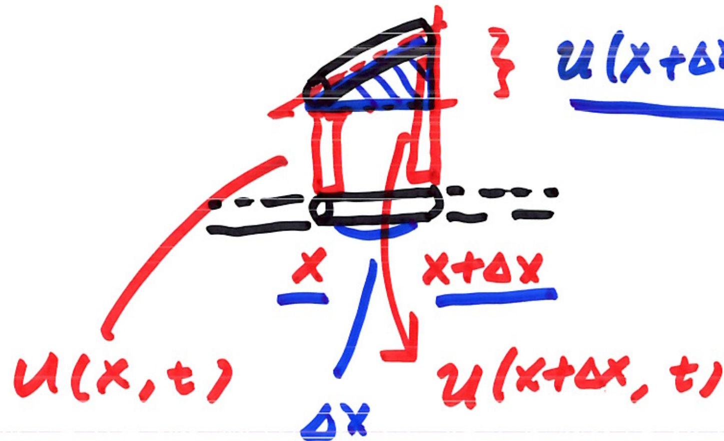
$$F = m \cdot a$$

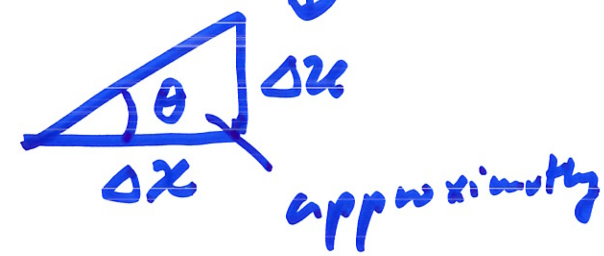
$$\cancel{A} \cdot \tau \left[\sin\theta(x+\Delta x, t) - \sin\theta(x,t)\right] = \underline{\underline{\rho \cdot \cancel{A} \cdot \Delta x}} \; \underline{\underline{\frac{\partial^2 u}{\partial t^2}}}$$

A : x-sec area

$$\Rightarrow \quad \frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho} \; \frac{\sin\theta(x+\Delta x, t) - \sin\theta(x,t)}{\Delta x} \quad \text{—} (*)$$

Small perturbation ($\theta \ll 1$)





$$\frac{\Delta u}{\Delta x} \approx \tan\theta$$

$\Delta x \to 0$ $$\frac{\partial u}{\partial x} \approx \tan\theta$$

If θ is small

$\Rightarrow$ $\tan\theta \sim \theta \sim \sin\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \approx \theta$$

small θ

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} \sim \sin\theta} \quad (**)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$\sin\theta \approx \theta$ for small θ

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$\cos\theta \approx 1$

plug (**) back to (*)

$$\Rightarrow \quad \frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho} \; \frac{\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x,\, t} - \left.\frac{\partial u}{\partial x}\right|_{x,\, t} \right]}{\Delta x}$$

$$\lim_{\Delta x \to 0} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho} \frac{\partial^2 u}{\partial x^2}}$$

1-D wave eq.

$$\frac{\tau}{\rho} > 0$$

let's call it

$$C^2 \equiv \frac{\tau}{\rho}$$

$$C \equiv \sqrt{\frac{\tau}{\rho}}$$

$$\frac{\partial^2 u}{\partial t^2} = \underbrace{C^2}_{\text{given positive const}} \frac{\partial^2 u}{\partial x^2}$$

rescale ...

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}}$$

$u(x, t)$

1-D wave

sep of var $u \sim G(x)H(t)$

$G'' = cG$ sin cos

$\ddot{H} = cH$ sin cos

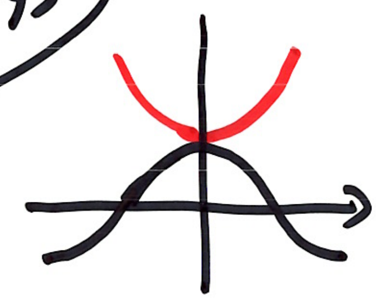
Compare w/ Laplace's eq.

$u(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \boxed{-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}}$$

sep of var

$u \sim G(x)H(y)$



$G(x) \sim$ sin cos $\qquad G'' = cG$

$\Rightarrow H(y) \sim$ sinh cosh $\qquad \ddot{H} = -cH$

Boundary condition

Hyperbolic type

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} \quad \text{---} \ (\star) \\[2ex] \text{(I)} \quad u(0,t)=0 \\ \text{(II)} \quad u(1,t)=0 \\ ?\ \text{(III)} \quad \underline{u(x,0)} = P(x) \quad \text{initial displacement} \\ ?\ \text{(IV)} \quad \underline{u_t(x,0)} = Q(x) \quad \text{'' ``velocity''} \end{cases}$$

Newton's 2nd law

$$\frac{d^2\vec{r}}{dt^2} = \frac{\vec{F}}{m}$$

$\underline{\vec{r}(0) = \Box}$ $\quad$ $\underline{\dot{\vec{r}}(0) = \Box}$

sketch of sol. (Nothing new in the technique for solution)

sep of var on (✱), (I), (II)

$u \sim G(x)H(t)$

$$\Rightarrow \begin{cases} \boxed{G'' = cG \quad G(0)=0 \quad G(1)=0} \\ \ddot{H} = cH \end{cases}$$

(Lec 4, 5,)

$c_n = -(n\pi)^2 \quad n = 1, 2, \dots$

$G_n(x) = \sin(n\pi x)$

$\Downarrow$

$$\ddot{H}_n = -(n\pi)^2 H_n$$

$$H_n(t) = A_n \cos(n\pi t) + B_n \sin(n\pi t)$$

$$u_n(x,t) = G_n(x) H_n(t)$$

Full sol:

$a_n A_n \to A_n$

$a_n B_n \to B_n$

$$* \quad u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left[A_n \cos(n\pi t) + B_n \sin(n\pi t) \right] \sin(n\pi x)$$

Use b.c. (III), (IV) to fix $\{A_n\}$, $\{B_n\}$