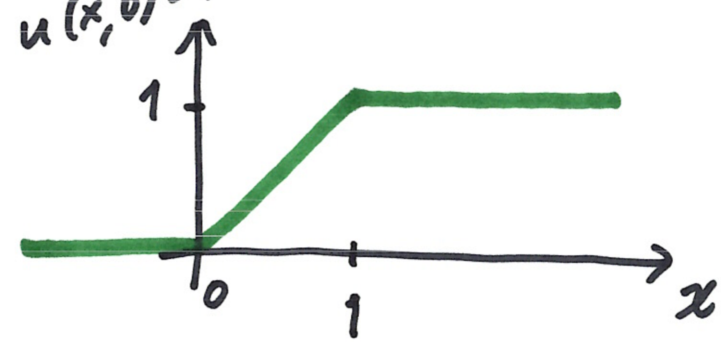
# Lecture 25

<u>MOC</u> → can be used to solve <u>some</u> nonlinear PDEs

**<u>Ex 1.</u>** $u(x,t) \quad -\infty < x < \infty \quad t \geq 0$

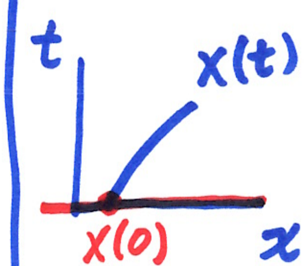
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad *$$




$$u(x,0) = P(x)$$

given

$$P(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$




$u(x,t) = u(x(t),t) \equiv u(t)$

MOC
$$\begin{cases} \dfrac{dx}{dt} = u = u(0) \Rightarrow x(t) = x(0) + u(0)t \\ \dfrac{du}{dt} = 0 \Rightarrow u(t) = u(0) \end{cases}$$

$$= x(0) + u(x(0),0)\cdot t$$
$$= x(0) + P(x(0))\cdot t$$

$$u(x(t),t) = u(x(0),0)$$
$$= P(x(0))$$

~~(i) If x~~ use short hand:

$$x(t) \longrightarrow x$$
$$x(0) \longrightarrow x_0$$

$$x = x_0 + p(x_0)\cdot t \qquad \text{——} \qquad (1)$$

$$u(x,t) = p(x_0) \qquad \text{——} \qquad (2)$$

As preparation, from b.c.

$$p(\underline{x(0)}_{x_0}) = \begin{cases} 0 & x(0)<0 \\ x(0) & 0 \le x(0) \le 1 \\ 1 & x(0) > 1 \end{cases}$$

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

(i) If

$$\begin{tikzcd}
x_0<0 \arrow[r, Rightarrow] \arrow[d] & p(x_0)=0 \arrow[r, Rightarrow, "(2)"] \arrow[d, Rightarrow, "(1)"] & \boxed{u(x,t)=0} \\
\boxed{x<0} \arrow[rr, bend right, dash, "(i)"'] & x=x_0 \arrow[l] &
\end{tikzcd}$$

(iii) If

$$\begin{tikzcd}
x_0>1 \arrow[r, Rightarrow] \arrow[d] & p(x_0)=1 \arrow[r, Rightarrow, "(2)"'] \arrow[d, Rightarrow, "(1)"] & \boxed{u(x,t)=1} \\
x-t>1 \arrow[d, Rightarrow] & x=x_0+t \arrow[l] & \\
\boxed{x>1+t} \arrow[rruu, bend right, dash, "(iii)"'] & &
\end{tikzcd}$$

(iii) If $0 \le x_0 \le 1 \Rightarrow p(x_0) = x_0$

$\Downarrow$ ①

$x = x_0 + x_0 t = x_0(1+t)$

$\Downarrow$

$x_0 = \dfrac{x}{1+t}$

② $u(x,t) = x_0$

$u(x,t) = \dfrac{x}{1+t}$

$0 \le \dfrac{x}{1+t} \le 1$

$\Downarrow$

$0 \le x \le 1+t$

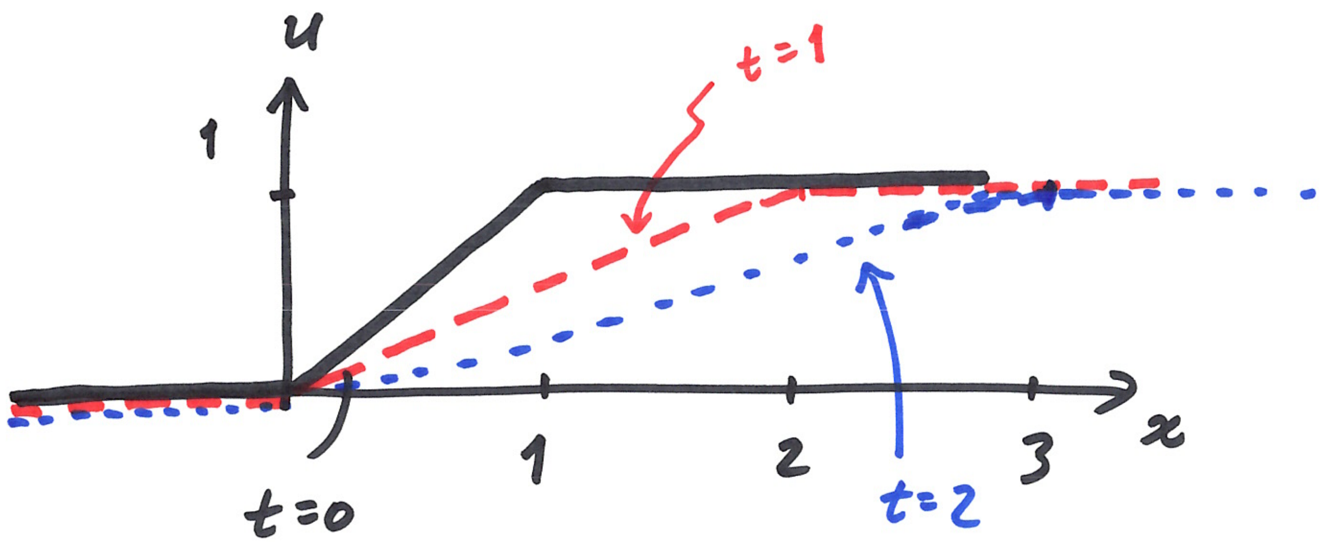
(ii)

Full sol:

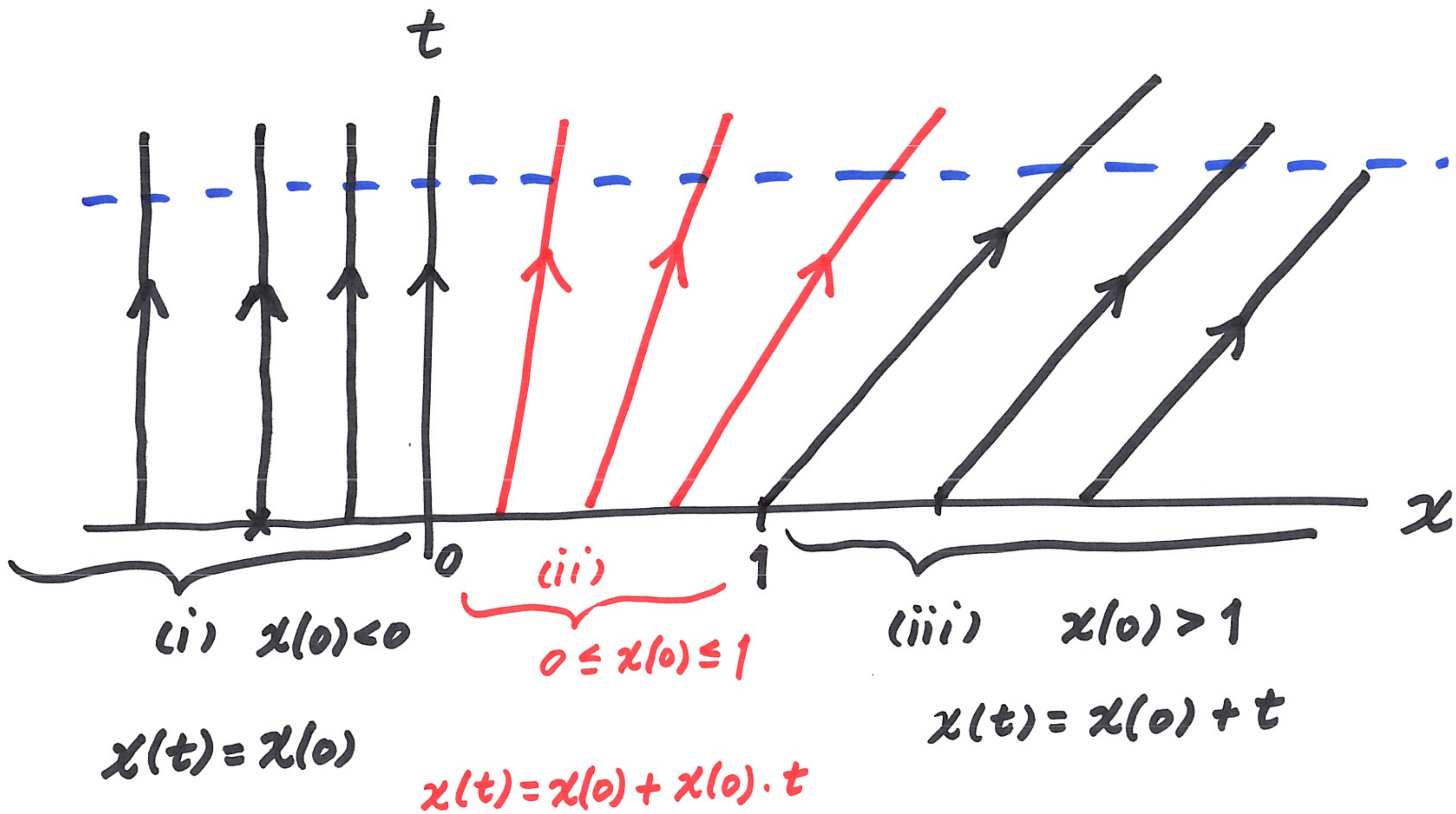
$$u(x,t) = \begin{cases} 0 & \text{if } x<0 \\ \dfrac{x}{1+t} & \text{if } 0 \le x \le 1+t \\ 1 & \text{if } x > 1+t \end{cases}$$

Compare w/ initial condition:

$$u(x,0) = p(x) = \begin{cases} 0 & \text{if } x<0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x>1 \end{cases}$$



# characteristics in $x$-$t$ plane

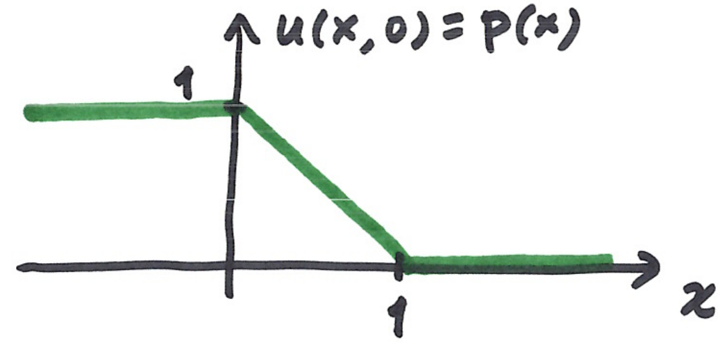


**Ex 2.** Same eq. as Ex 1

$-\infty < x < \infty \qquad t \geq 0$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = p(x)$$




given

$$p(x) = \begin{cases} 1 & x < 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

Repeat the first half of Ex. 1

Use shorthand

$X(t) \to x$

$X(0) \to x_0$

MOC : $\left.\begin{aligned} \frac{dx}{dt} &= u \\ \frac{du}{dt} &= 0 \end{aligned}\right\} \cdots \quad \cdots$

$$x = x_0 + p(x_0)t \quad \text{— (1)}$$

$$u(x,t) = p(x_0) \quad \text{— (2)}$$

$$p(x_0) = \begin{cases} 1 & x_0 < 0 \\ 1-x_0 & 0 \leq x_0 \leq 1 \\ 0 & x_0 > 1 \end{cases}$$

(i) If

$$\begin{tikzcd}
x_0 < 0 \arrow[r, Rightarrow] \arrow[d] & p(x_0) = 1 \arrow[r, Rightarrow, "②"] \arrow[d, Rightarrow, "①"] & u(x,t) = 1 \\
x - t < 0 \arrow[d, Rightarrow] & x = x_0 + t \arrow[l] & \\
x < t \arrow[urr, dash, bend right, "(i)"'] & &
\end{tikzcd}$$

(iii) If

$$\begin{tikzcd}
x_0 > 1 \arrow[r, Rightarrow] \arrow[d] & p(x_0) = 0 \arrow[r, Rightarrow, "②"] \arrow[d, Rightarrow, "①"] & u(x,t) = 0 \\
x > 1 \arrow[urr, dash, bend right, "(iii)"'] & x = x_0 \arrow[l] &
\end{tikzcd}$$

(ii) If $0 \le x_0 \le 1 \Rightarrow P(x_0) = 1 - x_0$

First consider $t < 1$

$\Downarrow$ ①

$$x = x_0 + (1 - x_0)t = (1-t)x_0 + t$$

$$\Downarrow$$

$$x_0 = \frac{x-t}{1-t}$$

$$0 \le \frac{x-t}{1-t} \le 1$$

$$\Downarrow$$

$$0 \le x - t \le 1 - t$$

$$t \le x \le 1$$

$\Rightarrow$ ②

$$u(x,t) = 1 - x_0$$

$$u(x,t) = 1 - \frac{x-t}{1-t}$$

$$\Downarrow$$

$$u(x,t) = \frac{1-x}{1-t}$$

$t \to 1$

?

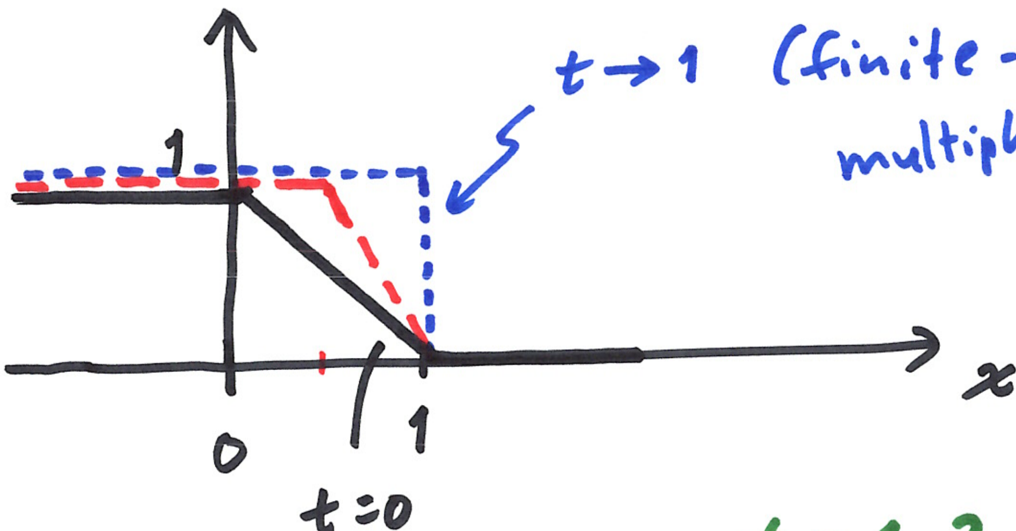
For $\underline{t<1}$

Full sol:

$$u(x,t) = \begin{cases} 1 & x < t \\ \dfrac{1-x}{1-t} & t \le x \le 1 \\ 0 & x > 1 \end{cases}$$

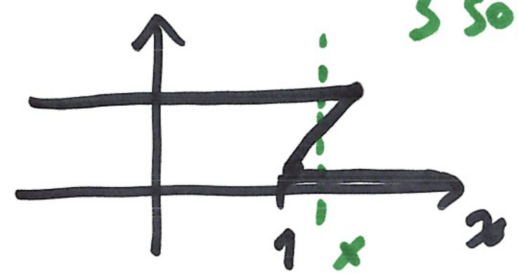
b.c. at $t=0$

$$u(x,0) = p(x) = \begin{cases} 1 & x < 0 \\ 1-x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$




$t \to 1$ (finite-time blowup)

multiple sol.!

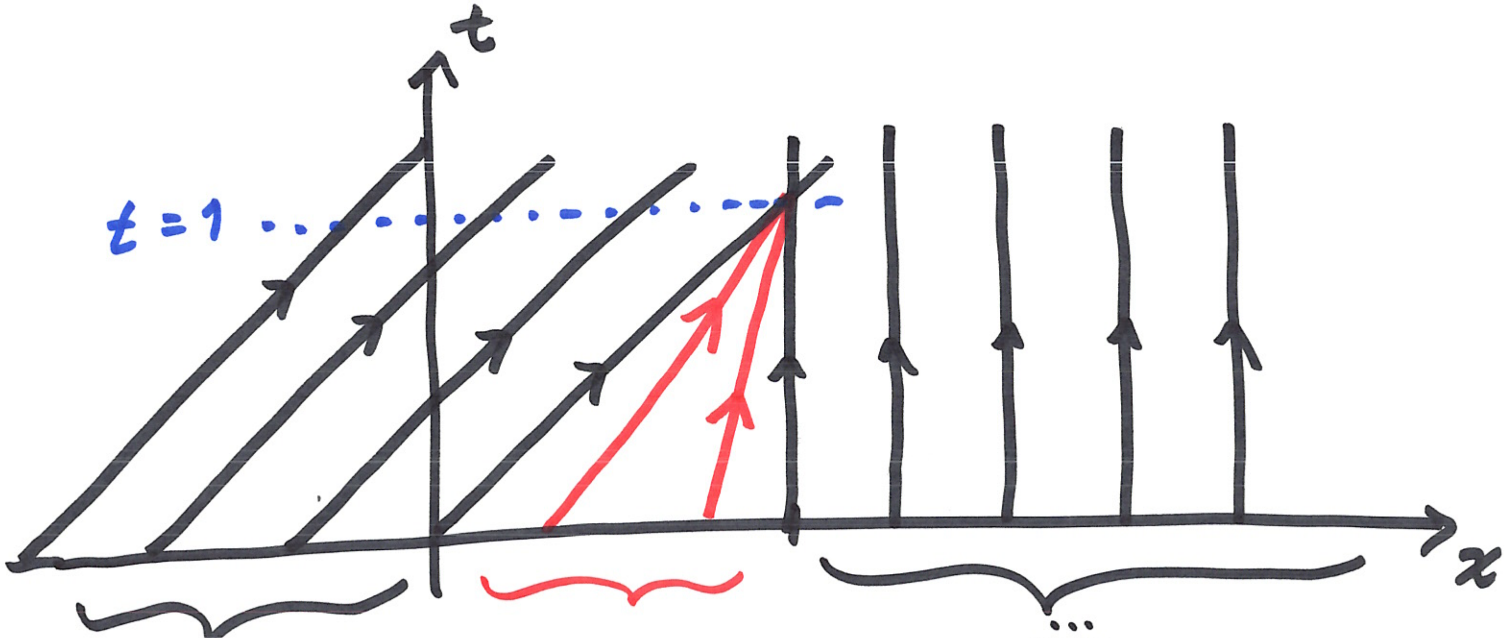



3 sol.

$\underline{t>1\ ?}$

depends on the physical problem (whether multiple sol. is acceptable)

# Characteristics in $x$-$t$ plane




(i) $x(t) = x(0) + t$

(ii) $x(t) = x(0) + [1 - x(0)]\,t$

(iii) $x(t) = x(0)$