Lecture 28

4/24

* PDEs in higher dimensions (3 or more variables)

All semester

PDE with 2 vars

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad u(x,t)$$

$$u(x,y)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0$$

$$\vdots$$

MOC

$u(x,t)$

$$* \quad \frac{\partial u}{\partial t} + \underline{A(x,t,u)}\,\frac{\partial u}{\partial x} = \underline{B(x,t,u)}$$

MOC

$$\begin{cases} \dfrac{dx}{dt} = A \\[2ex] \dfrac{du}{dt} = B \end{cases}$$

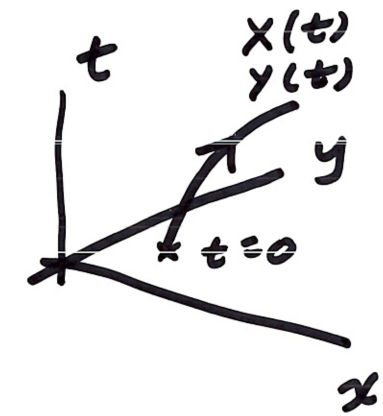
MOC PDE w/ 3 vars $u(x, y, t)$

$$\frac{\partial u}{\partial t} + A(x,y,t,u)\frac{\partial u}{\partial x} + B(x,y,t,u)\frac{\partial u}{\partial y} = C(x,y,t,u)$$

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dfrac{dx}{dt} = A \\[2ex] \dfrac{dy}{dt} = B \end{array}\right\} \text{trajectory} \\[12ex] \dfrac{du}{dt} = C \end{array}\right.$$

on it, LHS of PDE becomes

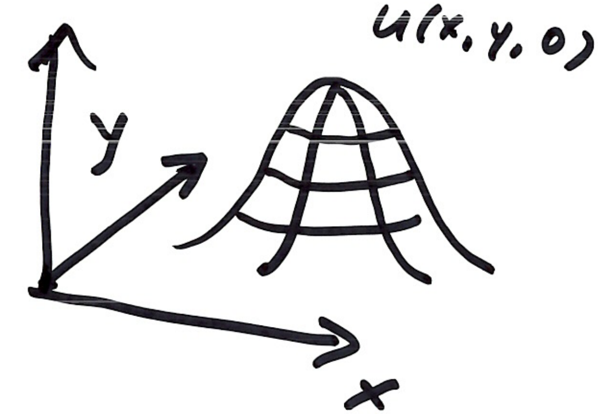
$$\underbrace{\frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{dy}{dt}\frac{\partial u}{\partial y}}_{\equiv \, \frac{du}{dt}}$$



trajectory in x-y-t space

$$u(x, y, t) = \underbrace{u(x(t), y(t), t)}_{u(t)}$$

Ex: $u(x,y,t)$ $\quad -\infty < x < \infty$, $\quad -\infty < y < \infty$, $\quad t \geq 0$



$$\begin{cases} \dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = t \\ u(x,y,0) = \underbrace{e^{-(x^2+y^2)}}_{p(x,y)} \end{cases}$$

MOC

$$\frac{dx}{dt} = x \qquad \underline{x(t)} = x(0)e^{t} \Longrightarrow x(0) = x(t)e^{-t}$$

$$\frac{dy}{dt} = y \qquad \underline{y(t)} = y(0)e^{t} \Longrightarrow y(0) = y(t)e^{-t}$$

$$\frac{du}{dt} = t \qquad \underline{u(t)} = \underline{u(0)} + \frac{t^2}{2}$$

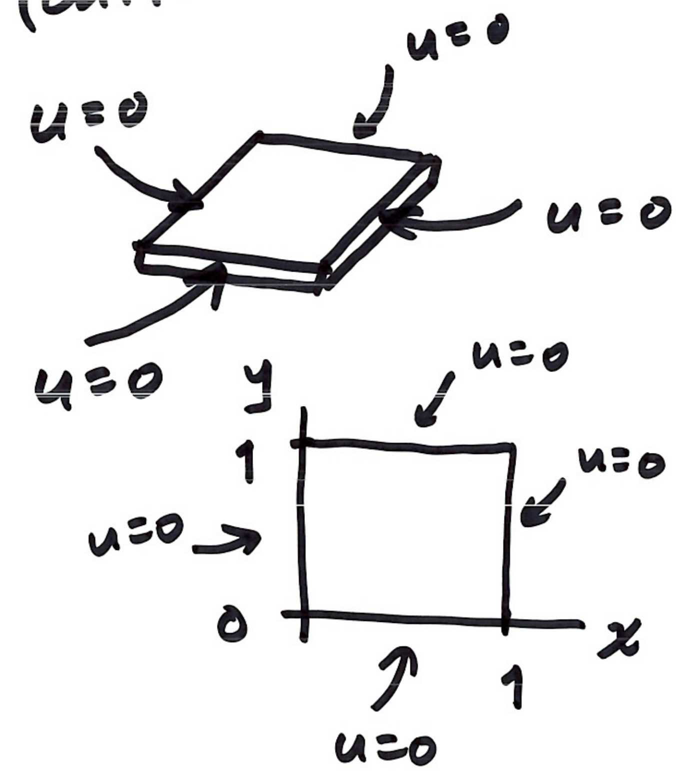
$$u(x(t), y(t), t) \qquad u(x(0), y(0), 0) = p(\underline{x(0)}, \underline{y(0)}) = p(x(t)e^{-t}, y(t)e^{-t})$$

Full sol: $\underline{u(x,y,t) = e^{-[(xe^{-t})^2 + (ye^{-t})^2]} + \frac{t^2}{2}}$ #

Old material

(Ch. 7)

time-dependent 2-D Heat eq. $u(x,y,t)$



$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{— } (*)$$

$u(0,y,t)=0$ — (i)

$u(1,y,t)=0$ — (ii)

$u(x,0,t)=0$ — (iii)

$u(x,1,t)=0$ — (iv)

$u(x,y,0) = P(x,y)$ — (v)

homogeneous subsystem

Sep. of var: $u(x,y,t) \sim R(x,y)\,S(t)$

$$\rightarrow (*) \quad R\frac{dS}{dt} = S\left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2}\right)$$

$$\rightarrow \underbrace{\frac{1}{S}\frac{dS}{dt}}_{\mathcal{S}(t)} = \underbrace{\frac{1}{R}\left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2}\right)}_{\mathcal{R}(x,y)} = \alpha$$

$$\frac{1}{S}\frac{dS}{dt} = \alpha$$

$$\underbrace{\frac{1}{R}\left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2}\right) = \alpha}$$

$$R(x,y) \sim G(x)H(y)$$

$$\cancel{R}\,\frac{1}{GH}\left[HG'' + G\ddot{H}\right] = \alpha$$

$$(\ )' \equiv \frac{d(\ )}{dx}$$

$$(\dot{\ }) \equiv \frac{d(\ )}{dy}$$

$$\Rightarrow \quad \frac{G''}{G} + \frac{\ddot{H}}{H} = \alpha$$

$$\underbrace{\frac{G''}{G}}_{G(x)} = \underbrace{\alpha - \frac{\ddot{H}}{H}}_{\mathcal{H}(y)} = \beta$$

$$\frac{dS}{dt} = \underline{\alpha}\, S$$

$$\frac{d^2G}{dx^2} = \underline{\beta}\, G$$

$$\frac{d^2H}{dy^2} = \underline{(\alpha - \beta)}\, H$$

$\underline{\alpha}$, $\underline{\beta}$

<u>s.o.v.</u> <u>b.c.</u> (i) – (iv)

$$\begin{cases} G'' = \beta G & G(0)=0,\ G(1)=0 \longrightarrow (\star) \\ \ddot{H} = (\alpha-\beta)H & H(0)=0,\ H(1)=0 \\ \dfrac{dS}{dt} = \alpha S & \end{cases}$$

$(\star)$: $\beta_n = \underline{-(n\pi)^2}$ $\quad G_n(x) = \sin(n\pi x)$

$$\ddot{H}_m = (\alpha_{m,n} - \beta_n) H_m$$

$$= (\underbrace{\alpha_{m,n} + (n\pi)^2}_{\delta_{mn}}) H_m$$

$\delta_{m,n} = -(m\pi)^2$

$H_m(y) = \sin(m\pi y)$

$\Rightarrow$ $\boxed{\alpha_{m,n} = -(n^2+m^2)\pi^2}$

$$\frac{dS_{mn}}{dt} = \alpha_{mn} S_{mn} = -(n^2+m^2)\pi^2$$

$$S_{m,n}(t) = S_{m,n}(0)e^{-(n^2+m^2)\pi^2 t}$$

$$u_{m,n}(x,y,t) = G_n(x)\,H_m(y)\,S_{m,n}(t)$$

$$= \sin(n\pi x)\,\sin(m\pi y)\,e^{-(n^2+m^2)\pi^2 t}$$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \underline{a_{m,n}}\,\sin(n\pi x)\,\sin(m\pi y)\,e^{-(n^2+m^2)\pi^2 t}$$

$a_{m,n}$: determined from b.c. (v)

$$p(x,y) = \sum_n \sum_m a_{m,n}\,\sin(n\pi x)\,\sin(m\pi y)$$

Apply orthogonality relation twice in x- & y-direction

$$a_{m,n} = 4\int_0^1\int_0^1 p(x,y)\,\sin(n\pi x)\,\sin(m\pi y)\,dx\,dy$$

#

Ch. 10 Fourier transform on $-\infty < x < \infty$

Ch. 3 Fourier series

periodic domain

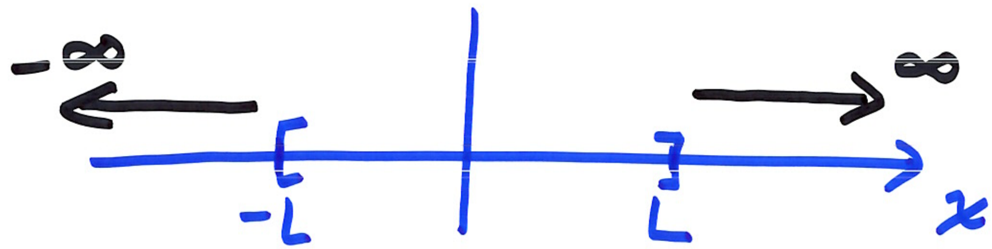
$x \in [-L, L]$

$[0, 2\pi]$

$[0, 1]$

$[-\pi, \pi]$

$\vdots$



F.S. $u(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\quad) + b_n \sin(\quad)$

$\| \downarrow$

$\int_{-\infty}^{\infty}$

<u>Sec 8.3</u>

$*$ $u(t)$ (*)

$$\frac{du}{dt} = -\alpha u + f(t), \quad u(0) = \boxed{}$$

Special case: $f(t) = 0 \Rightarrow \frac{du}{dt} = -\alpha u \Rightarrow u(t) = u(0)e^{-\alpha t}$

$\alpha = 0 \Rightarrow \frac{du}{dt} = f(t) \Rightarrow u(t) = u(0) + \int_0^t f(\hat{t})\,d\hat{t}$

$$u(5) = u(0) + \int_0^5 f(\hat{t})\,d\hat{t}$$

$$0 \longrightarrow 5 \quad \hat{t}$$

Strategy for solving (*)

$\hookrightarrow$ transforming it to

try $u(t) = v(t)\,\underline{B(t)}$ plug into $(*)$

"method of integrating factor"

$$\rightarrow \frac{dvB}{dt} = -\alpha vB + f$$

$$v\frac{dB}{dt} + B\frac{dv}{dt} = -\alpha vB + f$$

$$\Rightarrow B\frac{dv}{dt} = \underbrace{-\left[v\left(\frac{dB}{dt} + \alpha B\right)\right]}_{\text{if this vanishes we are good!}} + f$$

desire to have $\frac{dB}{dt} + \alpha B = 0$

$$\Longrightarrow B(t) = \cancel{B(0)}\, e^{-\alpha t}$$

don't care

So, Let

$$u(t) = v(t)\,e^{-\alpha t}$$

$$\Rightarrow e^{-\alpha t}\frac{dv}{dt} = f$$

$$\boxed{\frac{dv}{dt} = \underbrace{f e^{\alpha t}}_{F(t)}} \quad \text{— } (**)$$

Sol. of (**) $\quad v(t)=u(t)e^{\alpha t}$

$$v(t) = V(0) + \int_0^t F(\hat{t})\,d\hat{t}$$

$$= V(0) + \int_0^t f(\hat{t})e^{\alpha\hat{t}}\,d\hat{t}$$

$$\Rightarrow \quad u(t)e^{\alpha t} = u(0)\underbrace{e^{\alpha\cdot 0}}_{1} + \int_0^t f(\hat{t})\,e^{\alpha\hat{t}}\,d\hat{t}$$

$$u(t) = u(0)e^{-\alpha t} + e^{-\alpha t}\int_0^t f(\hat{t})\,e^{\alpha\hat{t}}\,d\hat{t}$$

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alternative

$$u(t) = u(0)e^{-\alpha t} + \int_0^t f(\hat{t})e^{-\alpha(t-\hat{t})}\,d\hat{t}$$

Ex:

$$\boxed{\frac{du}{dt} = -\underbrace{3}_{\alpha}u + \underbrace{e^{-t}}_{f(t)} \qquad u(0)=5}$$

$$u(t) = u(0)e^{-3t} + e^{-3t}\int_0^t \underbrace{e^{-\hat{t}}}_{f(\hat{t})} e^{3\hat{t}}\,d\hat{t}$$

$$= u(0)e^{-3t} + \left(e^{-3t}\right)\tfrac{1}{2}\left(e^{2t}-1\right)$$

$$= 5e^{-3t} + \tfrac{1}{2}\left(e^{-t} - e^{-3t}\right)$$

$$\begin{cases} \dfrac{dx}{dt} = u \\[2ex] \dfrac{du}{dt} = x \end{cases}$$

+)

$$\frac{d(x+u)}{dt} = x+u$$

$V \equiv x+u$

$$\frac{dv}{dt} = v \quad ✓$$

−)

$$\frac{dx}{dt} = u \qquad \frac{du}{dt} = x$$

$$\frac{d(x-u)}{dt} = -(x-u)$$

$$\frac{dv}{dt} = -V \quad ✓$$

$V \equiv x-u$

$$\frac{dx}{dt} = u$$

$$\frac{du}{dt} = x \longrightarrow \frac{d^2u}{dt} = u \quad ✓$$