Lecture 26 — 4/17

Using MOC to solve <u>higher-order</u> PDE

→ split into multiple first-order PDEs

Old ideas for $\underbrace{\text{ODE}}_{\text{higher-order}}$

$u(x)$

ODE

$$\begin{cases} u'''' + 7u''' + xu'' + \sin(x) = 0 \\ u(0) = A \\ u'(0) = B \\ u''(0) = C \\ u'''(0) = D \end{cases}$$

Let $V \equiv u'$

$W \equiv u''$ $(= V')$

$P \equiv u'''$ $(= W')$

$\cancel{Q \equiv u'''' \ (= P')}$

$$\begin{cases} P' = -7P - xW - \sin(x) \\ W' = P \\ V' = W \\ u' = V \end{cases}$$

$u(0) = A$
$V(0) = B$
$W(0) = C$
$P(0) = D$

PDE :

Ex: 1-D Wave eq. $u(x,t)$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

2-nd order PDE

(Not of MOC form)

$$\underbrace{\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)}_{\mathcal{L}} u = 0$$

$$\mathcal{L} u = 0$$

$$\underline{\mathcal{L}_1} \; \underline{\mathcal{L}_2}$$

1st order   1st order

2 ways

(i) $\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)u}_{w} = 0$

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial w}{\partial x} = 0 & \text{— MOC ✓} \\[2ex] \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = w & \text{— MOC ✓} \end{cases}$$

alternative

(ii) $\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u}_{w}=0$ will also work (in principle)

$$\begin{cases} \dfrac{\partial w}{\partial t}+\dfrac{\partial w}{\partial x}=0 \\ \dfrac{\partial u}{\partial t}-\dfrac{\partial u}{\partial x}=w \end{cases}$$

Sometimes, one way of decomposition works better than the other(s)

Ex: $u(x,t)$

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}-\frac{\partial u}{\partial x}+x$$

$$\begin{cases} \dfrac{\partial w}{\partial t}+\dfrac{\partial w}{\partial x}=w+x \\ \dfrac{\partial u}{\partial t}-\dfrac{\partial u}{\partial x}=w \end{cases} \checkmark$$

(ii) better $\Rightarrow \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u}_{w}=\underbrace{\left(\frac{\partial u}{\partial t}-\frac{\partial u}{\partial x}\right)}_{w}+x$

Ex: (relevent to HW5 Q4)

$-\infty < x < \infty \qquad t \geq 0$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + t \\[2ex] u(x,0) = x \\ u_t(x,0) = 2 \end{cases}$$

PDE: $\left(\dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}\right) u = t$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u}_{w} = t$$

$w(x,t) \equiv u_t(x,t) + u_x(x,t)$

at $t=0$

$$\begin{aligned} w(x,0) &\equiv u_t(x,0) + u_x(x,0) \\ &= 2 + \frac{\partial}{\partial x}(x) \\ &= 3 \end{aligned}$$

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial w}{\partial x} = t & — (1) \\[2ex] \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = w & — (2) \end{cases}$$

Need to solve (1) first → b.c. = ?

Solve ①

$$\boxed{\begin{gathered}\frac{\partial w}{\partial t} - \frac{\partial w}{\partial x} = t \\ w(x,0) = 3\end{gathered}}$$

MOC

$$\begin{cases} \dfrac{dx}{dt} = -1 & x(t) = X(0) - t \\ \dfrac{dw}{dt} = t & \Rightarrow w(t) = w(0) + t^2/2 \end{cases}$$

$$w(x(t),t) = w(x(0),0) + t^2/2$$
$$= 3 + t^2/2$$

$$\Downarrow$$

Full sol. for $w(x,t)$

$$\boxed{w(x,t) = 3 + t^2/2}$$

move to ②

② $$\boxed{\begin{gathered}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3 + t^2/2 \\ u(x,0) = x\end{gathered}}$$

b.c. for ② is just "$u(x,0) = x$"

MOC

$$\frac{dx}{dt} = 1 \Rightarrow \underline{x(t) = x(0) + t} \Rightarrow x(0) = x(t) - t$$

$$\frac{du}{dt} = 3 + \frac{t^2}{2} \Rightarrow u(t) = u(0) + 3t + \frac{t^3}{6}$$

$$u(x(t),t) = \underline{u(x(0),0)} + 3t + t^3/6$$
$$= x(0) + 3t + t^3/6$$
$$= (x(t) - t) + 3t + t^3/6$$

Full sol.

$$u(x,t) = x + 2t + t^3/6$$

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Further examples on MOC

Ex: $\quad -\infty < x < \infty \quad t \geq 0$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = x \\ u(x,0) = 0 \end{cases}$$

MOC:

coupled $\begin{cases} \dfrac{dx}{dt} = u & — ① \\ \dfrac{du}{dt} = x & — ② \end{cases}$

$$\boxed{\frac{d}{dt}\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix}}$$

✓ "decouple"

or

~~comp~~

✓ combine ①, ② into 1 eq. w/ 1 var

general remark (Not needed for our HW, exam questions) Dont worry about it!

$\vec{x}$ $\quad \dfrac{d\vec{x}}{dt} = \mathbb{A}\vec{x}$ $\quad$ diagonalization to "decouple"

λ_j eig. val. of $\mathbb{A}$

$\vec{e}_j$ " vector of $\mathbb{A}$ $\quad \boxed{\mathbb{A} = \mathbb{E}^{-1}\Lambda\mathbb{E}}$

$\mathbb{E} = (\vec{e}_j \ldots)$ $\quad \dfrac{d\vec{x}}{dt} = \mathbb{E}^{-1}\Lambda\mathbb{E}$ $\quad \vec{y} \equiv \mathbb{E}\vec{x}$

$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{pmatrix}$ $\quad \dfrac{d\vec{y}}{dt} = \Lambda\vec{y}$ ✓

Strategies :

(i)

$$\begin{cases} \dfrac{dx}{dt} = u & \text{①} \\ \dfrac{du}{dt} = x & \text{②} \end{cases}$$

+)

$$\frac{d(x+u)}{dt} = (x+u)$$

define $v \equiv x+u$

$$\frac{dv}{dt} = v \qquad v(t) = v(0)e^t$$

$$x(t)+u(t) = [x(0)+\underbrace{u(0)}_{=0}]e^t$$

$$= x(0)e^t$$

$$u(t) = x(t) - x(0)e^t$$

Use ①

$$\frac{dx}{dt} = x - x(0)e^t$$

(Sec 8.3)

$$x(t) = x(0)e^t + \int_0^t x(0)e^{\hat{t}}e^{(t-\hat{t})}\,d\hat{t}$$

⋮

keep going

⋮

✓

try to work out detail !

(ii) $$\frac{dx}{dt} = u \quad \text{— } ①$$

$$-)\quad \frac{du}{dt} = x \quad \text{— } ②$$

$$\frac{d(x-u)}{dt} = (u-x)$$

$$v \equiv x - u$$

$$\frac{dv}{dt} = -v$$

$$v(t) = v(0)e^{-t}$$

$$\vdots$$

work out detail

(iii)

$$\frac{dx}{dt} = u \quad \text{— } ①$$

$$\frac{du}{dt} = x \quad \text{— } ② \quad \Rightarrow \quad x(t) = \frac{du}{dt} = A\sinh(t) + B\cosh(t)$$

$$\frac{d}{dt}② \Rightarrow \frac{d^2u}{dt^2} = u \Rightarrow u(t) = A\cosh(t) + B\sinh(t)$$

From b.c. : "$u(x,0)=0$" $\Rightarrow$ $u(0)=0$

$$\underline{A=0} \Leftarrow 0 = u(0) = A\underbrace{\cosh(0)}_{1} + B\cancelto{0}{\sinh(0)}$$

$$\Rightarrow \quad \left.\begin{aligned} x(t) &= B\cosh(t) \\ u(t) &= B\sinh(t) \end{aligned}\right\} \quad \frac{u(t)}{x(t)} = \tanh(t)$$

$$u(t) = x(t)\tanh(t)$$

Full sol :

$$\boxed{u(x,t) = x\tanh(t)} \Leftarrow u(\underline{x(t)}, t) = \underline{x(t)} \cdot \tanh(t)$$