Lecture 16

3/13

Midterm: Wednesday in class (6:00-7:15 PM)

* Open book, open note
* Computer & cellphone NOT allowed.

Might use "shuffled" questions
(multiple versions)

$\cancel{*}$

Recap (Lec 15) Ch. 5

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$
+ periodic b.c. ✓

The "method" from Lecture 1-14

PDE + b.c.

~~Str~~ ↓ sep of var

Sturm-Liouville system

restrictive in the form of differential operator

& b.c. need to be homogeneous

S-L PDE + periodic b.c.

Exception

Linear PDE + periodic b.c.

* free some restrictions on the form of differential operator

* universal building blocks !!

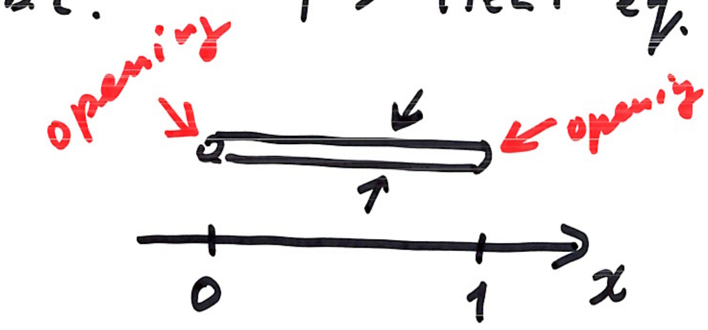
(Ch. 3)

Bypass separation of variables

System of PDE + periodic b.c.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{— (*)} \\ u(0,t) = u(1,t) & \text{(I)} \\ u_x(0,t) = u_x(1,t) & \text{(II)} \\ \cancel{u_{xx}(0,t) = u_{xx}(1,t)} \\ \cancel{u_{xxx}(0,t) = u_{xxx}(1,t)} \\ \vdots \\ u(x,0) = p(x) & \leftarrow \text{given} \end{cases}$$

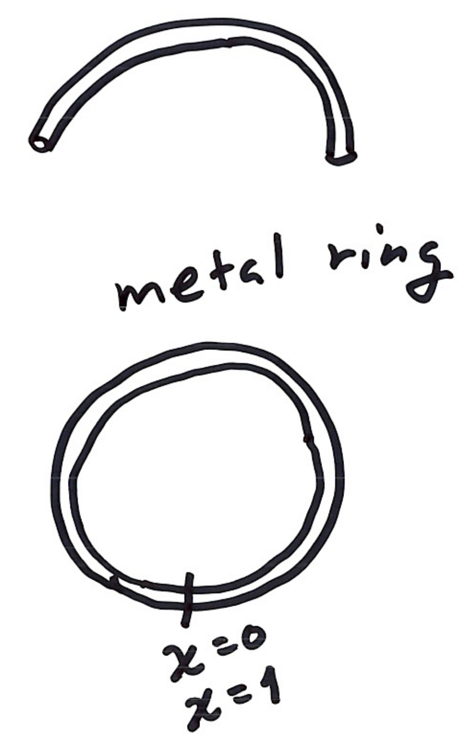
1-D Heat eq.



b.c.

$u(0,t) = \square$

$u(1,t) = \square$



Sep of var (one last time!)

$(*) \Rightarrow$ $\boxed{G'' = cG}$ $(\dot{H} = cH)$

$(\mathrm{I}) \longrightarrow$ $\boxed{G(0) = G(1)}$

$(\mathrm{II}) \longrightarrow$ $\boxed{G'(0) = G'(1)}$

$$u(x,t) \sim G(x)H(t)$$

b.c. (I) : $G(0)H(t) = G(1)H(t)$

$$\Rightarrow \quad \underline{[G(0) - G(1)]}\,\underline{H(t)} = 0$$

$\Rightarrow$ either $H(t) = 0$ X

or $\boxed{G(0) = G(1)}$

① If $\underline{c>0}$ $\quad G(x) = A\cosh(\sqrt{c}\,x) + B\sinh(\sqrt{c}\,x)$

$$\to G'(x) = \sqrt{c}\,A\sinh(\sqrt{c}\,x) + \sqrt{c}\,B\cosh(\sqrt{c}\,x)$$

b.c. (I) $\quad A\underbrace{\cosh(0)}_{1} + B\underbrace{\sinh(0)}_{0} = A\cosh(\sqrt{c}) + B\sinh(\sqrt{c})$

b.c. (II) $\quad \cancel{\sqrt{c}}A\underbrace{\sinh(0)}_{0} + \cancel{\sqrt{c}}B\underbrace{\cosh(0)}_{1} = \cancel{\sqrt{c}}A\sinh(\sqrt{c}) + \cancel{\sqrt{c}}B\cosh(\sqrt{c})$

$$\underbrace{\begin{pmatrix} 1-\cosh(\sqrt{c}) & -\sinh(\sqrt{c}) \\ -\sinh(\sqrt{c}) & 1-\cosh(\sqrt{c}) \end{pmatrix}}_{\mathbb{M}} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

goal: find c such that $\begin{pmatrix} A \\ B \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Need $\underline{\det(\mathbb{M}) = 0}$ in order to have

$\Rightarrow$ Need $[1-\cosh(\sqrt{c})]^2 - \sinh^2(\sqrt{c}) = 0$

skip some detail... only solution is $\underline{c=0}$ $\Rightarrow$ Contradiction!

2 if $c=0 \Rightarrow G''=0 \Rightarrow G(x) = Ax + B$

$\rightarrow G'(x) = A$

b.c. (I) $\quad A \cdot 0 + B = A \cdot 1 + B \Rightarrow A = 0$

b.c (II) $\quad A = A$

$\Rightarrow A = 0$ while B is undetermined ✓

$\Rightarrow c = 0$ is an eigenvalue. $\quad G_0(x) = \underline{\underline{B}}$

ok to set to 1

③ $c < 0$ $\quad G(x) = A\cos(\sqrt{-c}\,x) + B\sin(\sqrt{-c}\,x)$

$$\rightarrow G'(x) = -\sqrt{-c}\,A\sin(\sqrt{-c}\,x) + \sqrt{-c}\,B\cos(\sqrt{-c}\,x)$$

b.c. (I) $\quad A\underbrace{\cos(0)}_{1} + B\underbrace{\sin(0)}_{0} = A\cos(\sqrt{-c}) + B\sin(\sqrt{-c})$

$$-\cancel{\sqrt{-c}}\,A\underbrace{\sin(0)}_{0} + \cancel{\sqrt{-c}}\,B\underbrace{\cos(0)}_{1} = -\cancel{\sqrt{-c}}\,A\sin(\sqrt{-c}) + \cancel{\sqrt{-c}}\,B\cos(\sqrt{-c})$$

IM

$$\begin{pmatrix} 1-\cos(\sqrt{-c}) & -\sin(\sqrt{-c}) \\ \sin(\sqrt{-c}) & 1-\cos(\sqrt{-c}) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

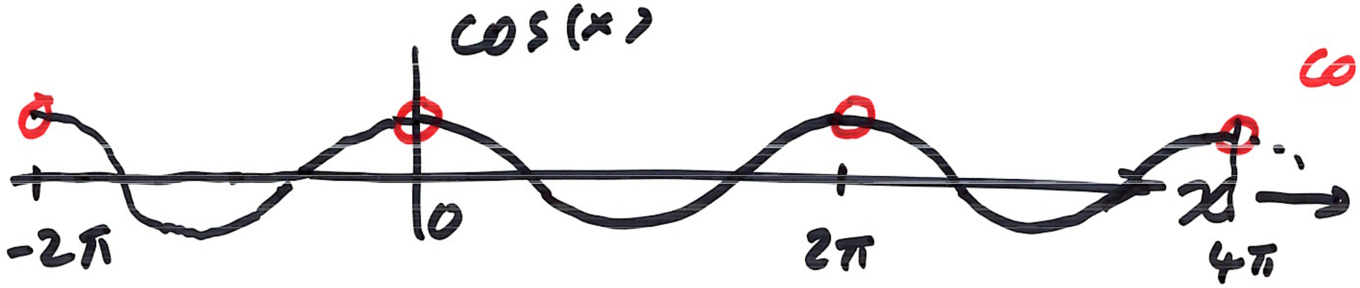
want $\det(IM) = 0$

want $\quad [1-\cos(\sqrt{-c})]^2 + \sin^2(\sqrt{-c}) = 0$

$$1 - 2\cos(\sqrt{-c}) + \underbrace{\cos^2(\sqrt{-c}) + \sin^2(\sqrt{-c})}_{1} = 0$$

$$\Rightarrow 2 - 2\cos(\sqrt{-c}) = 0$$

$$\Rightarrow \boxed{\cos(\sqrt{-c}) = 1}$$



$\cos(x) = 1$ at $x = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$

$$\Rightarrow \quad \sqrt{-c} = \cancel{0}, \ \pm 2n\pi \quad n = 1, 2, 3, \ldots$$

$$-c = (2n\pi)^2 \Rightarrow c_n = -(2n\pi)^2$$

$$\Rightarrow G_n(x) = A\cos(2n\pi x) + B\sin(2n\pi x)$$

At the end, the b.b.'s in x-dir are

$$\{ \underset{c=0}{1}, \ \underbrace{\cos(2n\pi x), \ \sin(2n\pi x)}_{n=1,2,3,\ldots} \}$$

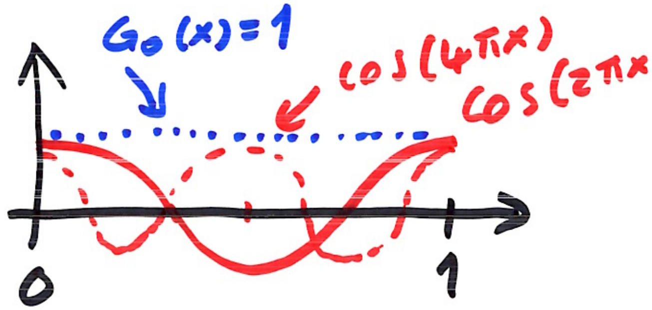
$[0, 1]$

Any periodic function $\underline{f(x)}$ on $[0,1]$

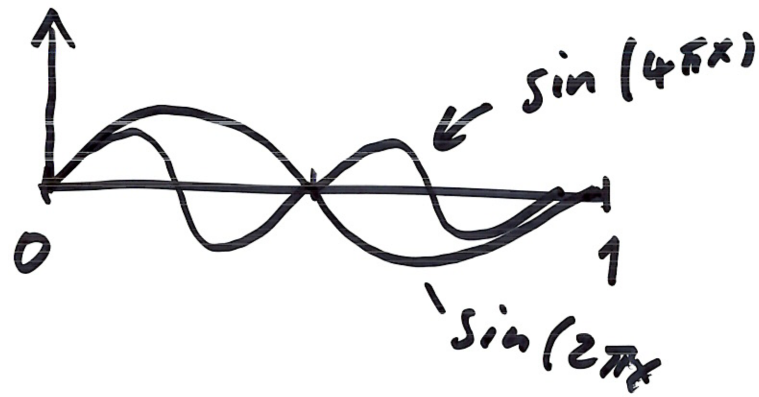
$$f(x) = a_0 \underbrace{G_0(x)}_{\text{"}1\text{"}} + a_1 \underbrace{G_1(x)}_{A_1\cos(2\pi x) + B_1\sin(2\pi x)} + a_2 \underbrace{G_2(x)}_{A_2\cos(4\pi x) + B_2\sin(4\pi x)} + \ldots$$

$$= a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(2n\pi x) + b_n \sin(2n\pi x)\right]$$

"Fourier series"



$G_0(x)=1$
$\cos(4\pi x)$
$\cos(2\pi x$
0
1



$\sin(4\pi x)$
$\sin(2\pi x$
0
1

for periodic domain $[0, \underline{\underline{L}}]$

b.b.'s $\left\{ 1, \cos\left(\frac{2n\pi x}{L}\right), \sin\left(\frac{2n\pi x}{L}\right) \right\}$

for periodic domain $[0, 2\pi]$

b.b.'s $\{ 1, \cos(nx), \sin(nx) \}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Recall (Lec 15):
orthogonality relation still holds!

$[0, L]$

$$\int_0^L \cos\left(\frac{2n\pi x}{L}\right)\cos\left(\frac{2m\pi x}{L}\right)dx$$

$= 0$ if $m \neq n$

$= \square$ if $m = n$

$\frac{L}{2}$?

$\vdots$

$\int_0^L \sin \ldots$ also.

From now on, consider periodic domain
$x \in [0, 2\pi]$

b.b.'s : $\{1, \cos(nx), \sin(nx)\}$

Claim: these are <u>universal</u> b.b.'s

e.g. they are still eigenvalues for

$$G''' = cG$$

$$\left.\begin{aligned} G(0) &= G(2\pi) \\ G'(0) &= G'(2\pi) \\ G''(0) &= G''(2\pi) \end{aligned}\right\} \text{periodic b.c.}$$

or, even $G''''' + 7G''' + 3G' = cG$

+ <u>periodic b.c.</u>

$\sin(nx)$

$[\sin(nx)]' \to n\,\underline{\cos(nx)}$

$[\sin(nx)]'' \to -n^2 \sin(nx)$ ✓

$[\sin(nx)]''' \to -n^3\,\underline{\cos(nx)}$

$[\sin(nx)]'''' \to n^4 \sin(nx)$ ✓

$\cos(nx)$: same story

BUT if we have both

$A \sin(nx) + B \cos(nx)$

"stay in the family"

$[A \sin(nx) + B \cos(nx)]''''' \to A_* \sin(nx) + B_* \cos(nx)$