$\boxed{\text{Discussion of sol. for HW3 + animation}}$

Lecture 24

HW3 solution

Q2(b)

① $\dfrac{\partial u}{\partial t} = B\dfrac{\partial^3 u}{\partial x^3} \rightarrow \dot{C}_n = -iBn^3 C_n$

② $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^5 u}{\partial x^5} \rightarrow \dot{C}_n = +iDn^5 C_n$

$u(x,t) = \sum_n C_n(t) e^{inx}$

$B > 0$

$D > 0$

① $C_n(t) = C_n(0)\, e^{-iBn^3 t}$

$u = \sum_n C_n(0)\, e^{i(nx - Bn^3 t)}$

$\sin(nx - Bn^3 t)$

$\cos(\quad " \quad)$

② $C_n(t) = C_n(0)\, e^{+iDn^5 t}$

$u = \sum_n C_n(0)\, e^{i(nx + \cancel{D^5} Dn^5 t)}$

$\sin(nx + \cancel{D^5} Dn^5 t)$

$\cos(\quad " \quad)$

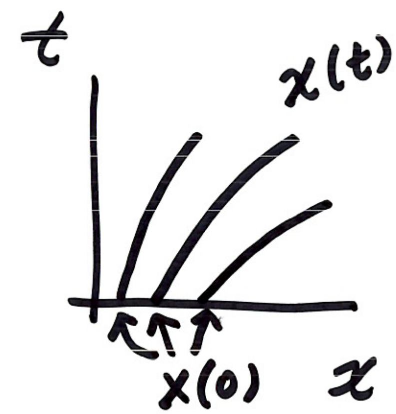
correction after class

"MOC form"

$u(x,t)$

Lec 23

✓ $$\frac{\partial u}{\partial t} + A(x,t,u)\frac{\partial u}{\partial x} = B(x,t,u)$$



- $\frac{dx}{dt} = A(x,t,u)$ trajectory

$\Rightarrow$ along the trajectory

LHS of PDE becomes $\underline{\frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}} \equiv \frac{du}{dt}$

$\Rightarrow$ whole PDE becomes

- $\frac{du}{dt} = B(x,t,u)$ ← proxy of the PDE on the trajectory

PDE → 2 families of ODEs

If a PDE does not have "MOC form"

<u>1-D Heat eq.</u> $\longrightarrow$ * cannot be solved by MOC

<u>1-D Wave eq.</u> $\longrightarrow$ * unless the PDE can be decomposed into multiple first-order PDEs, all w/ MOC form.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = 0$$

1

$$\rightarrow \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u}_{w} = 0$$

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial w}{\partial x} = 0 \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = w \end{cases}$$

$$\mathcal{L}\, u = 0$$

$$\underline{\mathcal{L}_1}\,\underline{\mathcal{L}_2}\, u = 0$$

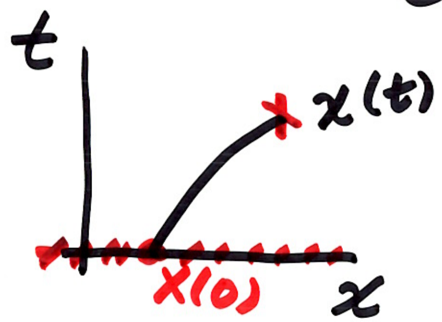
Ex: $u(x,t)$ $\qquad -\infty < x < \infty$

$t \geq 0$ $\qquad$ b.c. at $x \to \pm\infty$: we don't care

$$\begin{cases} \dfrac{\partial u}{\partial t} + t\,\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = p(x) \end{cases}$$

MOC

$$\begin{cases} \dfrac{dx}{dt} = t \Rightarrow \underline{x(t) = x(0) + \dfrac{t^2}{2}} \Rightarrow x(0) = x(t) - \dfrac{t^2}{2} \\ \dfrac{du}{dt} = 0 \Rightarrow u(t) = u(0) \end{cases}$$



$u(x,t) \equiv u(x(t),t) \equiv u(t)$

$$\underline{u(x(t),t)} = \underline{u(x(0),0)} = p(\underline{x(0)})$$

$$= \underline{p\left(x(t) - \frac{t^2}{2}\right)}$$

$$\underline{u(x,t) = p\left(x - \frac{t^2}{2}\right)} \quad \#$$

e.g. if $p(x) = e^{-x^2} \Rightarrow u(x,t) = e^{-\left[x - \frac{t^2}{2}\right]^2}$ #

Ex: $u(x,t)$ $\qquad -\infty < x < \infty$

$t \geq 0$

$$\frac{\partial u}{\partial t} + t x \frac{\partial u}{\partial x} = t u$$

$$u(x,0) = p(x)$$

$$\begin{cases} \dfrac{dx}{dt} = t x \Rightarrow x(t) = x(0) e^{t^2/2} \Rightarrow x(0) = x(t) e^{-t^2/2} \\ \dfrac{du}{dt} = t u \Rightarrow u(t) = u(0) e^{t^2/2} \end{cases}$$

$$u(\underline{x(t)}, t) = u(x(0), 0) e^{t^2/2}$$

$$= p(x(0)) e^{t^2/2}$$

$$= p(\underline{x(t)} e^{-t^2/2}) e^{t^2/2}$$

$$\Leftarrow$$

$u(x,t)$

$$= p(x e^{-t^2/2}) e^{t^2/2}$$

#

---

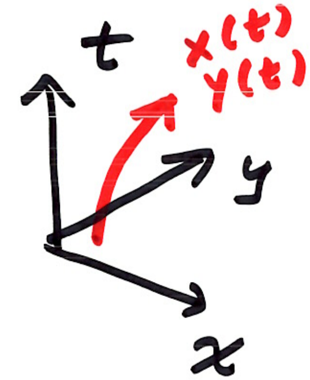
eqs. w/ 3 var.

on trajectory

$u(x, y, t) \longrightarrow u(x(t), y(t), t)$

$$\frac{\partial u}{\partial t} + A(x,y,t,u)\frac{\partial u}{\partial x} + B(x,y,t,u)\frac{\partial u}{\partial y} = C(x,y,t,u)$$

- $\left\{ \begin{array}{l} \frac{dx}{dt} = A \\ \frac{dy}{dt} = B \end{array} \right.$



on the trajectory, LHS of PDE becomes, $\underline{\frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{dy}{dt}\frac{\partial u}{\partial y}} \equiv \frac{du}{dt}$

$\Rightarrow$ PDE becomes

- $\frac{du}{dt} = C$