Lecture 18

(Lecture 17 = midterm, no lecture)

HW3 released, Due Apr. 4

↳ Q1 is from "wave eq." before midterm

Recap: (Lec 16): System on a <u>periodic domain</u>

<u>periodic b.c.</u> → "universal" building blocks

"Fourier series"

$0 \le x \le 2\pi$ system is periodic in x-dir

~~f(x)~~ $\{1, \cos(nx), \underbrace{, \sin}(nx)\}$ $n = 1, 2, 3, \dots$

$G'' = cG$

$G(0) = G(2\pi)$

$G'(0) = G'(2\pi)$

$0 \le x \le L$

$\left\{1, \cos\left(\frac{2n\pi x}{L}\right), \sin\left(\frac{2n\pi x}{L}\right)\right\}$

$n = 1, 2, 3, \dots$

Real F.S.

consider $0 \le x \le 2\pi$ from now on

$f(x)$ periodic on $[0, 2\pi]$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \qquad (*)$$

$\frac{2}{L}$

physical space ← Fourier space

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$(**)$

Fourier space (spectral space) ← physical space

Complex Fourier series

$\cos(nx)$, $\sin(nx)$ → into one family

~~$0 \le t \le$~~ $0 \le x \le 2\pi$

$$f(x) = a_0 \cdot 1 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Real Fourier series

claim: $[b.b.]'''''' \longrightarrow (\text{const}) \cdot (b.b.)$

"universal"

→ works for any linear combo of $\left(\frac{d}{dx}\right)^n$

~~e.g.~~ Ex: $0 \le x \le 2\pi$ periodic in x

$$\frac{\partial u}{\partial t} = 7 \frac{\partial^5 u}{\partial x^5} + 4 \frac{\partial^2 u}{\partial x^2} + 100\, u$$

Not necessary

~~sep of var~~

$$7G''''' + 4G'' + 100G = cG$$

$$G(0) = G(2\pi)$$
$$G'(0) = G'(2\pi)$$
$$\vdots$$
$$G''''(0) = G''''(2\pi)$$

Bypass sep of var

Complex F.S. (equivalent to real F.S. but easier to use for solving PDE)

simply rewrite (*)

Recall

$$\left\{\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right.$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \quad \checkmark$$

Recall $\frac{1}{i} = -i$

(*) $\rightarrow$ $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\left(\frac{e^{inx}+e^{-inx}}{2}\right) + b_n\left(\frac{e^{inx}-e^{-inx}}{2i}\right)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(C_n e^{inx} + C_n^* e^{-inx}\right)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{inx} + \sum_{n=-\infty}^{-1} C_n e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

Define (one of two options)

$$C_n \equiv \frac{a_n - ib_n}{2}$$

$$\left(\text{Then, } C_n^* = \frac{a_n + ib_n}{2}\right)$$

Define $C_0 \equiv a_0$

Define $C_{-n} \equiv C_n^*$

$-\infty \leftarrow$

$$f(x) = \dots + c_{-3}e^{-i3x} + c_{-2}e^{-i2x} + \cancel{c_{-1}e^{-ix} + c_0} +$$

$$+ \underbrace{c_{-1}e^{-ix}} + \underbrace{c_0}_{\text{real}} + \underbrace{c_1e^{ix}} + c_2e^{i2x} + c_3e^{i3x} + \dots$$

$\rightarrow +\infty$

$\oplus$ real

$$(**) \rightarrow \quad c_n = \frac{a_n - i b_n}{2} = \frac{1}{2}\left[\frac{1}{\pi}\int_0^{2\pi} f(x)\underline{\cos(nx)}\,dx - i\,\frac{1}{\pi}\int_0^{2\pi} f(x)\underline{\sin(nx)}\,dx\right]$$

$n \neq 0$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\,dx$$

Now, this works for $n=0$, too!

$$c_0 = a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

Summary: Complex F.S. on $[0, 2\pi]$

HW3 Q2

$$\begin{cases} f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \\ c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\,dx \end{cases} \qquad (c_{-n} = c_n^*)$$

complex conjugate

based on defining $c_n \equiv \frac{a_n - i b_n}{2}$

alternative: If we define $c_n \equiv \frac{a_n + i b_n}{2}$

$$\Rightarrow \begin{cases} f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} \\ c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{+inx}\,dx \end{cases}$$

$$\frac{d^N}{dx^N} \, \underset{\sim}{e^{inx}} = \underline{\underline{(in)^N}} \, \underset{\sim}{e^{inx}}$$

Ex: $\dfrac{\partial u}{\partial t} = \dfrac{\partial^3 u}{\partial x^3}$ $\quad u(x,t)$ $\quad 0 \le x \le 2\pi$

$t \ge 0$

periodic b.c. in x

$$\boxed{\begin{aligned} u(0,t) &= u(2\pi,t) \\ u_x(0,t) &= u_x(2\pi,t) \\ u_{xx}(0,t) &= u_{xx}(2\pi,t) \end{aligned}}$$

Not necessary to write them out

$u(x,0) = \sin(2x)$ — b.c. (☆)

$$\boxed{\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \end{aligned}}$$

bypass sep. of var.

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx} \quad \text{———} \quad (***)$$

plug to PDE:

$$\frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} C_n(t) e^{inx} = \frac{\partial^3}{\partial x^3} \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}$$

$(in)^3 = -in^3$

$$\Rightarrow \sum_n \frac{dC_n}{dt} e^{inx} = \sum_n C_n \cdot (in)^3 e^{inx}$$

$$\Rightarrow \frac{dC_n}{dt} = -in^3 C_n$$

$$\Rightarrow \quad C_n(t) = \underbrace{C_n(0)}_{?} e^{-in^3 t}$$

b.c. (*)

$$\underbrace{\sin(2x) = u(x,0) = \sum_{n=-\infty}^{\infty} C_n(0) e^{inx}}_{(***) \text{ by setting } t \text{ to } 0.}$$

Need to keep but

$$\sin(2x) = \frac{e^{i2x} - e^{-i2x}}{2i}$$

visual inspection:

All $C_n(0) = 0$

except: $C_2(0) = \frac{1}{2i}$ $\quad \underbrace{C_{-2}(0) = -\frac{1}{2i}}_{\text{Never need to compute}}$

Full solution:

$$u(x,t) = C_2(t) e^{i2x} + \boxed{C_{-2}(t) e^{-i2x}} \quad (= [C_2(0)]^*)$$

$$= C_2(t) e^{i2x} + \text{its complex conjugate}$$

$$= 2 \cdot \mathrm{Re}\{C_2(t) e^{i2x}\}$$

$$Z = X + iY$$
$$+)\ Z^* = X - iY$$
$$Z + Z^* = 2X$$
$$= 2\,\mathrm{Re}(Z)$$

$$u(x,t) = 2 \cdot \mathrm{Re}\left\{ c_2(t) e^{i2x} \right\}$$

$$c_2(t) = c_2(0) e^{-i2^3 t} \qquad c_2(0) = \frac{1}{2i} = -\frac{i}{2}$$

$$= -\frac{i}{2} e^{-i8t}$$

$$c_2(t) e^{i2x} = -\frac{i}{2} e^{i2x} e^{-i8t}$$

$$= -\frac{i}{2} e^{i(2x-8t)}$$

$$= -\frac{i}{2} \left[\cos(2x-8t) + i \sin(2x-8t) \right]$$

$$= \frac{1}{2} \sin(2x-8t) - \frac{i}{2} \cos(2x-8t)]$$

$$\Rightarrow u(x,t) = 2 \cdot \mathrm{Re}\left\{ c_2(t) e^{i2x} \right\}$$

$$= \sin(2x-8t)$$

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