Lecture 15

The "method" from Lec 1-14 for linear system with homogeneous PDE + some homogeneous b.c.'s

(Ch. 8: nonhomog. system)
(Ch. 12: Some nonlinear PDEs)

sep of var on homog. subsystem → solve eigenvalue prob → eigenfunc. as b.b.'s → eigenfunction expansion → orthogonality relation ↳ all coefficients

$\not\equiv$

is the system "separable"?

Are the eigenfunctions orthogonal?

Are the eigenfunctions "complete"?

✓ Orthogonality relation ?

Revisit (Lec 4-7) examples

Lec: 4-5 $G'' = cG$ 1

$G(0)=0 \quad G(1)=0$

$\{\sin(n\pi x)\}$ $n=1,2,3,\ldots$

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 0 & \text{if } n \neq m \end{cases}$$

etc.

Lec 6-7 2

$G'' = cG$

$G'(0)=0 \quad G'(1)=0$

$\{1,\ \cos(n\pi x)\ \ n=1,2,3,\ldots\}$

HW1-Q1b 3

$G'' = cG$

$G'(0)=0 \quad G(1)=0$

$\left\{\cos\left(\frac{n\pi x}{2}\right)\ \ n=1,3,5,7,\ldots\right\}$

$0 \le x \le 1$

Proof of orthogonality (without knowing ~~the~~ detail of eig. func.)

$$\boxed{\frac{d^2G}{dx^2} = cG} \quad + \quad \text{b.c.}$$

$$\boxed{\mathcal{L} \equiv \frac{d^2}{dx^2}}$$

Let's say $G_n(x)$ and $G_m(x)$ are two eigenfunctions

$$\mathcal{L}G = cG$$

$$\mathcal{J} \equiv \int_0^1 (G_n \mathcal{L} G_m - G_m \mathcal{L} G_n)\, dx$$

$$\equiv \int_0^1 (G_n G_m'' - G_m G_n'')\, dx$$

$$\equiv \int_0^1 (G_n G_m')' - \cancel{G_n' G_m'} - [(G_m G_n')' - \cancel{G_m' G_n'}]\, dx$$

$$\equiv \int_0^1 [G_n G_m' - G_m G_n']'\, dx$$

$$= G_n(1) G_m'(1) - G_m(1) G_n'(1)$$

$$- [G_n(0) G_m'(0) - G_m(0) G_n'(0)] \quad \text{— } (*)$$

$(AB)' = AB' + A'B$

$AB' = (AB)' - A'B$

$G_n G_m''$

$= G_n (G_m')'$

$= (G_n G_m')' - G_n' G_m'$

b.c. from ① : $\underline{G(0)=0}$ $\underline{G(1)=0}$ $\Rightarrow$ $\mathcal{J}=0$

b.c. from ② : $\underline{G'(0)=0}$ $\underline{G'(1)=0}$ $\Rightarrow$ $\mathcal{J}=0$

b.c. from ③ : $\underline{G'(0)=0}$ $\underline{G(1)=0}$ $\Rightarrow$ $\mathcal{J}=0$

More generally,
consider b.c. :

$$(\star)\ \begin{cases} \underline{\alpha G(0) + \beta G'(0) = 0} \\ \gamma G(1) + \delta G'(1) = 0 \end{cases} \Rightarrow \begin{matrix} G'(0) = -\frac{\alpha}{\beta} G(0) \\ G'(1) = -\frac{\gamma}{\delta} G(1) \end{matrix}$$

① : $\alpha = 1$ $\beta = 0$
$\gamma = 1$ $\gamma = 0$

③ : $\alpha = 0$ $\beta = 1$
$\gamma = 1$ $\delta = 0$
etc.

$$(*) = G_n(1) \cdot \cancel{\left(-\frac{\alpha}{\beta}\right)} \cdot G_m(1) - G_m(1) \cancel{\left(-\frac{\alpha}{\beta}\right)} G_n(1)$$

$$- [\cancel{\cdot}\ \cancel{\cdot}]$$

$$= 0$$

with $(\star)$ as b.c.

$$\Rightarrow \underline{\mathcal{J} = 0}$$

Second half of the story

✓ $\mathcal{L} G_n = c_n G_n \qquad \mathcal{L} G_m = c_m G_m$

$$\Rightarrow \quad \mathcal{I} = \int_0^1 (G_n \underline{\mathcal{L} G_m} - G_m \underline{\mathcal{L} G_n})\, dx$$

$$= \int_0^1 (G_n \underline{c_m G_m} - G_m \underline{c_n G_n})\, dx$$

$$= (c_m - c_n) \int_0^1 G_n G_m \, dx$$

with b.c. ($\star$), $\mathcal{I} = 0$

$$\Rightarrow \quad (c_m - c_n) \int_0^1 G_n G_m \, dx = 0$$

$$\Rightarrow \quad \int_0^1 G_n G_m \, dx = 0 \quad \text{if } c_m \neq c_n$$

$\#$

Further generalization:

Sturm-Liouville system $\quad G(x) \quad a \le x \le b$

$p(a) \neq 0$
$p(b) \neq 0$

$$\left\{ \begin{array}{l} \frac{d}{dx}\left[p(x)\frac{d}{dx}\right]G + q(x)\,G = c \cdot r(x) \cdot G \\ \alpha G(a) + \beta G'(a) = 0 \\ \gamma G(b) + \delta G'(b) = 0 \end{array} \right.$$

otherwise "singular S-L system"

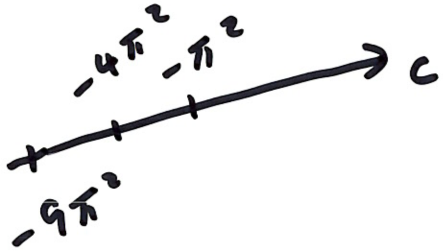
$G'' = cG$

special case w/

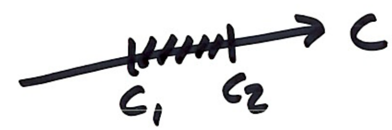
$p(x) = 1$
$q(x) = 0$
$r(x) = 1$

Sturm-Liouville Theorem: (Ch. 5)

S-L system has



* Discrete eigenvalues
* "Complete" eigenfunctions
  basis
* orthogonal eigenfunctions



S-L ~~system~~ Theorem : "scope" or "limit" of our "method"

If sep. of var $\longrightarrow$ S-L system $\Longrightarrow$ OK

Ex: $3 \le x \le 5 \quad t \ge 0$

$$\begin{cases} \dfrac{\partial u}{\partial t} = e^{2x}\dfrac{\partial^2 u}{\partial x^2} + 2e^{2x}\dfrac{\partial u}{\partial x} + \sin(x)\,u & \text{— } (*) \\ u(3,t) + 7\,u_x(3,t) = 0 & \text{— (I)} \\ 4\,u(5,t) + 16\,u_x(5,t) = 0 & \text{— (II)} \\ u(x,0) = \Phi(x) \end{cases}$$

homog. sub sys

sep of var:

$u \sim G(x)H(t)$

$\dot{H} = cH$

$$\underline{e^{2x}G'' + 2e^{2x}}\,G' + \sin(x)\,G = cG$$

$$\frac{d}{dx}\left(e^{2x}\frac{d}{dx}\right)G + \sin(x)\,G = cG \quad \checkmark \qquad \underline{\text{S-L}}$$

b.c.: (I) $\quad G(3) + 7G'(3) = 0 \quad \checkmark$

(II) $\quad 4G(5) + 16G'(5) = 0 \quad \checkmark$

Sketch of proof

$[AB]' = AB' + A'B$

$AB' = [AB]' - A'B$

$$\mathcal{L} \equiv \frac{d}{dx} p(x) \frac{d}{dx} + q(x)$$

$$\mathcal{J} \equiv \int_a^b (G_n \mathcal{L} G_m - G_m \mathcal{L} G_n)\, dx$$

$$\equiv \int_a^b \underbrace{G_n [(p G_m')'}_{} + \cancel{q G_m}] - G_m[(p G_n')' + \cancel{q G_n}]\, dx$$

$$= \int_a^b \underbrace{(G_n p G_m')' - \cancel{G_n' p G_m'}}_{} - [(G_m p G_n')' - \cancel{G_m' p G_n'}]\, dx$$

$$= \int_a^b [G_n p G_m' - G_m p G_n']'\, dx$$

$$= p(b)[G_n(b) G_m'(b) - G_m(b) G_n'(b)]$$
$$- p(a)[G_n(a) G_m'(a) - G_m(a) G_n'(a)]$$

with S-L b.c. $\Rightarrow \mathcal{J} = 0$

but

$$\mathcal{L} G_m = \cancel{C_m G_m}\ C_m r G_m$$
$$\mathcal{L} G_n = C_n r G_n$$

$$\mathcal{I} \equiv \int_a^b \left(G_n \underline{C_m} r G_m - G_m \underline{C_n} r G_n \right) dx$$

$$= (C_m - C_n) \int_a^b G_n G_m r(x)\, dx$$

$$\Rightarrow \quad \int_a^b G_n G_m \underline{\underline{r(x)}}\, dx = 0 \quad \text{if} \quad C_m \neq C_n$$

Special case (exception) X S-L

periodic b.c.

1-D Heat eq

$u(0,t) = -$

$u(1,t) = -$

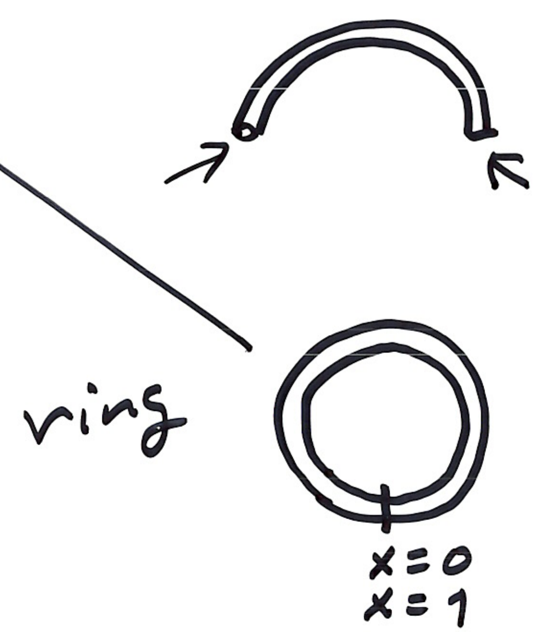
everything is the same at $x=0, 1$

$$\checkmark \begin{cases} u(0,t) = u(1,t) \\ u_x(0,t) = u_x(1,t) \end{cases}$$

$u_{xx}(0,t) = u_{xx}(1,t)$

$u_{xxx}(0,t) = u_{xxx}(1,t)$

$\vdots$



class of PDE + b.c.

reducible to S-L system by sep of var

$\downarrow$

custom-made eigenfunc.

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

outside the limit!

exception

periodic b.c.

+ more general differential operator

Ch. 3

Universal building blocks (depend on b.c. only)

bypass sep. of var.

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$u(0,t) = u(1,t)$

$u_x(0,t) = u_x(1,t)$

$\vdots$