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Positive periodic solutions of singular systems of first order ordinary differential equations ☆

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ABSTRACT

The existence and multiplicity of positive periodic solutions for first non-autonomous singular systems are established with superlinearity or sublinearity assumptions at infinity for an appropriately chosen parameter. The proof of our results is based on the Krasnoselskii fixed point theorem in a cone.

1. Introduction

A variety of population dynamics and physiological processes can be described as the following equation

$$x'(t) = -a(t)x(t) + \lambda b(t)f(x(t)). \tag{1.1}$$

Periodic solutions of the type problems have attracted much attention, see e.g. [6,8,10,12] and references therein.

On the other hand, recently, there are a considerable interest in the existence of positive periodic solutions of singular systems of the second order differential equations, see Chu et al. [2], Franco and Webb [4], Jiang et al. [5], and the author [11] and references therein. It has been shown that many results for nonsingular systems still valid for singular cases. In particular, the author [11] demonstrates that the Krasnoselskii fixed point theorem on compression and expansion of cones can be effectively used to deal with singular problems. In fact, by choosing appropriate cones, the singularity of the systems is essentially removed and the associated operator becomes well-defined for certain ranges of functions even there are negative terms.

Agarwal and O'Regan [1] provided some results on solutions of singular first order differential equations. Chu and Nieto [3] showed the existence of periodic solutions for singular first order differential equations with impulses based on a nonlinear alternative of Leray–Schauder. The results in [1,3] for first order differential equations deal with a single equation. In this paper, by employing the Krasnoselskii fixed point theorem on compression and expansion of a cone, we shall establish the existence and multiplicity of positive ω-periodic solutions for the following singular non-autonomous n-dimensional system

$$x_i'(t) = -a_i(t)x_i(t) + \lambda b_i(t)f_i(x_1(t), \ldots, x_n(t)), \quad i = 1, \ldots, n, \tag{1.2}$$

where $\lambda > 0$ is a positive parameter. Our results give an almost complete structure of the existence of positive periodic solutions of (1.2) with an appropriately chosen parameter. Our results further show that there are analogous results between the first order and second ordinary differential equations.

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First we make assumptions for (1.2). Let $\mathbb{R} = (-\infty, \infty), \mathbb{R}_+ = [0, \infty), \mathbb{R}_+^n = \Pi_{i=1}^n \mathbb{R}_+$ and for any $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$, $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$.

(H1) $a_i, b_i \in C(\mathbb{R}, [0, \infty))$ are ω-periodic functions such that $\int_0^\omega a_i(t)dt > 0, \int_0^\omega b_i(t)dt > 0, \quad i = 1, \ldots, n.$
(H2) $f_i : \mathbb{R}_+^n \setminus \{0\} \to (0, \infty)$ is continuous, $i = 1, \ldots, n$

Our main results are:

Theorem 1.1. *Let* (H1), (H2) *hold. Assume that* $\lim_{\|\mathbf{u}\| \to 0} f_i(\mathbf{u}) = \infty$ *for some* $i = 1, \ldots, n$.

(a). If $\lim_{\|\mathbf{u}\| \to \infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = 0, \quad i = 1, \ldots, n$, *then, for all* $\lambda > 0$, *(1.2) has a positive periodic solution.*
(b). If $\lim_{\|\mathbf{u}\| \to \infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|} = \infty \quad$ *for* $\quad i = 1, \ldots, n$, *then, for all sufficiently small* $\lambda > 0$, *(1.2) has two positive periodic solutions.*
(c). There exists a $\lambda_0 > 0$ *such that (1.2) has a positive periodic solution for* $0 < \lambda < \lambda_0$.

We now give an example for Theorem 1.1. Consider the following system of two equations

$$\begin{cases} \dot{x} = -a_1(t)x + \lambda\left(\sqrt{x^2 + y^2}\right)^{-\alpha} + \lambda\left(\sqrt{x^2 + y^2}\right)^{\beta}, \\ \dot{y} = -a_2(t)y + \lambda\left(\sqrt{x^2 + y^2}\right)^{-\alpha} + \lambda\left(\sqrt{x^2 + y^2}\right)^{\beta}, \end{cases} \tag{1.3}$$

with $\alpha, \beta > 0, \ a_1 \geqslant 0, a_2 \geqslant 0$ are ω-periodic continuous in t. Corollary 1.2 is an application of Theorem 1.1. Since we use the summation norm in our theorems, we only need to note the following inequality

$$\sqrt{x^2 + y^2} \leqslant |x| + |y| \leqslant \sqrt{2}\sqrt{x^2 + y^2}.$$

Corollary 1.2. *Assume that* a_1, a_2 *satisfy* (H1). *Let* $\alpha > 0, \beta > 0, \lambda > 0$.

(a). If $0 < \beta < 1$, *then, for all* $\lambda > 0$, *(1.3) has a positive periodic solution.*
(b). If $\beta > 1$, *then, for all sufficiently small* $\lambda > 0$, *(1.3) has two positive periodic solutions.*
(c). There exists a $\lambda_0 > 0$ *such that (1.3) has a positive periodic solution for* $0 < \lambda < \lambda_0$.

Remark 1.3. As discussed in [11], we can extend Theorem 1.1 to the following singular systems with possible negative e_i,

$$x_i'(t) = -a_i(t)x_i(t) + \lambda b_i(t)f_i(x_1(t), \ldots, x_n(t)) + \lambda e_i(t), \quad i = 1, \ldots, n, \tag{1.4}$$

where $e_i(t), \ i = 1, \ldots, n$, are continuous ω-periodic functions. When $e_i(t)$ takes negative values, we shall need a stronger condition on $b_i(b_i > 0)$.

Such a result can be proved in the same way as in [11]. We will not give a detailed proof here. The idea to deal with negative e_i is to split $b_i(s)f_i(x(s)) + e_i(t)$ into the two terms $\frac{1}{2}b_i(s)f_i(x(s))$ and $\frac{1}{2}b_i(s)f_i(x(s)) + e_i(t)$. The first term is always nonnegative and used to carry out the estimates of the operator. We will make the second term $\frac{1}{2}b_i(s)f_i(x(s)) + e_i(t)$ nonnegative by choosing appropriate domains of f_i. This is possible because $\lim_{\|\mathbf{u}\| \to 0} f_i(x) = \infty$ or $\lim_{\|\mathbf{u}\| \to \infty} f_i(x) = \infty$. The choice of the even split of $b_i(s)f_i(x(s))$ here is not necessarily optimal in terms of obtaining maximal λ-intervals for the existence of periodic solutions of the systems.

Remark 1.4. O'Regan and the author [8], and the author [10] established the existence, multiplicity and nonexistence of positive periodic solution of the first order ODE

$$x_i'(t) = a_i(t)g_i(x(t))x_i(t) - \lambda b_i(t)f_i(x(t - \tau(t))), \quad i = 1, \ldots, n, \tag{1.5}$$

where g_i are positive bounded functions and $\tau \in C(\mathbb{R}, [0, \infty))$ is a ω-periodic function. These results can also be extended to (1.5) if f_i has a singularity at zero.

2. Preliminaries

We recall some concepts and conclusions of an operator in a cone. Let E be a Banach space and K be a closed, nonempty subset of E. K is said to be a cone if (i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geqslant 0$ and (ii) $u, -u \in K$ imply $u = 0$. The following well-known result of the fixed point theorem is crucial in our arguments.

Lemma 2.1 *([7]). Let X be a Banach space and $K(\subset X)$ be a cone. Assume that Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$\mathcal{T} : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K,$$

be completely continuous operator such that either

(i) $\|\mathcal{T}u\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_1$ and $\|\mathcal{T}u\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_2$; or
(ii) $\|\mathcal{T}u\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_1$ and $\|\mathcal{T}u\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_2$.

Then $\mathcal{T}$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We now introduce some notation. For $r > 0$, let

$$\sigma = \min_{i=1,\dots,n}\{\sigma_i\} > 0, \quad \text{where } \sigma_i = e^{-\int_0^\omega a_i(t)dt}, \quad i = 1,\dots,n,$$
$$M(r) = \max\{f_i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n, \sigma r \leqslant \|\mathbf{u}\| \leqslant r, \quad i = 1,\dots,n\} > 0,$$
$$m(r) = \min\{f_i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n, \sigma r \leqslant \|\mathbf{u}\| \leqslant r, \quad i = 1,\dots,n\} > 0,$$
$$\Gamma = \sigma \min_{n=1,\dots,n}\left\{\frac{\int_0^\omega b_i(s)ds}{\sigma_i^{-1}-1}\right\} > 0, \ \chi = \sum_{i=1}^n \frac{\sigma_i^{-1}}{\sigma_i^{-1}-1}\int_0^\omega b_i(s)ds > 0.$$

In order to apply Lemma 2.1 to (1.2), let X be the Banach space defined by

$$X = \{\mathbf{u}(t) \in C(\mathbb{R}, \mathbb{R}^n) : \mathbf{u}(t+\omega) = \mathbf{u}(t), \quad t \in \mathbb{R}, \quad i = 1,\dots,n\},$$

with a norm $\|\mathbf{u}\| = \sum_{i=1}^n \sup_{t\in[0,\omega]}|u_i(t)|$, for $\mathbf{u} = (u_1,\dots,u_n) \in X$. For $\mathbf{u} \in X$ or $\mathbb{R}_+^n$, $\|\mathbf{u}\|$ denotes the norm of $\mathbf{u}$ in X or $\mathbb{R}_+^n$, respectively.

Define

$$K = \{\mathbf{u} = (u_1,\dots,u_n) \in X : u_i(t) \geqslant \sigma_i \sup_{t\in[0,\omega]}|u_i(t)|, \quad i = 1,\dots,n, \ t \in [0,\omega]\}.$$

It is clear K is cone in X and $\min_{t\in[0,\omega]}\sum_{i=1}^n|u_i(t)| \geqslant \sigma\|u\|$ for $\mathbf{u} = (u_1,\dots,u_n) \in K$. For $r > 0$, define $\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| < r\}$.It is clear that $\partial\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| = r\}$. Let $\mathbf{T}_\lambda : K \setminus \{0\} \to X$ be a map with components $\left(T_\lambda^1,\dots,T_\lambda^n\right)$:

$$T_\lambda^i\mathbf{u}(t) = \lambda\int_t^{t+\omega} G_i(t,s)b_i(s)f_i(\mathbf{u}(s))ds, \quad i = 1,\dots,n, \tag{2.1}$$

where

$$G_i(t,s) = \frac{e^{\int_t^s a_i(\theta)d\theta}}{\sigma_i^{-1}-1}$$

satisfying

$$\frac{1}{\sigma_i^{-1}-1} \leqslant G_i(t,s) \leqslant \frac{\sigma_i^{-1}}{\sigma_i^{-1}-1} \quad \text{for} \quad t \leqslant s \leqslant t+\omega.$$

Lemma 2.2. *Assume* (H1)–(H2) *hold. Then $\mathbf{T}_\lambda(K \setminus \{0\}) \subset K$ and $\mathbf{T}_\lambda : K \setminus \{0\} \to K$ is compact and continuous.*

Proof. If $\mathbf{u} = (u_1,\dots,u_n) \in K \setminus \{0\}$, then $\min_{t\in[0,\omega]}\sum_{i=1}^n|u_i(t)| \geqslant \sigma\|\mathbf{u}\| > 0$, and then T_λ^i is defined. In view of the definition of K, for $\mathbf{u} \in K \setminus \{0\}$, we have, $i = 1,\dots,n$,

$$\left(T_\lambda^i\mathbf{u}\right)(t+\omega) = \lambda\int_{t+\omega}^{t+2\omega} G_i(t+\omega,s)b_i(s)f_i(\mathbf{u}(s))ds = \lambda\int_t^{t+\omega} G_i(t,s)b_i(s)f_i(\mathbf{u}(s))ds = \left(T_\lambda^i\mathbf{u}\right)(t).$$

It is easy to see that $\int_t^{t+\omega} b_i(s)f_i(\mathbf{u}(s))ds$ is a constant because of the periodicity of $b_i(t)f_i(\mathbf{u}(t))$. One can show that, for $\mathbf{u} \in K \setminus \{0\}$and $t \in [0,\omega], \ i = 1,\dots,n$,

$$T_\lambda^i\mathbf{u}(t) \geqslant \frac{1}{\sigma_i^{-1}-1}\lambda\int_t^{t+\omega} b_i(s)f_i(\mathbf{u}(s))ds = \sigma_i\frac{\sigma_i^{-1}}{\sigma_i^{-1}-1}\lambda\int_0^\omega b_i(s)f_i(\mathbf{u}(s))ds \geqslant \sigma_i \sup_{t\in[0,\omega]}|T_\lambda^i\mathbf{u}(t)|.$$

Thus $\mathbf{T}_\lambda(K \setminus \{0\}) \subset K$ and it is easy to show that $\mathbf{T}_\lambda : K \setminus \{0\} \to K$ is compact and continuous. $\square$

Lemma 2.3. *Assume that* (H1)–(H2) *hold. Then* $\mathbf{u} \in K \setminus \{0\}$*is a positive periodic solution of* (1.2) *if and only if it is a fixed point of* $\mathbf{T}_\lambda$ *in* $K \setminus \{0\}$

Proof. If $\mathbf{u} = (u_1, \ldots, u_n) \in K \setminus \{0\}$ and $\mathbf{T}_\lambda \mathbf{u} = \mathbf{u}$, then, for $i = 1, \ldots, n$,

$$u_i'(t) = \frac{d}{dt}\left(\lambda \int_t^{t+\omega} G_i(t,s)b_i(s)f_i(\mathbf{u}(s))ds\right) = \lambda G_i(t,t+\omega)b_i(t+\omega)f_i(\mathbf{u}(t+\omega) - \lambda G_i(t,t)b_i(t)f_i(\mathbf{u}(t)) - a_i(t)T_\lambda^i u(t)$$
$$= \lambda[G_i(t,t+\omega) - G_i(t,t)]b_i(t)f_i(\mathbf{u}(t)) - a_i(t)T_\lambda^i u(t) = -a_i(t)u_i(t) + \lambda b_i(t)f_i(\mathbf{u}(t)).$$

Thus $\mathbf{u}$ is a positive ω-periodic solution of (1.2). On the other hand, if $\mathbf{u} = (u_1, \ldots, u_n)$ is a positive ω-periodic function, then $\lambda b_i(t)f_i(\mathbf{u}(t)) = a_i(t)u_i(t) + u_i'(t)$ and

$$T_\lambda^i \mathbf{u}(t) = \lambda \int_t^{t+\omega} G_i(t,s)b_i(s)f_i(\mathbf{u}(s))ds = \int_t^{t+\omega} G_i(t,s)(a_i(s)u_i(s) + u_i'(s))ds = \int_t^{t+\omega} G_i(t,s)a_i(s)u(s)ds + \int_t^{t+\omega} G_i(t,s)u_i'(s)ds$$
$$= \int_t^{t+\omega} G_i(t,s)a_i(s)u(s)ds + G_i(t,s)u(s)|_t^{t+\omega} - \int_t^{t+\omega} G_i(t,s)a_i(s)u_i(s)ds = u_i(t).$$

Thus, $\mathbf{T}_\lambda \mathbf{u} = \mathbf{u}$, Furthermore, in view of the proof of Lemma 2.2, we also have $u_i(t) \geqslant \sigma_i \sup_{t\in[0,\omega]} u_i(t)$ for $t \in [0,\omega]$. That is, $\mathbf{u}$ is a fixed point of $\mathbf{T}_\lambda$ in $K \setminus \{0\}$. □

Lemma 2.4. *Assume that* (H1)–(H2) *hold. For any* $\eta > 0$ *and* $\mathbf{u} = (u_1, \ldots, u_n) \in K \setminus \{0\}$, *if there exists a* f_i *such that* $f_i(\mathbf{u}(t)) \geqslant \sum_{j=1}^n u_j(t)\eta$ *for* $t \in [0,\omega]$, *then* $\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda\Gamma\eta\|\mathbf{u}\|$.

Proof. Since $\mathbf{u} \in K \setminus \{0\}$ and $f_i(\mathbf{u}(t)) \geqslant \sum_{j=1}^n u_j(t)\eta$ for $t \in [0,\omega]$, we have

$$(T_\lambda^i \mathbf{u})(t) \geqslant \frac{1}{\sigma_i^{-1} - 1}\lambda \int_0^\omega b_i(s)f_i(\mathbf{u}(s))ds \geqslant \frac{1}{\sigma_i^{-1} - 1}\lambda \int_0^\omega b_i(s)\sum_{j=1}^n u_j(s)\eta ds \geqslant \frac{1}{\sigma_i^{-1} - 1}\lambda \int_0^\omega b_i(s)ds \sum_{j=1}^n \sigma_j \sup_{t\in[0,\omega]} u_j(t)\eta$$
$$\geqslant \lambda \min_{i=1,\ldots,n}\{\sigma_i\} \frac{\int_0^\omega b_i(s)ds}{\sigma_i^{-1} - 1}\eta\|\mathbf{u}\|.$$

Thus $\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda\Gamma\eta\|\mathbf{u}\|$. □

Let $\hat{f}_i : [1,\infty) \to \mathbb{R}_+$ be the function given by

$$\hat{f}_i(\theta) = \max\{f_i(u) : u \in \mathbb{R}_+^n \quad \text{and} \quad 1 \leqslant |u| \leqslant \theta\}, \quad i = 1, \ldots, n.$$

It is easy to see that $\hat{f}_i(\theta)$ is a nondecreasing function on $[1,\infty)$. The following lemma is essentially the same as [11, Lemma 3.6] and [9, Lemma 2.8].

Lemma 2.5 ([9,11]). *Assume* (H1) *holds. If* $\lim_{|x|\to\infty} \frac{f_i(x)}{|x|}$ *exists (which can be infinity), then* $\lim_{\theta\to\infty} \frac{\hat{f}_i(\theta)}{\theta}$ *exists and* $\lim_{\theta\to\infty} \frac{\hat{f}_i(\theta)}{\theta} = \lim_{|x|\to\infty} \frac{f_i(x)}{|x|}$.

Lemma 2.6. *Assume that* (H1)–(H2) *hold. Let* $r > \frac{1}{\sigma}$ *and if there exists an* $\varepsilon > 0$ *such that*

$$\hat{f}_i(r) \leqslant \varepsilon r, \quad i = 1, \ldots, n,$$

then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \lambda\chi\varepsilon\|\mathbf{u}\|$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_r$.

Proof. From the definition of $\mathbf{T}$, for $\mathbf{u} \in \partial\Omega_r$, we have

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \sum_{i=1}^n \frac{\sigma_i^{-1}}{\sigma_i^{-1} - 1}\lambda \int_0^\omega b_i(s)f_i(\mathbf{u}(s))ds \leqslant \sum_{i=1}^n \frac{\sigma_i^{-1}}{\sigma_i^{-1} - 1}\lambda \int_0^\omega b_i(s)\hat{f}_i(r)ds \leqslant \sum_{i=1}^n \frac{\sigma_i^{-1}}{\sigma_i^{-1} - 1}\lambda \int_0^\omega b_i(s)ds \quad \varepsilon\|\mathbf{u}\|$$
$$= \lambda\chi\varepsilon\|\mathbf{u}\|. \quad \square$$

In view of the definitions of $m(r)$ and $M(r)$, it follows that $M(r) \geqslant f_i(\mathbf{u}(t)) \geqslant m(r)$ for $t \in [0,\omega]$, $i = 1, \ldots, n$ if $\mathbf{u} \in \partial\Omega_r$, $r > 0$. Thus it is easy to see that the following two lemmas can be shown in similar manners as in Lemmas 2.4 and 2.6.

Lemma 2.7. *Assume* (H1)–(H2) *hold. If* $\mathbf{u} \in \partial\Omega_r,\ r > 0$, *then* $\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda \frac{\Gamma}{\sigma} m(r)$.

Lemma 2.8. *Assume* (H1)–(H2) *hold. If* $\mathbf{u} \in \partial\Omega_r,\ r > 0$, *then* $\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \lambda \chi M(r)$.

3. Proof of Theorem 1.1

Part (a). From the assumptions, there is an $r_1 > 0$ such that

$$f_i(\mathbf{u}) \geqslant \eta \|\mathbf{u}\|,$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n_+$ and $0 < \|\mathbf{u}\| \leqslant r_1$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_1}$, then

$$f_i(\mathbf{u}(t)) \geqslant \eta \sum_{i=1}^{n} u_i(t), \quad \text{for} \quad t \in [0, 1].$$

Lemma 2.4 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda \Gamma \eta \|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_1}.$$

We now determine Ω_{r_2}. Since $\lim_{\|\mathbf{u}\| \to \infty} \frac{f_i(x)}{\|\mathbf{u}\|} = 0, \quad i = 1, \ldots, n$ it follows from Lemma 2.5 that $\lim_{\theta \to \infty} \frac{\hat{f}_i(\theta)}{\theta} = 0, \quad i = 1, \ldots, n$. Therefore there is an $r_2 > \max\{2r_1, \frac{1}{\sigma}\}$ such that

$$\hat{f}_i(r_2) \leqslant \varepsilon r_2, \quad i = 1, \ldots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \chi < 1.$$

Thus, we have by Lemma 2.6 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \lambda \varepsilon \chi \|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

By Lemma 2.1, it follows that $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (1.2). □

Part (b). Fix a number $r_1 > 0$. Lemma 2.8 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_1}, \quad 0 < \lambda < \lambda_0.$$

In view of $\lim_{\|\mathbf{u}\| \to 0} f_i(x) = \infty$, there is a positive number $r_2 < r_1$ such that

$$f_i(\mathbf{u}) \geqslant \eta \|\mathbf{u}\|$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n_+$ and $0 < \|\mathbf{u}\| \leqslant r_2$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Then

$$f_i(\mathbf{u}(t)) \geqslant \eta \sum_{i=1}^{n} u_i(t),$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_2},\ t \in [0, 1]$. Lemma 2.4 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda \Gamma \eta \|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

On the other hand, since $\lim_{\|\mathbf{u}\| \to \infty} \frac{f_i}{\|\mathbf{u}\|} = \infty$, there is an $\widehat{H} > 0$ such that

$$f_i(\mathbf{u}) \geqslant \eta \|\mathbf{u}\|,$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n_+$ and $\|\mathbf{u}\| \geqslant \widehat{H}$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Let $r_3 = \max\left\{2r_1, \frac{\widehat{H}}{\sigma}\right\}$. If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_3}$, then

$$\min_{0 \leqslant t \leqslant \omega} \sum_{i=1}^{n} u_i(t) \geqslant \sigma \|\mathbf{u}\| = \sigma r_3 \geqslant \widehat{H},$$

which implies that

$$f_i(\mathbf{u}(t)) \geqslant \eta \sum_{i=1}^{n} u_i(t) \quad \text{for} \quad t \in [0, \omega].$$

It follows from Lemma 2.4 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda \Gamma \eta \|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.1 that $\mathbf{T}_\lambda$ has two fixed points $\mathbf{u}_1$ in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ and $\mathbf{u}_2 \in \Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ such that

$$r_2 < \|\mathbf{u}_1\| < r_1 < \|\mathbf{u}_2\| < r_3.$$

Consequently, (1.2) has two positive solutions for $0 < \lambda < \lambda_0$.

Part (c). Fix a number $r_1 > 0$. Lemma 2.7 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_1}, \quad 0 < \lambda < \lambda_0.$$

In view of $\lim_{x \to 0} f_i(x) = \infty$, there is a positive number $r_2 < r_1$ such that

$$f_i(\mathbf{u}) \geqslant \eta \|\mathbf{u}\|,$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $0 < \|\mathbf{u}\| \leqslant r_2$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Then

$$f_i(\mathbf{u}(t)) \geqslant \eta \sum_{i=1}^{n} u_i(t),$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_2},\ t \in [0, 1]$. Lemma 2.4 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda \Gamma \eta \|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

It follows from Lemma 2.1 that $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$. Consequently, (1.2) has a positive solution for $0 < \lambda < \lambda_0$.

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