Lecture 19

Recap (Lec 18) F.S. expansion   PDE + periodic b.c.

complex   universal b.b.   Bypass sep. of var.

Ex from Lec 18:

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \quad \text{— } (*) \qquad 0 \le x \le 2\pi, \; t \ge 0$$

b.c. in x-dir: periodic

b.c. in t-dir:

$u(x,0) = \sin(2x)$

universal b.b. F.S.

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx} \quad \text{— } (\#)$$

periodic function

$f(x) \qquad 0 \le x \le 2\pi$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\,dx$$

plug into $(*)$: extract the ODE for $C_n(t)$:

$$\Rightarrow \quad \frac{dC_n}{dt} = (in)^3 C_n = -in^3 C_n \xrightarrow[\text{solve}]{} C_n(t) = C_n(0) e^{-in^3 t}$$

take (✱) and set t to 0:

$$\underbrace{\sin(2x) = u(x,0)}_{\text{b.c.}} = \underbrace{\sum_{n=-\infty}^{\infty} C_n(0)\, e^{inx}}_{(✱)\text{ with } t=0}$$

$$\sin(2x) = \frac{e^{i2x} - e^{-i2x}}{2i}$$

$C_n(0)$ is just the Fourier coefficient of $u(x,0)$

visual comparison

$\Downarrow$

All $C_n(0) = 0$, except $\begin{cases} C_2(0) = \frac{1}{2i} \\ C_{-2}(0) = \frac{-1}{2i} \end{cases}$ ✓

Never need to process $[C_{-2}(t) e^{-i2x}]$

$\Leftarrow$ $C_n(t) = 0$ for all n, except

$$C_2(t) = C_2(0)\, e^{-i2^3 t} = \frac{1}{2i} e^{-i8t}$$

$\Rightarrow$ Full solution:

$$\begin{aligned} u(x,t) &= C_2(t)\, e^{i2x} + \text{c.c.} \\ &= 2 \cdot \text{Re}\{C_2(t) e^{i2x}\} \\ &= 2 \cdot \text{Re}\{\tfrac{-i}{2} e^{i(2x-8t)}\} \\ &= \sin(2x-8t) \quad \# \end{aligned}$$

$$\boxed{\begin{aligned} &e^{i(2x-8t)} \\ &= \cos(2x-8t) \\ &\quad + i\sin(2x-8t) \end{aligned}}$$

$$\begin{array}{rl} & Z = X + iY \\ +) & Z^* = X - iY \\ \hline & 2X = 2\,\text{Re}(Z) \end{array}$$

HW3-Q2 with $U=0$, $K=0$, $B=1$, $D=0$

Ex:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^3 u}{\partial x^3} & 0 \le x \le 2\pi, \quad t \ge 0 \\ \text{periodic in } x & \\ u(x,0) = p(x) & \leftarrow \text{given (e.g., HW3-Q2)} \end{cases}$$

$$p(x) = \frac{[1-\cos(x)]^8}{256}$$

$$\boxed{\frac{dC_n}{dt} = -in^3 C_n}$$

Repeat first half of previous example:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}, \qquad \begin{cases} C_n(t) = C_n(0) e^{-in^3 t} & n \neq 0 \\ C_0(t) = \underline{C_0(0)} & \end{cases}$$

$$\Rightarrow \quad u(x,t) = \sum_{n=-\infty}^{\infty} C_n(0) e^{i(nx - n^3 t)}$$

at $t=0$

$$\sum_{n=-\infty}^{\infty} C_n(0) e^{inx} = P(x)$$

$$\Rightarrow \quad C_n(0) = \frac{1}{2\pi}\int_0^{2\pi} p(x) e^{-inx}\, dx$$

← This is good for any n, including $\underline{n=0}$

In Matlab, "i" is $\sqrt{-1}$

Matlab coding:

$$C_n(0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{P(x)e^{-inx}} dx = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} P(x)\cos(nx)dx}_{\text{trapz}} - i\,\underbrace{\frac{1}{2\pi}\int_0^{2\pi} P(x)\sin(nx)dx}_{\text{trapz}}$$

even better,
trapz actually recognizes complex func.

$$C \leftarrow \text{trapz}(\underline{x}, \underline{P(x)e^{-inx}})$$

$$\parallel$$

$$\underline{[\text{Re}\,C, \text{Im}\,C]}$$

Simplest way

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(0)e^{i(nx-n^3t)}$$

$$\cancel{= C_0 t}$$

$$= C_0(0) + \sum_{n=1}^{\infty} C_n(0)e^{i(nx-n^3t)}$$

$$+ \sum_{n=-\infty}^{-1} C_n(0)e^{i(nx-n^3t)}$$

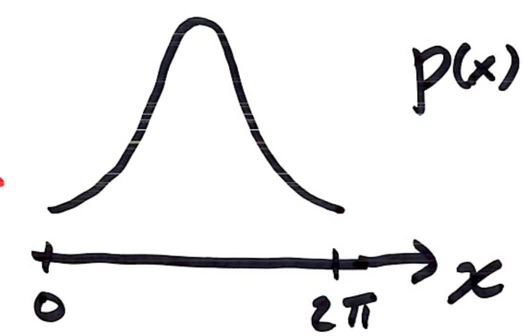
$$= C_0(0) + 2\cdot \text{Re}\left\{\sum_{n=1}^{\infty} C_n(0)e^{i(nx-n^3t)}\right\}$$

C.C. of each other

HW3-Q2 $U=0,\ K=0,\ B=1,\ D=0$

$$p(x) = \frac{[1-\cos(x)]^8}{256}$$

say, want to plot $u(x,t)$ at $t=0.2$



real
imag

```
clear
dx = 2*pi/1000;
x = [0 : dx : 2*pi];
p = ((1 - cos(x)).^8)/256;
N = 50;
for n=1:N
  c(n) = trapz(x, p.*exp(-i*n*x))/(2*pi);
end
c00 = trapz(x,p)/(2*pi);
```

$C_n(0)$ / Not to be confused with $C_n(t)$

$C_0(0)$

```
t=0.2
u = 0*x;      or      u = 0*x + i*0*x;
for n=1:N
  u = u + c(n)*exp(i*(n*x-(n^3)*t));
end
u = c00 + 2*real(u);
plot(x,u)
```

HW3 Q3 $\quad 0 \le x \le 2\pi$, $t \ge 0$ $\quad$ periodic in x

$$\frac{\partial u}{\partial t} = 9t\,\frac{\partial^3 u}{\partial x^3} + \frac{\partial^4 u}{\partial x^4} + t\,\frac{\partial^5 u}{\partial x^5} - u$$

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\, e^{inx}$$

$$\Downarrow$$

$$\dot{C}_n = 9t\cdot(in)^3 C_n + (in)^4 C_n + (in)^5\cdot t\, C_n - C_n$$

$$(\dot{\ }) \equiv \frac{d(\ )}{dt}$$

$$\frac{dC_n}{dt} = [\underbrace{-i\,9n^3 t + n^4 + i n^5 t - 1}_{f(t)}]\, C_n$$

Solve $C_n(t)$ ...

$u(x,t)$

$0 \le x \le 2\pi$

periodic

$$u(x,t) = \sum_n C_n(t)\, e^{inx}$$

$\cos(nx) + i\sin(nx)$

$$\left|e^{inx}\right| \le 1$$

$$\frac{\partial u}{\partial t} = \frac{\partial^N u}{\partial x^N}$$

$$\frac{dC_n}{dt} = (in)^N C_n$$

$$C_n(t) = C_n(0)\, e^{(in)^N t}$$

$$= C_n(0)\, e^{\underline{(i^N)} \cdot n^N t}$$

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

$N = 2, 6, 10, 14, \ldots$

$$C_n(t) = C_n(0)\, e^{-n^2 t}$$

diffusive

$N = 4, 8, 12, 16, \ldots$

$$C_n(t) = C_n(0)\, e^{+n^4 t}$$

anti-diffusive

if N is odd $\quad i^N \to \pm i$

$$\boxed{C_n(t) = C_n(0)\, e^{\pm i (n^N t)}}$$

$$u(x,t) = \sum_n C_n(t)\, \underline{e^{inx}}$$

Oscillatory in space & time

Wave eqs.

$N = 3, 5$

3	5
7	9
11	13
15	17
⋮	⋮

$$\underbrace{e^{\pm i(n^N t)} \quad e^{inx}}$$

$$e^{i(nx \pm n^N t)}$$

$+ \qquad -$

HW3-Q2(b)

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^4 u}{\partial x^4} \qquad \boxed{\begin{matrix} A>0 \\ B>0 \end{matrix}}$$

$A>0 \quad B=0 \Rightarrow$ diffusion eq.

$A=0 \quad B>0 \Rightarrow$ anti-diffusion

what if $A>0 \quad B=0$?

$$\Rightarrow \quad \dot{C}_n = \underbrace{(-An^2 + Bn^4)}_{S(n)}$$