Lecture 27

4/19

Final Exam : Focus on material after midterm

25 pts   Ch. 3, Ch. 8, Ch. 12

* Open book, open note
* 4 Qs   *
  * Solve as many Qs as you like
  * Instructor will pick your "best 3" sols. to determine the grade
* Only need to solve 3 Qs perfectly to earn 25 pts.

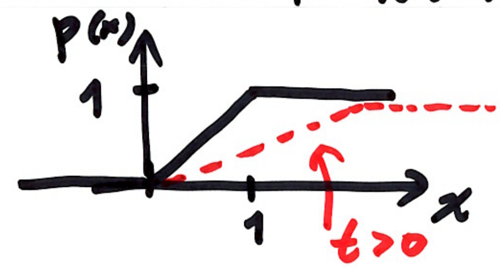
Qualitative behavior of sol. of some nonlinear PDEs (in the context of MOC)

Ex:
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad -\infty < x < \infty$$
$$t \geq 0$$
$$u(x,0) = p(x)$$

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

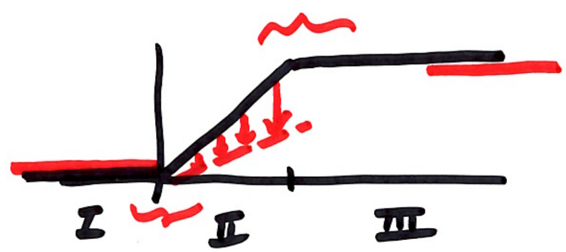
Previous lecture:

Ex 1
$$p(x) = \begin{cases} 0 & x<0 \\ x & 0\le x\le 1 \\ 1 & x>1 \end{cases}$$



Ex 2
$$p(x) = \begin{cases} 1 & x<0 \\ 1-x & 0\le x\le 1 \\ 0 & x>1 \end{cases}$$

Ex 1



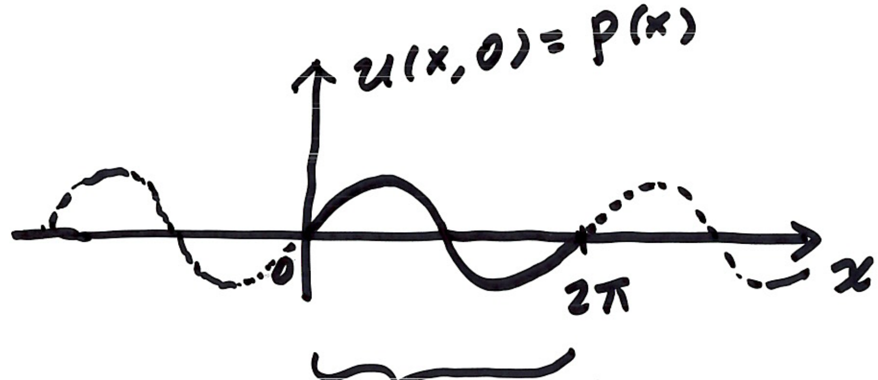
$t=0$	I	II	III
u	0	x	1
u_x	0	1	0
$-u\cdot u_x$	0	$-x$	0

Ex 3. $\cancel{\frac{\partial u}{\partial t} =}$ $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$ $\quad -\infty < x < \infty$

$t \geq 0$

$u(x,0) = p(x)$

Consider $p(x) = \sin(x)$



(No analytic sol.)

suffices to consider sol. over 1 period

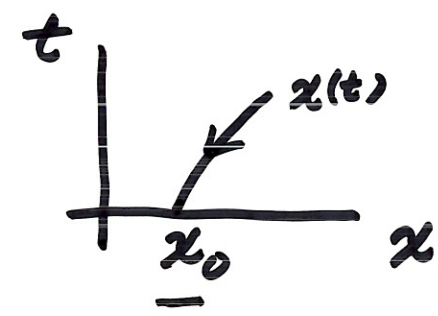
Goal: Obtain some qualitative understanding of — —

MOC:
$$\begin{cases} \frac{dx}{dt} = u \equiv u(0) \Rightarrow x(t) = x(0) + \underline{u(0)}\, t \\ \qquad\qquad\qquad\qquad\quad = x(0) + p(x(0)) \cdot t \\ \frac{du}{dt} = 0 \Rightarrow \underline{u(t) = u(0)} \end{cases}$$

$$u(x(t),t) = u(x(0),0)$$
$$= p(x(0))$$

As before, use short hand:
$x(t) \to x$
$x(0) \to x_0$

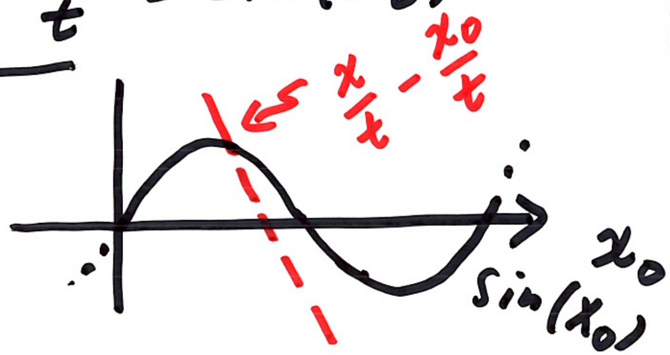
$$\begin{cases} x = x_0 + p(x_0)\,t & \text{— ①} \\ u(x,t) = p(x_0) & \text{— ②} \end{cases}$$



$$\begin{cases} x = x_0 + \sin(x_0)\,t & \text{— ①} \\ u(x,t) = \sin(x_0) & \text{— ②} \end{cases}$$

①: "given (x, t) solve x_0"

$$\frac{x}{t} - \frac{x_0}{t} = \sin(x_0)$$



$u(\overset{x}{0.1}, \overset{t}{0.3}) = ?$

① ~~$0.1 = x_0 + \sin(0.1)\cdot(0.3)$~~

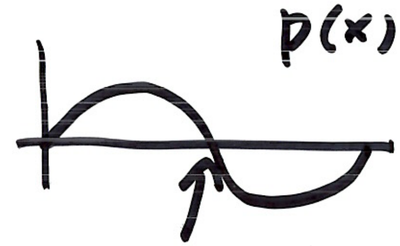
$0.1 = x_0 + \sin(x_0)\cdot 0.3$

solve x_0

plug it into

$= \sin(x_0)$

Consider the behavior of solution in the vicinity of $x = \pi$



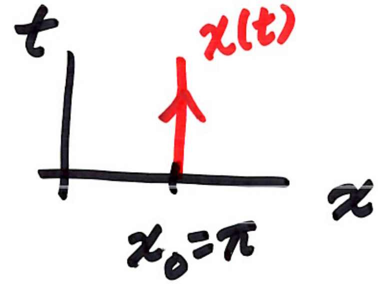
* Claim : <u>nodal point at $x = \pi$</u> <u>never move</u>

$x(0)$

From ① : Choose $x_0 = \pi$

$\hookrightarrow$ $x = \pi + \sin(\pi) \cdot t$ ← all

$x(t)$ $= \pi + 0 = \underline{\underline{\pi}}$



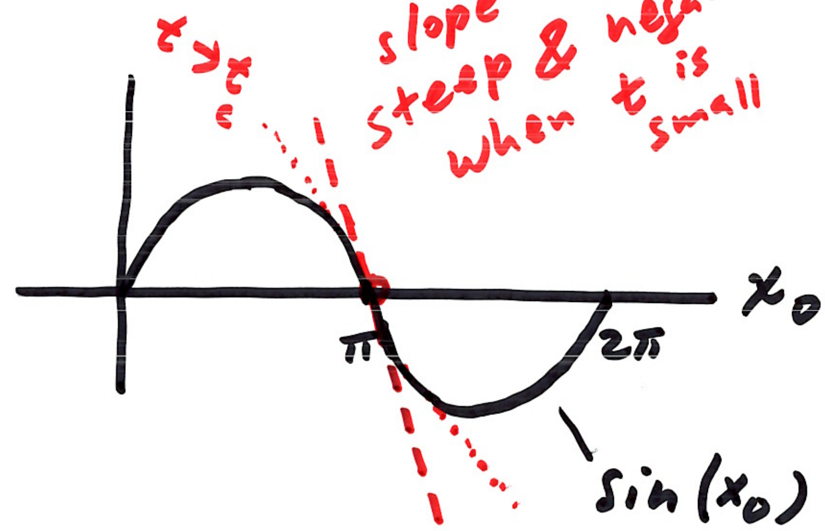
$u(\pi, t) = 0$ for all time

(is $\underline{\underline{\text{a}}}$ sol.)

↑ not necessarily the "only" sol.

From ① $\underbrace{\frac{x}{t} - \frac{x_0}{t}}_{\text{consider}} = \sin(x_0)$

the line that runs through $x_0 = \pi$



$\frac{x}{t} - \frac{x_0}{t}$

slope $= -\frac{1}{t}$

t small:

1 intersection

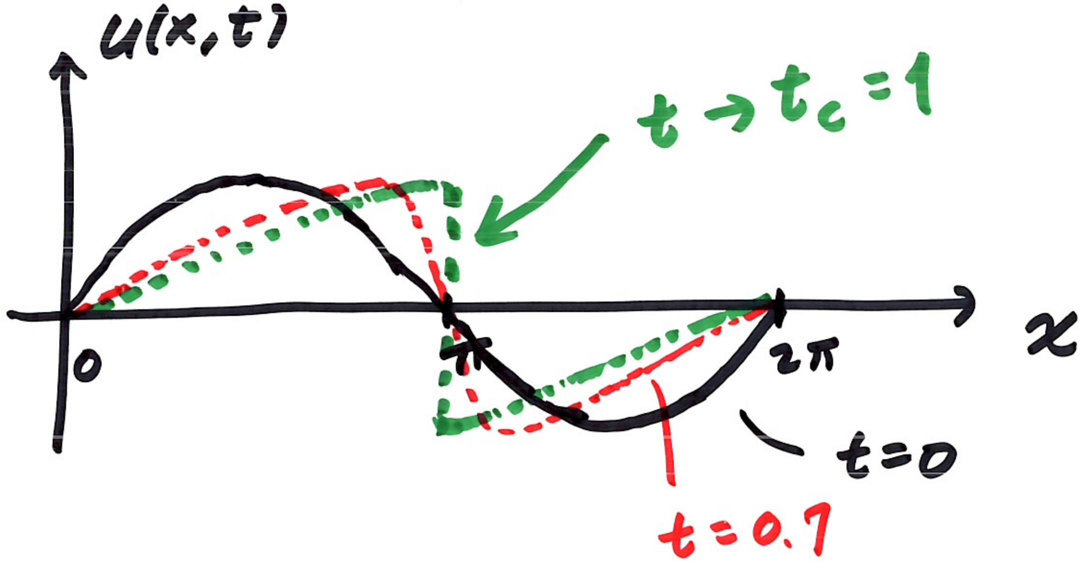
$\Rightarrow$ 1 sol. of x_0 for given (x, t)

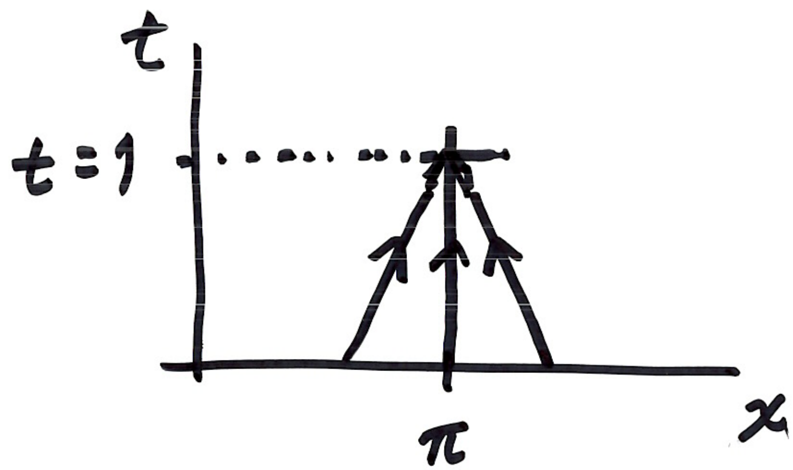
increasing $t \longrightarrow \underline{\underline{t_c}} \longleftarrow$ when the line $\frac{x}{t} - \frac{x_0}{t}$ becomes the <u>tangent</u> of $\sin(x_0)$ at $x_0 = \pi$

"finite-time blowup" — $\frac{t_c = 1}{\text{infinite many}}$ $\Big\}$ $\Rightarrow$ $-\frac{1}{t} = \left.\frac{d\sin(x_0)}{dx_0}\right|_{x_0=\pi}$

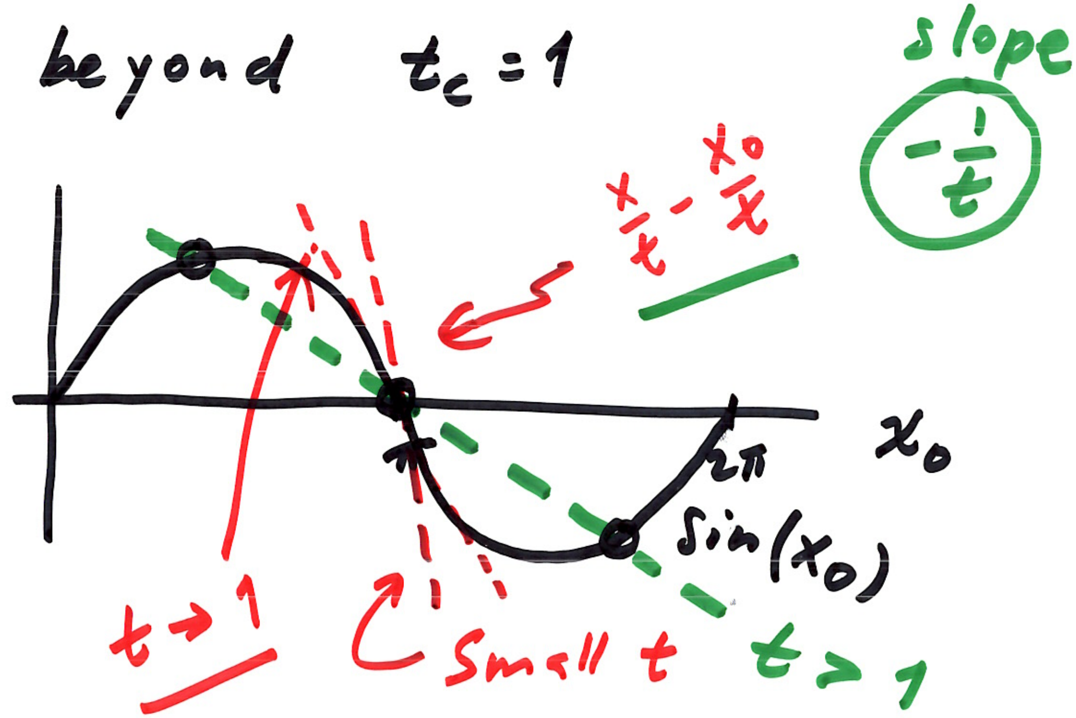
$= \cos(\pi) = -1$

Sol. in physical space




t
t = 1
π
x

increase t beyond $t_c = 1$

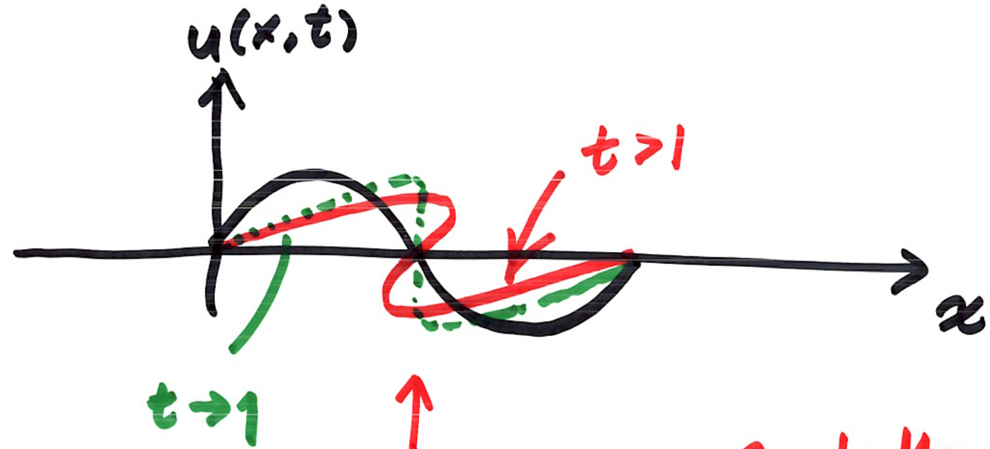


$t > 1$

3 sol. of x_0
for a given pair
of (x, t)

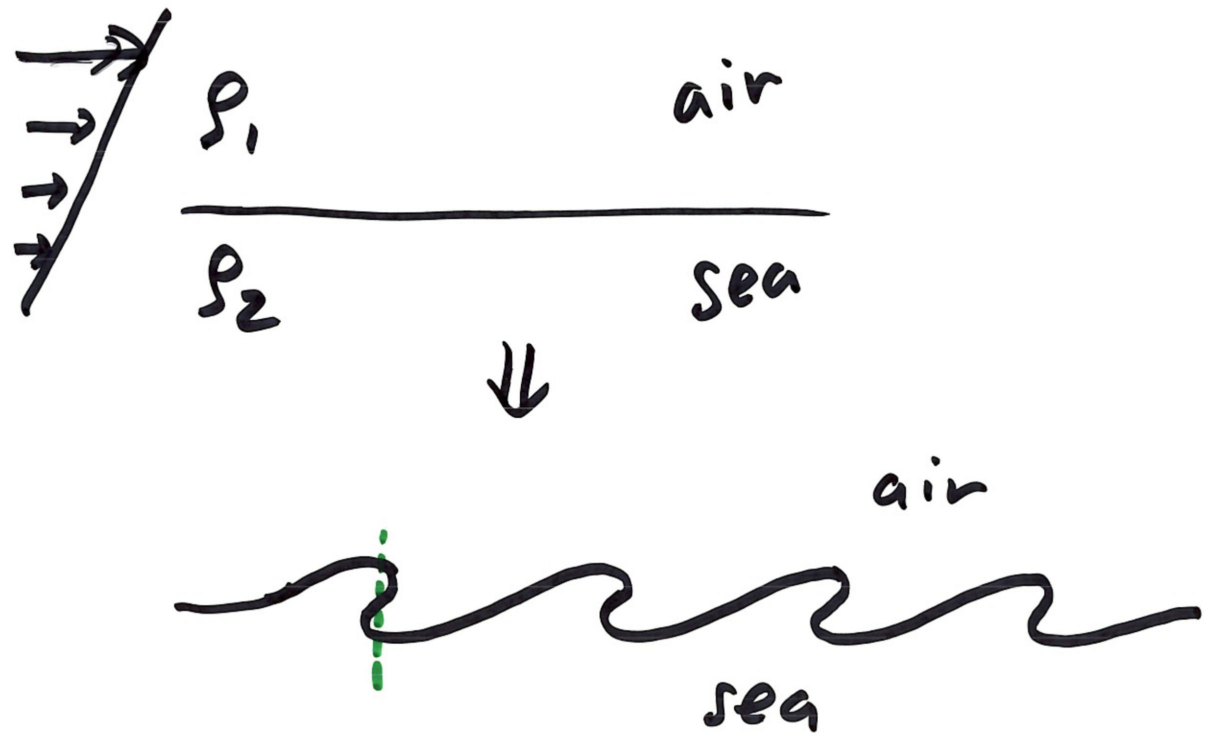
$u(x,t) = \sin(x_0)$ → 3 different values of u for the same (x, t)

physical space

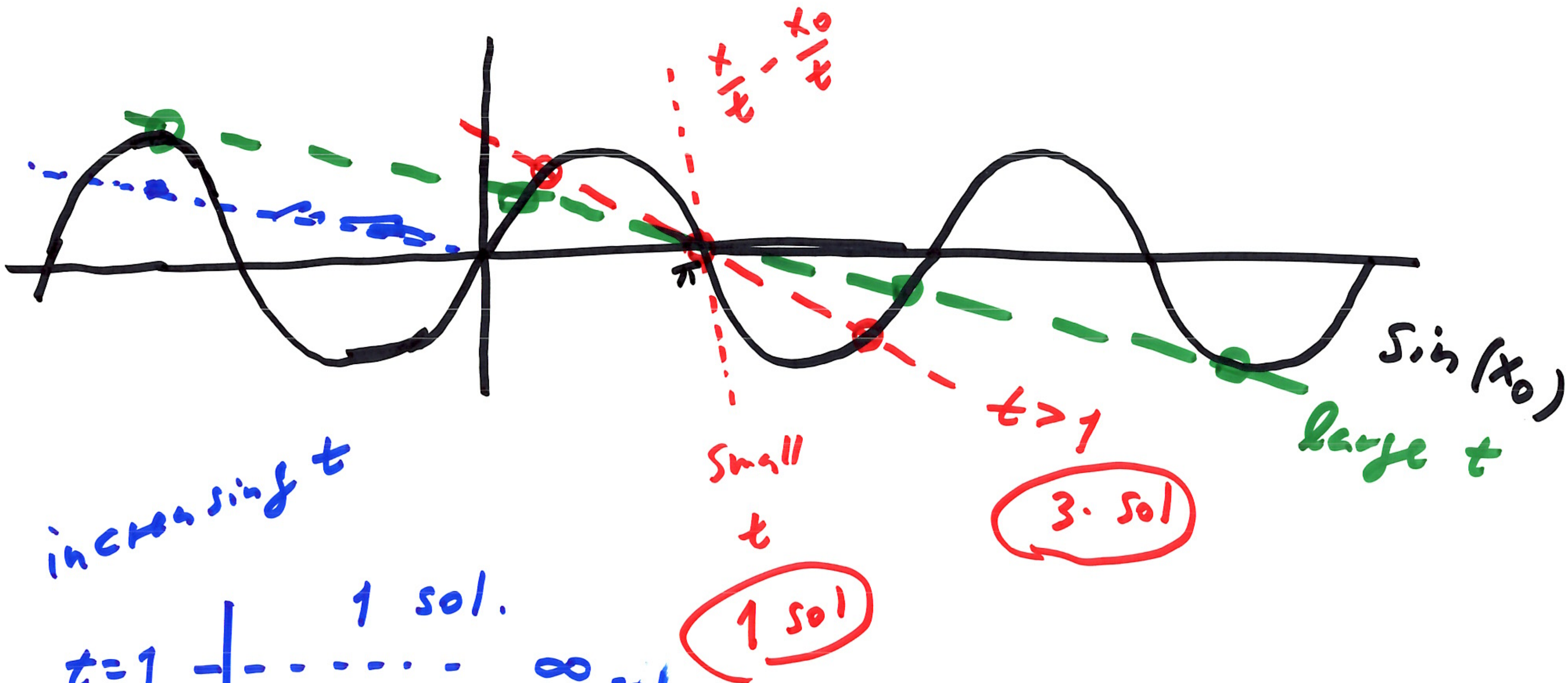


at $x=\pi$: 3 different values of u

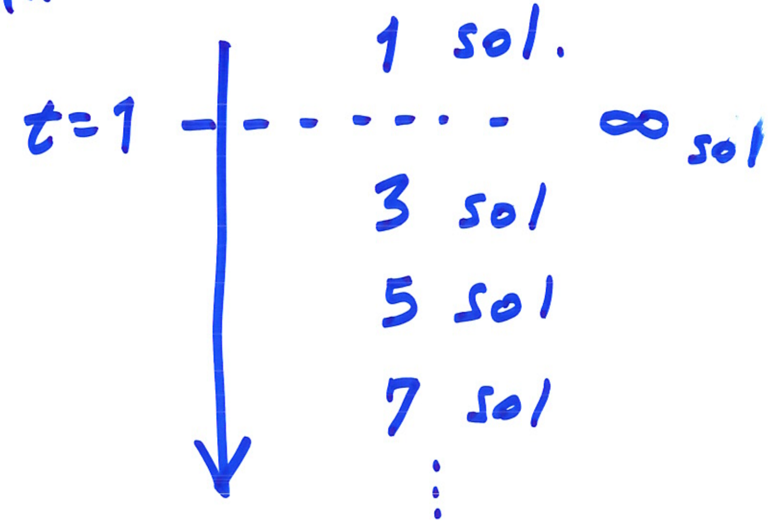
real phenomena:
shear flow
different density

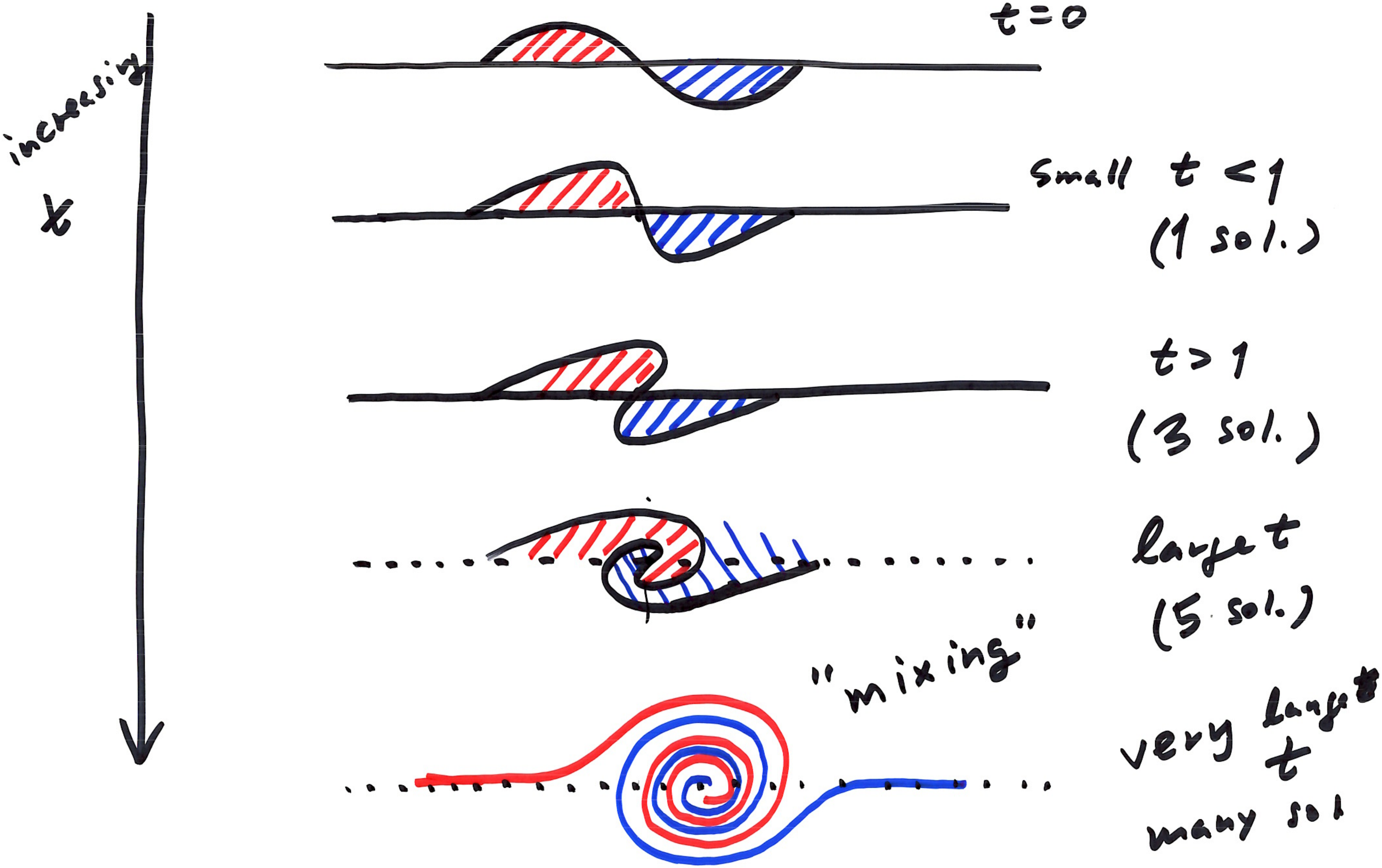


Further increasing t ?



increasing t




increasing
t
$t=0$
Small $t<1$
(1 sol.)
$t>1$
(3 sol.)
large t
(5 sol.)
"mixing"
very large t
many sol