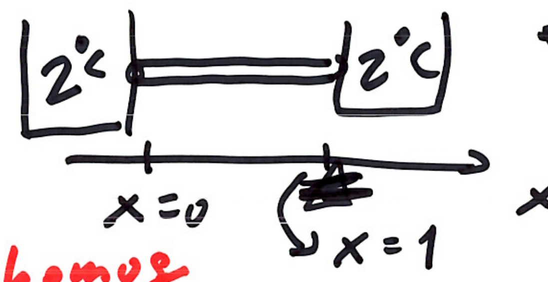
Lecture 20

3/27

Ch. 8 <u>Nonhomogeneous</u> systems

- type 1 * nonhomog. PDE + homog. b.c.
- type 2 * homog. PDE + nonhomog. b.c.
- type 3 * both

type 2 **Ex 1.** 1-D Heat eq.



$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 2 \\ u(1,t) = 2 \\ u(x,0) = p(x) \end{cases}$$

Nonhomog b.c.

try sep. of var.

$u \sim G(x)H(t)$

b.c. 1: $G(0)H(t) = 2$

$G(0) = \dfrac{2}{H(t)}$

~~G(0)H~~

$G(1)H(t) = 2$

$G(1) = \dfrac{2}{H(t)}$

sep of var

$$\begin{cases} \underline{G'' = cG} + \begin{cases} G(0) = 2/H(t) \\ G(1) = 2/H(t) \end{cases} \\ \dot{H} = cH \end{cases}$$

will not work

$$\underline{\underline{\hat{u} \equiv u - 2}}$$

$$\left\{ \begin{aligned} & \frac{\partial \hat{u}}{\partial t} = \frac{\partial^2 \hat{u}}{\partial x^2} \\ & \hat{u}(0,t) = 0 \\ & \hat{u}(1,t) = 0 \\ & \hat{u}(x,0) = \underline{p(x) - 2} \end{aligned} \right.$$

$\hat{p}(x)$

$\hat{u} \equiv u - 2$

PDE: $\quad \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \Rightarrow \dfrac{\partial(\hat{u}+2)}{\partial t} = \dfrac{\partial^2(\hat{u}+2)}{\partial x^2}$

$$\Downarrow$$

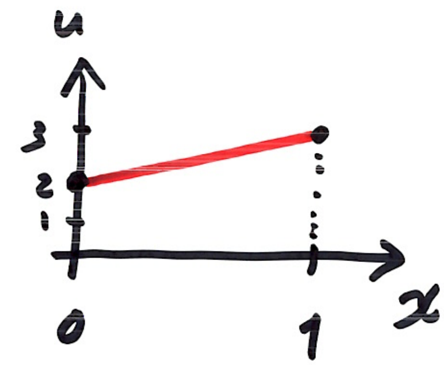
$$\underline{\frac{\partial \hat{u}}{\partial t} = \frac{\partial^2 \hat{u}}{\partial x^2}}$$

Ex 2

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{— } (*) \\ u(0,t) = 2 & \text{— (i)} \\ u(1,t) = 3 & \text{— (ii)} \\ u(x,0) = p(x) & \text{— (iii)} \end{cases}$$

departure from steady sol.

$$\underline{\underline{\hat{u} \equiv u - \underline{U_s(x)}}}$$



$\underline{U_s(x)} = \underbrace{2+x}$

happens to be steady sol.

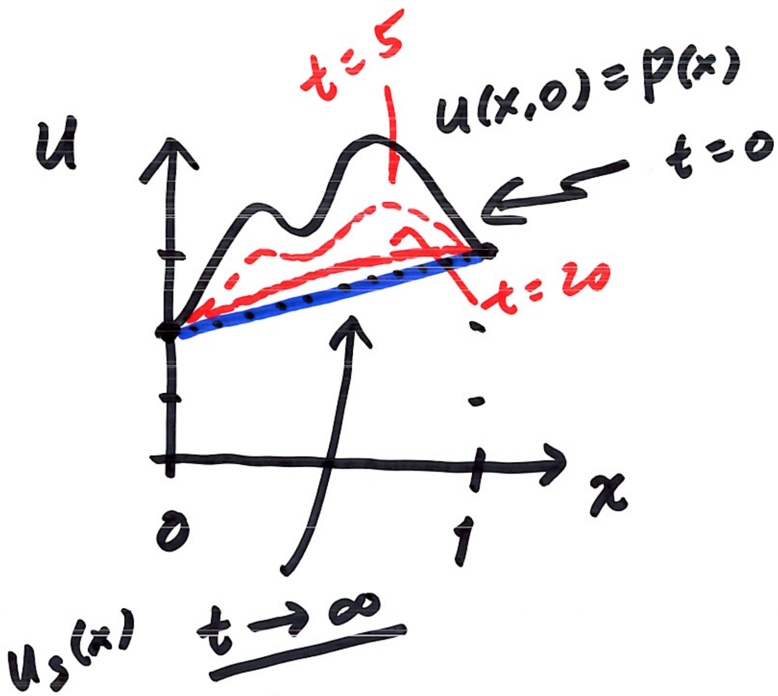
✓ (i) $\quad \hat{u}(0,t) \equiv u(0,t) - U_s(0) = 2 - 2 = 0$

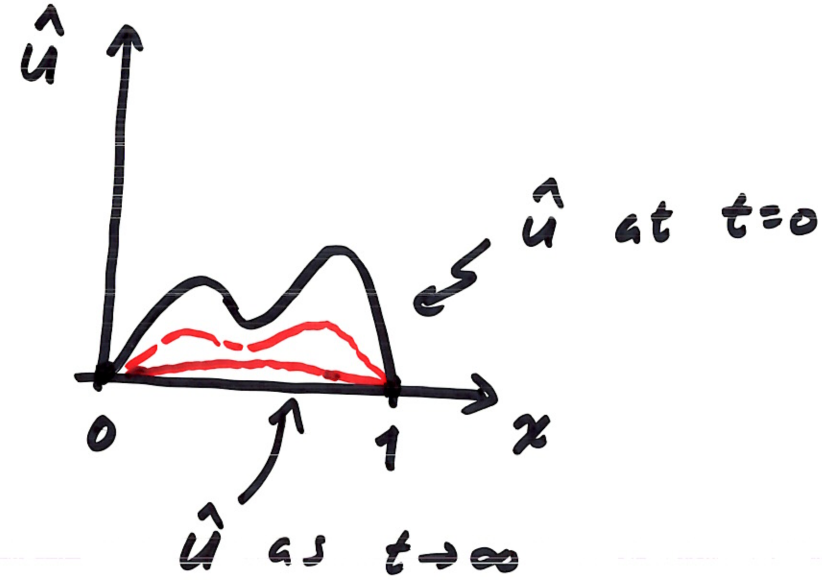
✓ (ii) $\quad \hat{u}(1,t) = \dots = 0$

$$* \to \quad \frac{\partial\, \hat{u} + (2+x)}{\partial t} = \frac{\partial^2\, \hat{u} + (2+x)}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial \hat{u}}{\partial t} = \frac{\partial^2 \hat{u}}{\partial x^2} \quad ✓$$

$$\text{(iii)} \Rightarrow \quad \hat{u}(x,0) = p(x) - U_s(x) = \underbrace{p(x) - (2+x)}_{\hat{p}(x)}$$


u
t=5
u(x,0)=p(x)
t=0
t=20
0
1
x
$u_s(x)$ $\underline{t\to\infty}$


$\hat{u}$
$\hat{u}$ at t=0
0
1
x
$\hat{u}$ as $t\to\infty$

Ex 2 General strategy to "homogenize" b.c. (i), (ii)

could work, even if b.c. (i), (ii) depend on t

$$\begin{cases} u(0,t) = e^{-t} \\ u(1,t) = 3 \end{cases}$$

$$\hat{u}(x,t) \equiv u(x,t) - f(x,t)$$

need $\begin{cases} f(0,t) = 2 \\ f(1,t) = 3 \end{cases}$

✓ for example $f(x,t) \equiv 2 + x^3$

b.c. (i) $\rightarrow$ $\hat{u}(0,t) = 0$ ✓

b.c. (ii) $\rightarrow$ $\hat{u}(1,t) = 0$ ✓

b.c. (iii) $\rightarrow$ $\hat{u}(x,0) = p(x) - (2 + x^3)$ ✓

PDE?

$$\frac{\partial(\hat{u} + f)}{\partial t} = \frac{\partial^2(\hat{u} + f)}{\partial x^2}$$

$$\frac{d^2 f}{dx^2} = \frac{d^2}{dx^2}(2 + x^3) = \underline{6x}$$

$$\Rightarrow \boxed{\frac{\partial \hat{u}}{\partial t} = \frac{\partial^2 \hat{u}}{\partial x^2} + 6x}$$

nonhomogeneous PDE !

we turn type 2 $\longrightarrow$ type 1

has a systematic approach for us to use

Steady sol. is the "optimal" choice of ~~the~~ "$f(x,t)$"

→ it already satisfies the b.c. and PDE !

Homogeneous vs. nonhomogeneous PDE

$u(x,t)$

homog. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + t^2 u + \sin(t)\, u$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - t^2 - \sin(t) \right) u = 0$$

$\boxed{\mathcal{L} u = 0}$ homog. PDE

nonhomog. PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \underbrace{e^{-t}\sin(x)}_{Q(x,t)}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u = Q(x,t)$$

$$\boxed{\mathcal{L}\, \underline{u} = Q}$$

unknown that we want to solve

given, known
"imposed"
(just like those nonhomog. b.c.'s!)

Analogy in linear algebra

$\mathbb{A}\vec{x} = 0$ homog.

$\mathbb{A}\vec{x} = \vec{b}$ nonhomog.

unknown given

homog. PDE

$$\mathcal{L} u = 0$$

if u_1, u_2 are two sols.

$$\Downarrow$$

$A u_1 + B u_2$ is still a sol.

$$\begin{array}{rr} & A \mathcal{L} u_1 = 0 \\ +) & B \mathcal{L} u_2 = 0 \\ \hline & \mathcal{L}[A u_1 + B u_2] = 0 \end{array}$$

nonhomog PDE

$$\mathcal{L}u = Q$$

say u_1 and u_2 are sols. $\Rightarrow$ $u_1 + u_2$ no longer a sol.

$$\mathcal{L}u_1 = Q$$

$$+)\ \mathcal{L}u_2 = Q$$

$$\mathcal{L}[u_1 + u_2] = Q + Q = 2Q \neq Q$$

nonhomog PDE

sys 1: $\mathcal{L}u = Q_1$

$\downarrow$

u_1

sys 2: $\mathcal{L}u = Q_2$

$\downarrow$

u_2

$$\Rightarrow \quad \mathcal{L}(u_1+u_2) = Q_1 + Q_2$$

$u_1 + u_2$ is the sol. of

$$\mathcal{L}u = \underline{Q_1 + Q_2}$$

$Q \rightarrow Q_1 + Q_2$

homog. vs nonhomog. b.c.

$u(x,t)$ $\qquad 0 \le x \le 1$

homog

$$5\,u(0,t) + 3\,u_x(0,t) = 0$$

u_1 is a sol.

u_2 is a sol.

$$v \equiv u_1 + u_2$$

$$5\,v(0,t) + 3\,v_x(0,t) = 0 \checkmark$$

nonhomog. b.c.

$$u(0,t) = 2$$

u_1 is a sol.

u_2 is a sol.

$$\begin{array}{rl} & u_1(0,t) = 2 \\ +) & u_2(0,t) = 2 \\ \hline & v(0,t) = 2+2 = 4 \neq 2 \end{array}$$

$v \equiv u_1 + u_2$

×

$$u(0,t) = 5$$

$$\begin{array}{rl} & u_1(0,t) = 2 \\ +) & u_2(0,t) = 3 \\ \hline & v(0,t) = 5 \end{array}$$

$\underline{v \equiv u_1} + u_2$

type 1 : nonhomog. PDE + <u>homog. b.c.</u>

Ex:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \underbrace{Q(x,t)}_{\text{given}} \\ u(0,t)=0 \\ u(1,t)=0 \\ u(x,0)=p(x) \end{cases}$$

1-D Heat eq. w/ heat source/sink imposed

homog. subsystem

sep of var $\rightarrow$ $G''=cG \quad G(0)=0 \quad G(1)=0$

$\cancel{\dot{H}=cH}$

$\{\sin(n\pi x)\}$ STOP!

expand both $u(x,t)$ and $Q(x,t)$ in $\{\sin(n\pi x)\}$!

will turn the nonhomog <u>PDE</u> $\rightarrow$ nonhomog. <u>ODE</u>