Lecture 23

4/5

Recap : (Lec 22)

Method of Characteristics (MOC)

Ch. 12

All ~~other~~ methods (until now)

$$\text{PDE} \rightarrow \text{ODEs} \rightarrow \text{solve}$$

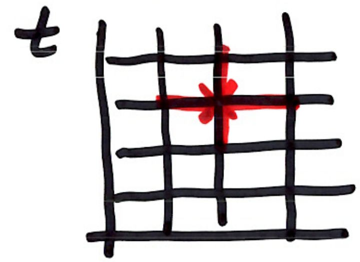
physical space

spectral (Fourier) space

$u(x,t)$

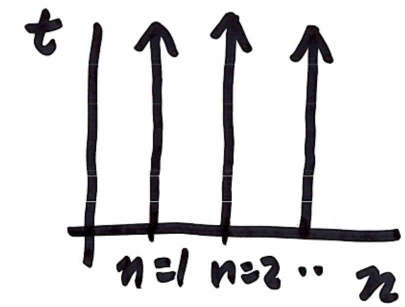
PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$



x & t "entangled" "coupled"

n & t "decoupled"

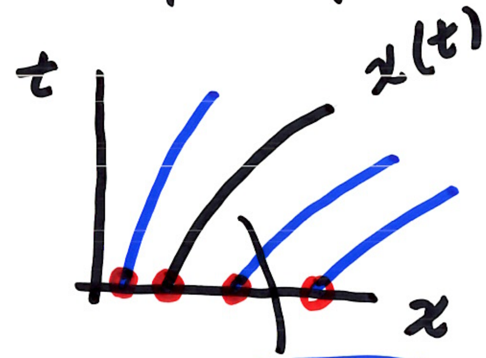


$$\sum_n \underline{a_n(t)} G_n(x) \ldots$$

MOC : decouple x, t without leaving physical space!

x depends on t

$u(x,t)$



PDE

b.c.

$$\frac{\partial u}{\partial t} = U \frac{\partial u}{\partial x}$$

$u(x,0) = p(x)$,

HW3
Q2 (a)
case (i)

x, t independent

magic trajectory in x-t plane

on which, PDE $\rightarrow$ ODE

characteristic

$u(x,t)$
in the context
of PDE

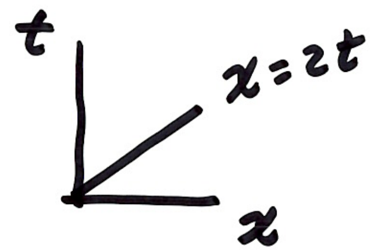
x & t are
independent

along a trajectory $x(t)$
in x-t plane, x is
related to t

then, $u(x,t) = u(x(t),t)$

think of it as $\underline{u(t)}$

$u(x,t)$
$= e^{x}\sin(t)$



$u(x,t)$
$= u(x(t),t)$
$= \underline{e^{2t}\sin(t)}$

Calculus: $f(x,y)$
$\hookrightarrow f(x(t),y(t))$

$x \equiv x(t)$
$y \equiv y(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Ex: (HW3 Q2(a)(i))

$u(x,t)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = -10\,\dfrac{\partial u}{\partial x} & \text{— (*)} \quad \underline{t \ge 0} \\ u(x,0) = p(x) \end{cases}$$

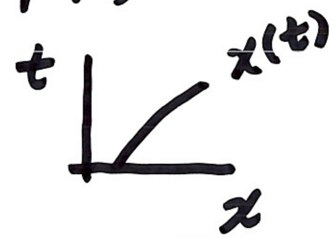
To simplify, let's consider infinite domain in x

$-\infty < x < \infty$

b.c. in x-direction: (at $x \to \pm\infty$) "we don't care"

along $x(t)$

$\underline{u(x(t),t)}$



$$\frac{du}{dt} = \underline{\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}}\,\underbrace{\frac{dt}{dt}}_{=1}$$

(*): $\underline{\dfrac{\partial u}{\partial t} + 10\,\dfrac{\partial u}{\partial x}} = 0$ — (**)

If we choose $\frac{dx}{dt} = 10$ to define the trajectory, then along the trajectory the LHS of (**) $\rightsquigarrow \frac{du}{dt}$

$\Rightarrow$ The PDE collapses into ODE : $\underline{\dfrac{du}{dt} = 0}$

along the trajectory $\underline{\dfrac{dx}{dt} = 10}$

Equivalent to saying that we turned the PDE into 2 families of ODE:

$$\begin{cases} \dfrac{dx}{dt} = 10 & \text{— eq. for trajectory} \\ \dfrac{du}{dt} = 0 & \text{— "proxy" of original PDE on the trajectory, which is now just an ODE!} \end{cases}$$

$\dfrac{dx}{dt} = 10 \Rightarrow x(t) = \underline{x(0)} + 10t$

$\dfrac{du}{dt} = 0 \Rightarrow \underline{u(t)} = \underline{u(0)}$ — (*)

Recall $u(x, t)$ — in general

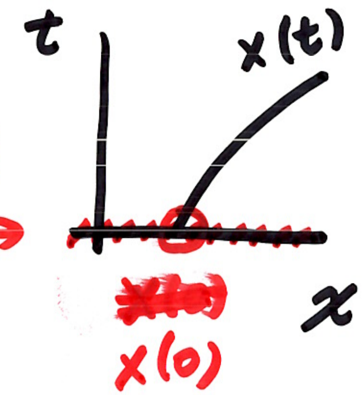
$u(t) \equiv u(x(t), t)$ — on a trajectory $x \equiv x(t)$

in (☆) $u(t)$ is understood as $u(x(t), t)$

$u(0)$ " as $u(x(0), 0)$

(☆): $u(x(t), t) = u(x(0), 0)$

the original b.c. for PDE is given on this line →



since $x(t) = x(0) + 10t$

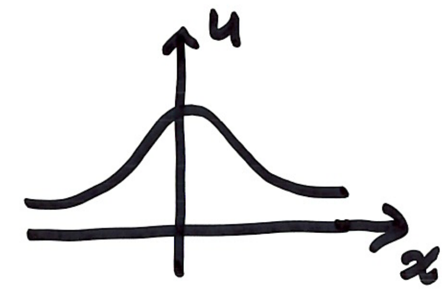
$\Rightarrow x(0) = x(t) - 10t$

$$u(x(t), t) = \underline{u(x(0), 0)} \underset{\uparrow \text{ b.c.}}{=} P(x(0)) = P(x(t) - 10t)$$

$$\Rightarrow \underline{u(x,t) = P(x - 10t)} \quad \#$$

Full solution!!

Ex: if $u(x,0) = p(x) = e^{-x^2}$ ✓



$$\Rightarrow \underline{u(x,t) = e^{-(x-10t)^2}} \quad \checkmark \quad \#$$