Lecture 29

4/26

Review for final exam ~~(5~~ (25 pts)

- Open book, open note * focus: material "after midterm"

<u>4Qs</u> → solve as many Qs as you like.

(Instructor will pick your "best 3" sol. to determine the grade)

* Correct solutions for 3 Qs → perfect score.

Ch. 3 PDE in periodic domain → (F.S.)

Ch. 8 Nonhomogeneous system

Ch. 12 Method of Characteristics

Ch. 3 System with periodic b.c.

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

periodic in x

$u(x,0) = \underline{p(x)}$

$0 \le x \le 2\pi$

$\underline{\text{periodic}}$ in x

$f(x)$

Complex F.S.

$$f(x) = \boxed{\sum_{n=-\infty}^{\infty} c_n e^{inx}}$$

$$\equiv$$

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$= c_0 + 2 \cdot \mathrm{Re}\left[\sum_{n=1}^{\infty} c_n e^{inx}\right]$$

Need to keep

Never need to process

$$\cdots + c_{-2} e^{-i2x} + \underbrace{c_{-1} e^{-i1x}} + \underset{\text{real}}{c_0} + \left[\underbrace{c_1 e^{i1x}} + c_2 e^{i2x} + \ldots\right]$$

c.c. of each other

$\underline{\text{Sum}}$ = real

$$\boxed{\frac{\partial^N}{\partial x^N}}\left(e^{inx}\right) \rightarrow \boxed{(in)^N}\left(e^{inx}\right)$$

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$\vdots$

$i^{96} = 1$

$i^{97} = i$

Ex:

$$\frac{\partial^2 u}{\partial t^2} = 7\,\frac{\partial^5 u}{\partial x^5} + t\,\frac{\partial^6 u}{\partial x^6}$$

+ periodic b.c.

$0 \le x \le 2\pi$

F.S. $\downarrow$

$$\boxed{\ddot{C}_n = (in)^5 \cdot 7\,C_n + t\,(in)^6 C_n}$$

C_{-n}

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$$

Ch. 8 nonhomog sys

$u(x,t)$

Ex:

$$\underbrace{\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u}_{\mathcal{L}u} + \underbrace{\sin(t)}_{\substack{\text{forcing} \\ Q}}$$

$$\overbrace{\mathcal{L}}\, u = 0 \quad \text{homog.}$$

$$\mathcal{L} u = \underline{Q(x,t)} \quad \text{nonhomog}$$

unknown

known, given

$u(x,0) = 0$

$u(x,0) = 3$ nonhomog.

<u>type 1</u> : nonhomog. PDE + homog. b.c.
(or periodic b.c.)

type 2 : homog. PDE + nonhomog. b.c.

type 3 : nonhomog PDE + nonhomog b.c.

strategy : same → remove

$\hat{u}(x,t) = u(x,t) - \underset{\sim\sim}{f(x,t)}$

⇓

transform system to <u>type 1</u>

or, even <u>homog. sys.</u>

Ex: HW4-Q4 ← can be achieved if $f(x,t)$ is chosen as the <u>steady sol.</u>

<u>type 1</u> : nonhomog. PDE + homog. b.c. (but not periodic)

Ex :

$$\begin{cases} \boxed{\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ &u(x,0)=0 \\ &u(x,1)=0 \end{aligned}} + Q(x,t) \\ u(x,0) = P(x) \end{cases}$$

$0 \le x \le 1$

sep of var on homog. subsystem

→ b.b. in x-dir $\{\sin(n\pi x)\}$

— STOP —

expand u, Q

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(n\pi x)$$

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin(n\pi x)$$

$$\boxed{\frac{da_n}{dt} = -(n\pi)^2 a_n + q_n(t)}$$

Sec 8.3

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$\frac{d^2 a_n}{d t^2} = -(n\pi)^2 a_n + q_n(t)$$

Ex:

HW1 Q1(a)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad 0 \le x \le 1$$

$$\cancel{u_x(x,0)=0}$$

$$\cancel{u_x t}$$

$$u_x(0,t)=0$$

$$u_x(1,t)=0$$

$$u(x,0)=p(x)$$

$\rightarrow$ b.b. $\{1,\ \cos(n\pi x)\}$

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(n\pi x)$$

$$* \quad Q(x,t) = q_0(t) + \sum_{n=1}^{\infty} q_n(t)\cos(n\pi x)$$

$$\boxed{\frac{da_0}{dt} = q_0}$$

$$\frac{da_n}{dt} = -(n\pi)^2 a_n + q_n \qquad n>0$$

Ex

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad 0 \le x \le 2\pi \\ \text{periodic in } x \\ u(x,0) = p(x) \end{cases}$$

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\, e^{inx}$$

$$* \quad Q(x,t) = \sum_{n=-\infty}^{\infty} q_n(t)\, e^{inx}$$

$$\boxed{\frac{dC_0}{dt} = q_0}$$

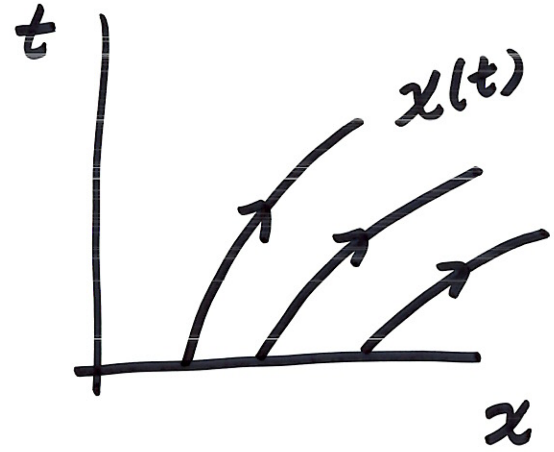
$$\frac{dC_n}{dt} = -n^2 C_n + q_n$$

$n > 0$

Ch. 12 MOC

$u(x, t)$

$$\frac{\partial u}{\partial t} = -10\,\frac{\partial u}{\partial x}$$



HW3
Q2(a)
case (i)

MOC form 1st-order PDE

✓ $\dfrac{\partial u}{\partial t} + \underline{A(x,t,u)}\,\dfrac{\partial u}{\partial x} = \underline{B(x,t,u)}$

$\dfrac{dx}{dt} = A$ eq. for traj

LHS of PDE $\Leftarrow$ follow it

becomes $\dfrac{\partial u}{\partial t} + \dfrac{dx}{dt}\dfrac{\partial u}{\partial x} \equiv \dfrac{du}{dt}$

$\Rightarrow$ PDE becomes $\dfrac{du}{dt} = B$

$u(x, t)$

follow ~~on~~ a trajectory

$\underline{u(X(t), t)}$

$\equiv$

$\underline{u(t)}$

$u(x,t)$

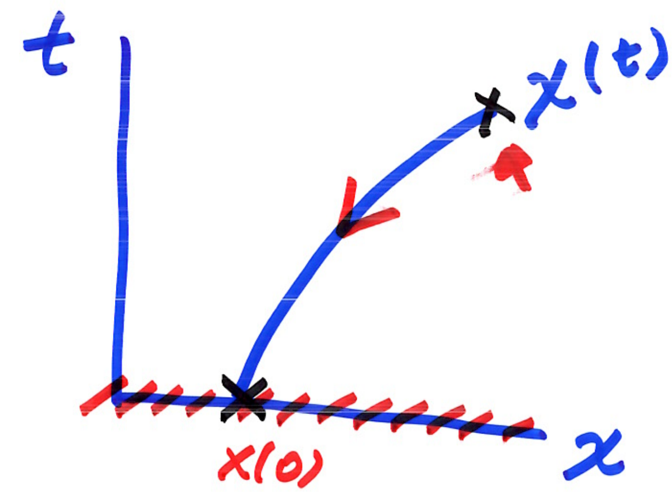
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = t$$

$$u(x,0) = p(x)$$

MOC

$$\left\{ \begin{array}{l} \dfrac{dx}{dt} = u \longleftarrow \text{trajectory } \underline{x(t)} \\[2ex] \underline{\dfrac{du}{dt} = t} \longleftarrow \end{array} \right.$$

proxy of the original PDE on the trajectory



$u(0) \equiv u(x(0),0) = p(x(0))$

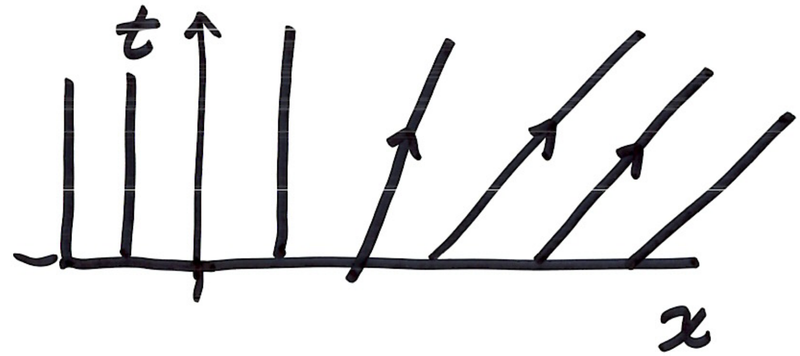
$\longrightarrow$ relation between $u(t)$ & $\underline{u(0)}$

$\downarrow$

$\underline{\underline{\text{b.c}}}$

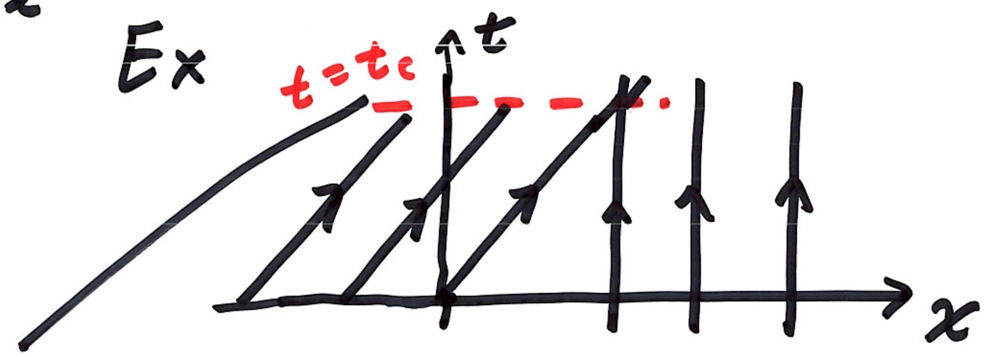
Use of Characteristics

Ex



Ex

"finite-time blowup"



$t > t_c$ multiple sol.

Higher-order PDEs

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = x$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u}_{w} = x$$

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial w}{\partial x} = x \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = w \end{cases}$$

MOC ✓

MOC ✓

HW5 Q4

alternative

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u}_{w} = x$$

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = x$$

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = w$$

same final answer