Lecture 22

* Nonhomogeneous systems

Lec 20, 21 → **type 1**: nonhomog PDE, homog b.c.

type 2: homog. PDE, nonhomog. b.c.

type 3: nonhomog PDE, nonhomog b.c.

≠

type 0: homog. PDE, homog. b.c. or periodic b.c

key: use a "transformation" (e.g. $\hat{u}(x,t) = u(x,t) - f(x,t)$) to make b.c. homog.

$\Rightarrow$ either type 1 or type 0 → solve ✓

optimal choice of $f(x,t)$ is steady sol.

guaranteed to $\Rightarrow$ type 0

Ex 1 type 1 (Lec 21) ~~Lec 00, t~~ $0 \le x \le 1$

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}} + Q(x,t)$$

$$\boxed{\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}} \longrightarrow \text{b.b. in } x$$

$u(x,0) = P(x)$

sep of var (only halfway through) → STOP

$G_n(x) = \sin(n\pi x)$ ~~$G_n(x) = \sin(n\pi x)$~~

Key

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} \underline{a_n(t)} \sin(n\pi x) \\ Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin(n\pi x) \end{cases}$$

$$\frac{da_n}{dt} = -(n\pi)^2 a_n + q_n(t)$$

Solve ✓ ⟶ (Lec 21 Sec 8.3)

Ex 2
type 1

$0 \le x \le 1$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$u_x(0,t) = 0$$

$$u_x(1,t) = 0$$

$$u(x,0) = p(x)$$

<u>b.c.</u>

Hw 1 - Q1a

b.b. in x

$$\rightarrow \{ \underline{1}, \cos(n\pi x) \}$$

$$\begin{cases} u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos(n\pi x) \\ Q(x,t) = q_0(t) + \sum_{n=1}^{\infty} q_n(t) \cos(n\pi x) \end{cases}$$

$$\rightarrow \frac{d a_0}{dt} = q_0(t)$$

$$\frac{d a_n}{dt} = -(n\pi)^2 a_n + q_n(t) \qquad n \neq 0$$

Ex 3 "type 1" — periodic b.c.

$0 \le x \le 2\pi$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + Q(x,t) \\ \text{b.c. in } x : \text{ periodic} \\ u(x,0) = p(x) \end{cases}$$

universal b.b. w/ F.S.

$$\left.\begin{aligned} u(x,t) &= \sum_{n=-\infty}^{\infty} C_n(t)\, e^{inx} \\ Q(x,t) &= \sum_{n=-\infty}^{\infty} g_n(t)\, e^{inx} \end{aligned}\right\}$$

Key $\rightarrow$ *

$$\frac{dC_n}{dt} = -n^2 C_n + g_n(t), \quad n \neq 0$$

$$\underline{\frac{dC_0}{dt} = g_0(t)}$$

Ex 4

type 1

"forced" 1-D wave eq.

$$\left\{\begin{array}{l} \boxed{\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(1,t) = 0 \end{array}} + Q(x,t) \\ u(x,0) = p(x) \\ u_t(x,0) = \cancel{Q(x)}\ R(x) \end{array}\right.$$

$0 \le x \le 1$

$G_n(x) = \sin(n\pi x)$

$$\left\{\begin{array}{l} u(x,t) = \sum_n a_n(t) \sin(n\pi x) \\ Q(x,t) = \sum_n q_n(t) \sin(n\pi x) \end{array}\right.$$

$$\frac{d^2 a_n}{dt^2} = -(n\pi)^2 a_n + q_n(t)$$

2nd-order nonhomog. ODE

"Wronskian"

Ex. 5

type 3

$$0 \le x \le \pi \qquad t \ge 0$$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 2\sin(x) \\ u(0,t) = 1 \\ u_x(\pi,t) = -2 \\ u(x,0) = 1 + 2\sin(x) + \sin\left(\frac{x}{2}\right) \end{cases}$$

wish $\hat{u}(x,t) = u(x,t) - \underline{f(x,t)}$

at ~~the minimum,~~ need $\left.\begin{array}{l} f(0,t) = 1 \\ f_x(\pi,t) = -2 \end{array}\right\}$

optimal choice

$$f(x,t) = u_s(x)$$

(steady sol.)

steady sol $u_s(x)$ satisfies

$$\begin{cases} \dfrac{d^2 u_s}{dx^2} + 2\sin(x) = 0 \\ u_s(0) = 1 \quad ✓ \quad \text{— (i)} \\ u_s'(\pi) = -2 \quad ✓ \quad \text{— (ii)} \end{cases}$$

$$u_s'' = -2\sin(x)$$

$$u_s' = 2\cos(x) + A$$

$$u_s = 2\sin(x) + Ax + B$$

b.c. (i) $\Rightarrow B = 1$

b.c. (ii)

$\cancel{2\sin}$ $2\underbrace{\cos(\pi)}_{-1} + A = -2 \Rightarrow A = 0$

$$\Rightarrow u_s(x) = \underline{1 + 2\sin(x)}$$

$$u_s(x) = 1 + 2\sin(x)$$

$$\hat{u}(x,t) = u(x,t) - u_s(x)$$

Cancels out nonhomog. term in PDE!

$$\Rightarrow \quad \text{type 0} \left\{ \begin{aligned} &\frac{\partial \hat{u}}{\partial t} = \frac{\partial^2 \hat{u}}{\partial x^2} \\ &\hat{u}(0,t) = 0 \\ &\hat{u}_x(\pi,t) = 0 \\ &\hat{u}(x,0) = \sin\left(\frac{x}{2}\right) \end{aligned} \right.$$

$$\xrightarrow{\text{solve}} \quad \hat{u}(x,t) = \sin\left(\frac{x}{2}\right) e^{-\frac{t}{4}}$$

Full sol. $u(x,t) = \hat{u}(x,t) + u_s(x)$

$$= 1 + 2\sin(x) + \sin\left(\frac{x}{2}\right) e^{-\frac{t}{4}}$$

#

Ch. 12 Method of Characteristics (MOC)

All other methods (up to now): linear PDEs

MOC can be used to solve some nonlinear PDEs

MOC is particularly efficient for solving some linear PDEs

* The method is applicable only to PDEs of a certain form.

HW3 Q2

$$\boxed{\frac{\partial u}{\partial t} = U\frac{\partial u}{\partial x}} + K\frac{\partial^2 u}{\partial x^2} + \dots$$

Case $K=0, B=0, D=0$

All methods we learned so far

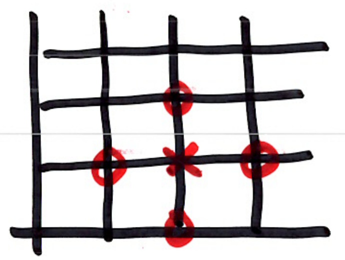
PDE

$\downarrow$

$\{\text{ODEs}\}$

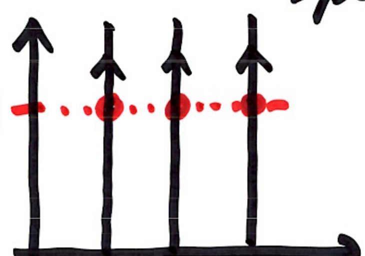
$u(x,t)$

physical space



$\Downarrow$

spectral space



"wavenumber"

$$\frac{\partial u}{\partial t} = \boxed{\frac{\partial^2 u}{\partial x^2}} + \text{b.c.'s}$$

physical space

$-(n\pi)^2$

"x" $\leftrightarrow$ "n"

$$u = \sum_n a_n^{(t)} G_n(x)$$

spectral space (Fourier space)

solve

$$\boxed{\frac{da_n}{dt} = -(n\pi)^2 a_n}$$

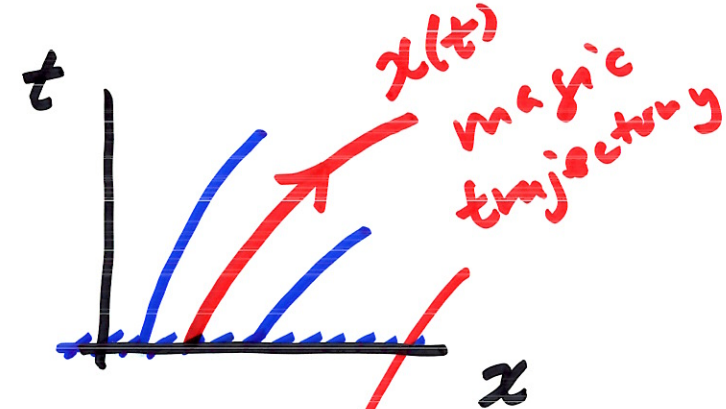
$a_n(t) \xleftarrow{\text{solve}}$

Conceptual ...

MOC

PDE $\longrightarrow$ ODE

without leaving the physical space!



$u(x,t)$

PDE x, t

Such that, PDE collapses to ODE

$$\frac{\partial u}{\partial t} = U \frac{\partial u}{\partial x}$$

$U = \text{const}$

HW3 Q2 case (i)

$u(x,0) = p(x)$