# Lecture 21



HW4 — due Tue 4/18
HW5 — will due Fri 4/28

Final Exam – Wed 5/3 4:50-6:40pm

Recap (Lec 20) *

"type 0" homog PDE
homog b.c.

Nonhomog. systems

type 1 $\left\{ \begin{array}{l} \text{nonhomog. PDE} \\ \text{homog. b.c.} \end{array} \right.$

$\hat{u}(x,t) = u(x,t) - f(x,t)$

type 2 $\left\langle \begin{array}{l} \text{homog. PDE} \\ \text{nonhomog. b.c.} \end{array} \right.$

type 3 $\left\langle \begin{array}{l} \text{nonhomog. PDE} \\ \text{nonhomog. b.c.} \end{array} \right.$

} same strategy: ~~fi~~ first use a "transformation" to change b.c. to homog.

type 2, type 3 → type 1 at worst

→ type 0 if we are lucky

guaranteed, if $f(x,t) = u_s(x)$

<u>type 1</u>

Ex: 1-D Heat eq. w/ (imposed) heat source/sink

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \underbrace{Q(x,t)}_{\text{given}} \quad (*)$$

$\cancel{u(x,}$

$u(0,t)=0$

$u(1,t)=0$ — homog. subsystem

$u(x,0)=P(x)$

sep of var. etc.

$$\boxed{G''=cG \quad G(0)=0 \quad G(1)=0}$$

$\cancel{\dot{H}=cH}$ will not use

$G_n(x)=\sin(n\pi x)$

plug into full PDE

Key: Expand both $u(x,t)$ and $Q(x,t)$ in $\{\sin(n\pi x)\}$

$$u(x,t)=\sum_{n=1}^{\infty} a_n(t)\sin(n\pi x) \quad (\star)$$

$$\underbrace{Q(x,t)}_{\text{given}}=\sum_{n=1}^{\infty} \underbrace{g_n(t)}_{\text{known}}\sin(n\pi x)$$

$$g_n(t)=2\int_0^1 Q(x,t)\sin(n\pi x)\,dx$$

(*) : $$\frac{\partial}{\partial t} \sum_{n=1}^{\infty} a_n(t) \sin(n\pi x)$$

$$= \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} a_n(t) \underline{\sin(n\pi x)} + \underline{\sum_{n=1}^{\infty} q_n(t) \sin(n\pi x)}$$

$$\sum_{n=1}^{\infty} \underline{\underline{\frac{da_n}{dt}}} \cdot \sin(n\pi x) = \sum_{n=1}^{\infty} \underline{-(n\pi)^2 a_n(t)} \sin(n\pi x) + \sum_{n=1}^{\infty} \underline{q_n}(t) \sin(n\pi x)$$

need $a_n(0)$ as b.c. $\leftarrow$ $$\boxed{\frac{da_n}{dt} = -(n\pi)^2 a_n + q_n(t)}$$ (**) nonhomog. <u>ODE</u>

If $\underline{Q(x,t)=0}$ $\Rightarrow$ $q_n(t) = 0$ for all $n$

$$\Rightarrow \frac{da_n}{dt} = -(n\pi)^2 a_n \Rightarrow \underline{a_n(t) = \underline{a_n(0)} e^{-(n\pi)^2 t}}$$

Note: set $\underline{\cancel{t_0}^{\,t} \text{ to } 0}$ in (*): $$\sum_{n=1}^{\infty} a_n(0) \sin(n\pi x) = P(x) \Rightarrow a_n(0) = 2\int_0^1 P(x) \sin(n\pi x)\, dx$$

side note

$**$ belongs to the prototype of nonhomog ODE

$u(t)$ $\qquad \dfrac{du}{dt} = -\alpha u + \underset{\text{given}}{f(t)}$ — $(***)$ $\qquad$ b.c. $\quad u(0) = \underset{\text{given}}{\square}$

Special case:

(i) If $f(t) = 0$

$$\frac{du}{dt} = -\alpha u$$

$$\frac{du}{u} = -\alpha\, dt$$

$$\int_{t=0}^{t=t} d\ln u = -\int_{t=0}^{t=t} \alpha\, dt$$

$$u(t) = u(0)\, e^{-\alpha t} \quad ✓$$

(ii) If $\alpha = 0$

$$\frac{du}{dt} = f(t) \qquad \int_0^t du = \int_{\hat{t}=0}^{\hat{t}=t} f(\hat{t})\, d\hat{t}$$

$$u(t) = u(0) + \int_{\hat{t}=0}^{\hat{t}=t} f(\hat{t})\, d\hat{t} \quad ✓$$

(iii) general case? $\quad$ try to reduce it to (i) or (ii)!

$$\frac{du}{dt} = -\alpha u + f(t) \quad \text{——} \quad (***)$$

try $\quad u(t) = V(t)\,\underline{B(t)}$ $\qquad$ change of var

$u \rightarrow V$

$$\cancel{\frac{d\,VB}{dt} = -\alpha} \qquad \frac{d(VB)}{dt} = -\alpha V B + f$$

$$\underline{V\frac{dB}{dt}} + B\frac{dV}{dt} = \underline{-\alpha V B} + f$$

$$B\frac{dV}{dt} + V\Big[\underbrace{\frac{dB}{dt} + \alpha B}_{\text{if this vanishes}}\Big] = f$$

if this vanishes ~~m~~ mission accomplished!

"desirable" $B(t)$ satisfies $\quad \dfrac{dB}{dt} + \alpha B = 0 \;\Rightarrow\; B(t) = e^{-\alpha t}$

do not care about the const

Thus, if we define $u(t) = V(t)\,\underbrace{e^{-\alpha t}}_{B(t)}$ $\quad V(t) = u(t)e^{+\alpha t}$ $\quad u(0) = V(0)$

then $\dfrac{dV}{dt} = \dfrac{f(t)}{B(t)} = \underbrace{f(t)e^{+\alpha t}}$

$F(t) \equiv f(t)e^{+\alpha t}$

Solve $\dfrac{dV}{dt} = F(t)$

$$V(t) = V(0) + \int_{\hat{t}=0}^{\hat{t}=t} F(\hat{t})\,d\hat{t}$$

Sec 8.3

$$u(t)e^{\alpha t} = u(0) + \int_{\hat{t}=0}^{\hat{t}=t} f(\hat{t})e^{\alpha \hat{t}}\,d\hat{t}$$

$$\boxed{u(t) = u(0)e^{-\alpha t} + e^{-\alpha t}\int_{\hat{t}=0}^{\hat{t}=t} f(\hat{t})e^{\alpha \hat{t}}\,d\hat{t}}$$

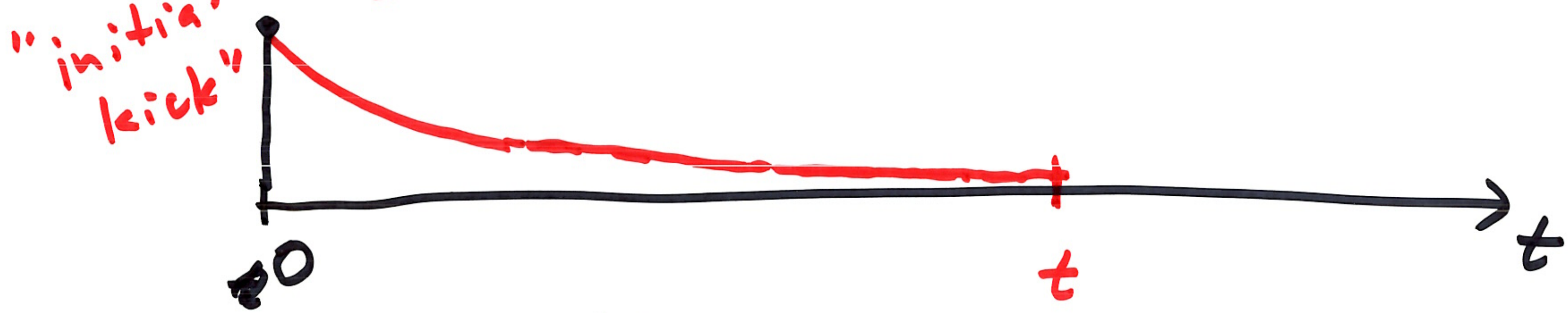
Special case (i) $\underline{f(t) = 0}$ $\quad u(t) = u(0)\, e^{-\alpha t}$ ✓

" (ii) $\alpha = 0$ $\quad u(t) = u(0) + \int_{\hat{t}=0}^{\hat{t}=t} f(\hat{t})\, d\hat{t}$ ✓

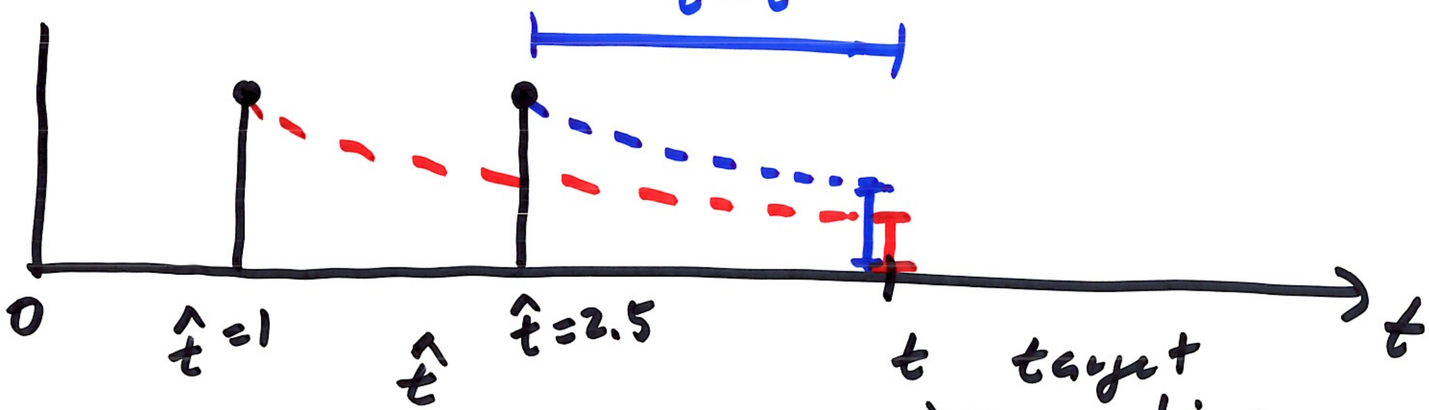
**Sec 8.3** alternative :

$$u(t) = u(0)\,e^{-\alpha t} + \int_{\hat{t}=0}^{\hat{t}=t} f(\hat{t})\,e^{-\alpha(t-\hat{t})}\,d\hat{t}$$

No $f(t)$




$f(t) \neq 0$



getting back to the main story line:

$$\frac{da_n}{dt} = -(n\pi)^2 a_n + q_n(t) \qquad a_n(0) = \boxed{\phantom{x}}$$

$$\underline{\underline{a_n(t)}} = \underbrace{a_n(0)\, e^{-(n\pi)^2 t} + \int_{\hat{t}=0}^{\hat{t}=t} q_n(\hat{t})\, e^{-(n\pi)^2 (t-\hat{t})}\, d\hat{t}}_{} \quad \#$$

$$u(x,t) = \sum_{n=1}^{\infty} \underline{a_n(t)} \sin(n\pi x)$$