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# EXISTENCE AND NONEXISTENCE OF POSITIVE RADIAL SOLUTIONS FOR QUASILINEAR SYSTEMS

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Abstract. The paper deals with the existence and nonexistence of positive radial solutions for the weakly coupled quasilinear system $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)+\lambda f(v)=0$, $\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)+\lambda g(u)=0$ in $B$, and $u=v=0$ on $\partial B$, where $p>1$, $B$ is a finite ball, $f$ and $g$ are continuous and nonnegative functions. We prove that there is a positive radial solution for the problem for various intevals of $\lambda$ in sublinear cases. In addition, a nonexistence result is given. We shall use fixed point theorems in a cone.

## 1. Introduction.

In this paper we consider the existence and nonexistence of positive radial solutions for the weakly coupled quasilinear elliptic system

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)+\lambda f(v)=0 \ \text{ in } \ \Omega \\ \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)+\lambda g(u)=0 \ \text{ in } \ \Omega \\ u=v=0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where $p>1$, $\Omega$ denotes the finite ball $B=\{x\in\mathbb{R}^N : |x|<1,\ N\geq 2\}$ and $\lambda>0$ is a parameter.

(1.1) is a generalization of the following boundary value problem

$$\begin{cases} \Delta u+\lambda f(u)=0 \ \text{ in } \ \Omega \\ u=0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{1.2}$$

(1.2) has received extensive investigation in the past several decades. Lions [7] discussed, under various combinations of superlinearity or sublinearity of $f$ at infinity and $f(0)=0$ or $f(0)>0$, the existence and nonexistence of positive solutions of (1.2) in a general bounded regular domain in $\mathbb{R}^N$. The results of [7] are also interpreted in terms of bifurcation diagrams. [2, 3, 4, 8, 9] obtained some existence and uniqueness results of elliptic systems. In this paper we shall establish the existence and nonexistence of positive radial solutions of the weakly coupled quasilinear elliptic system (1.1) in sublinear cases. First, let $\varphi(t)=|t|^{p-2}t$ and introduce the notation

$$f_0=\lim_{v\to 0^+}\frac{f(v)}{\varphi(v)},\quad f_\infty=\lim_{v\to\infty}\frac{f(v)}{\varphi(v)},$$

and

$$g_0=\lim_{u\to 0^+}\frac{f(u)}{\varphi(u)},\quad g_\infty=\lim_{u\to\infty}\frac{g(u)}{\varphi(u)}.$$

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We shall show that if (1.1) is sublinear, or $f_0 = g_0 = \infty$ and $f_\infty = g_\infty = 0$, then (1.1) has a positive solution for all $\lambda > 0$. In addition, we shall prove that (1.1) has a positive solution for small enough $\lambda > 0$ if $f_0 = g_0 = \infty$ regardless of the behavior of $f, g$ at $\infty$. A typical model in this case is $f(u) = e^u$.

We now turn to the main results of this paper. Our main results are:

**Theorem 1.1.** Assume $f, g : [0, \infty) \to [0, \infty)$ are continuous.
(a). If $f_0 = g_0 = \infty$ and $f_\infty = g_\infty = 0$, then for all $\lambda > 0$ (1.1) has a positive radial solution.
(b). If $f_0 = g_0 = \infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ (1.1) has a positive radial solution.
(c). If $f_0, g_0 < \infty$ and $f_\infty, g_\infty < \infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ (1.1) has no positive radial solution.

Note that (c) is a special case of Theorem 1.3 in [4]. But its proof here is different from that in [4]. In particular, here we are able to give an explicit formula to calculate the interval $(0, \lambda_0)$ for which (1.1) does not have a positive radial solution.

2. **Preliminaries.** A radial solution of (1.1) can be considered as a solution of the system

$$\begin{cases} \left(r^{N-1}\varphi(u'(r))\right)' + \lambda r^{N-1} f(v(r)) = 0, \quad 0 < r < 1 \\ \left(r^{N-1}\varphi(v'(r))\right)' + \lambda r^{N-1} g(u(r)) = 0, \quad 0 < r < 1 \\ u'(0) = v'(0) = u(1) = v(1) = 0. \end{cases} \tag{2.3}$$

We shall treat classical solutions of (2.3), namely a vector-valued function $(u, v)$ with $u, v \in C^1[0, 1]$, and $\varphi(u'), \varphi(v') \in C^1(0, 1)$, which satisfies (2.3). A solution $(u(r), v(r))$ is positive if $u(r), v(r) \geq 0$, for all $r \in (0, 1)$ and there is at least one nontrivial component of $(u, v)$. In fact, it is easy to prove that such a nontrivial component of $(u, v)$ is positive on $(0, 1)$.

The following well-known result of the fixed point index is crucial in our arguments.

**Lemma 2.1.** ([1, 5, 6]). Let $E$ be a Banach space and $K$ a cone in $E$. For $r > 0$, define $K_r = \{u \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is a compact operator and $\partial K_r = \{u \in K : \|x\| = r\}$.

(i) If there exists a $x_0 \in K \setminus \{0\}$ such that

$$x - Tx \neq t x_0, \text{ for all } x \in \partial K_r \text{ and } t \geq 0,$$

then

$$i(T, K_r, K) = 0.$$

(ii) If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$ and $Tx \neq x$ for $x \in \partial K_r$, then

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.1 to (2.3), let $X$ be the Banach space $C[0, 1] \times C[0, 1]$ and, for $(u, v) \in X$,

$$\|(u, v)\| = \sup_{t \in [0,1]} |u(t)| + \sup_{t \in [0,1]} |v(t)|.$$

Define $K$ be a cone in $X$ by

$$K = \{(u, v) \in X : u(t), v(t) \geq 0, \ t \in [0, 1]\}.$$

Also, define, for $r$ a positive number, $\Omega_r$ by

$$\Omega_r = \{(u,v) \in K : \|(u,v)\| < r\}.$$

Note that $\partial\Omega_r = \{(u,v) \in K : \|(u,v)\| = r\}$.

Let $\mathbf{T}_\lambda : K \to X$ be a map with components $(A_\lambda, B_\lambda)$, which are defined by

$$\begin{aligned} A_\lambda(u,v)(r) &= \int_r^1 \varphi^{-1}\Big(\frac{1}{s^{N-1}} \int_0^s \tau^{N-1}\lambda f(v(\tau))d\tau\Big)ds,\ r \in [0,1], \\ B_\lambda(u,v)(r) &= \int_r^1 \varphi^{-1}\Big(\frac{1}{s^{N-1}} \int_0^s \tau^{N-1}\lambda g(u(\tau))d\tau\Big)ds,\ r \in [0,1]. \end{aligned} \tag{2.4}$$

It is straightforward to verify that (2.3) is equivalent to the fixed point equation

$$\mathbf{T}_\lambda(u,v) = (u,v) \quad \text{in} \quad K.$$

**Lemma 2.2.** Assume $f, g : [0,\infty) \to [0,\infty)$ are continuous. Then $\mathbf{T}_\lambda(K) \subset K$ and $\mathbf{T}_\lambda : K \to K$ is a compact operator and continuous.

*Proof.* It is clear that $\mathbf{T}_\lambda(K) \subset K$. We now show that $\mathbf{T}_\lambda$ is compact. Let $(u_m, v_m)_{m\in\mathbb{N}}$ be a bounded sequence in $K$ and let $R > 0$ be such that $\|(u_m, v_m)\| \le R$ for all $m \in \mathbb{N}$. Hence, by the definition of $\mathbf{T}_\lambda$, we have,

$$(A_\lambda(u_m, v_m))'(r) = \begin{cases} -\varphi^{-1}(\frac{1}{r^{N-1}} \int_0^r \tau^{N-1}\lambda f(v_m)(\tau))d\tau), & 0 < r < 1, \\ 0, & r = 0, \end{cases}$$

and

$$(B_\lambda(u_m, v_m))'(r) = \begin{cases} -\varphi^{-1}(\frac{1}{r^{N-1}} \int_0^r \tau^{N-1}\lambda g(u_m)(\tau))d\tau), & 0 < r < 1, \\ 0, & r = 0. \end{cases}$$

Then it is easy to see that both $(\mathbf{T}_\lambda(u_m, v_m))_{m\in\mathbb{N}}$ and $((\mathbf{T}_\lambda(u_m, v_m))')_{m\in\mathbb{N}}$ are uniformly bounded sequences. It follows from the Arzela-Ascoli theorem that there exists a $\mathbf{v} \in K$ and a subsequence of $\mathbf{T}_\lambda(u_m, v_m)$ converging to $\mathbf{v}$ in $X$.

It remains to show the continuity of $\mathbf{T}_\lambda$. Let's take a sequence $(u_m, v_m)_{m\in\mathbb{N}}$ in $K$ converging to $(u,v) \in K$ in $X$. Note that $\varphi^{-1}$ and $f, g$ are continuous. It is not hard to see that the Dominated Convergence Theorem guarantees that

$$\lim_{m\to\infty} T_\lambda(u_m, v_m)(r) = T_\lambda(u,v)(r) \tag{2.5}$$

for each $r \in [0,1]$. Moreover, the compactness of $A_\lambda$ implies that $A_\lambda(u_m, v_m)(r)$ converges uniformly to $A_\lambda(u,v)(r)$ on $[0,1]$. Suppose this is false. Then there exists $\varepsilon_0 > 0$ and a subsequence $(u_{m_j}, v_{m_j})_{j\in\mathbb{N}}$ of $(u_m, v_m)_{m\in\mathbb{N}}$ such that

$$\sup_{r\in[0,1]} |A_\lambda(u_{m_j}, v_{m_j})(r) - A_\lambda(u,v)(r)| \ge \varepsilon_0,\ j \in \mathbb{N}, \tag{2.6}$$

Now, it follows from the compactness of $A_\lambda$ that there exists a subsequence of $(u_{m_j}, v_{m_j})_{j\in\mathbb{N}}$ (without loss of generality assume the subsequence is $(u_{m_j}, v_{m_j})_{j\in\mathbb{N}}$) such that $(A_\lambda(u_{m_j}, v_{m_j}))_{j\in\mathbb{N}}$ converges uniformly to $y_0 \in C[0,1]$. Thus, from (2.6), we easily see that

$$\sup_{r\in[0,1]} |y_0(r) - A_\lambda(u,v)(r)| \ge \varepsilon_0. \tag{2.7}$$

On the other hand, from the pointwise convergence (2.5) we obtain

$$y_0(r) = A_\lambda(u,v)(r),\ r \in [0,1].$$

This is a contradiction to (2.7). In the same way we can show that $B_\lambda(u_m, v_m)(r)$ converges uniformly to $B_\lambda(u,v)(r)$ on $[0,1]$. Therefore $\mathbf{T}_\lambda$ is continuous. □

Define two new functions $\hat{f}(t) : [0, \infty) \to [0, \infty)$ and $\hat{g}(t) : [0, \infty) \to [0, \infty)$ by

$$\hat{f}(t) = \max\{f(v) : 0 \le v \le t\}, \quad \hat{g}(t) = \max\{g(u) : 0 \le u \le t\}$$

Note that $\hat{f}_0 = \lim_{t\to 0^+} \frac{\hat{f}(t)}{\varphi(t)}$ and $\hat{f}_\infty = \lim_{t\to\infty} \frac{\hat{f}(t)}{\varphi(t)}$, and $\hat{g}_0 = \lim_{t\to 0^+} \frac{\hat{g}(t)}{\varphi(t)}$ and $\hat{g}_\infty = \lim_{t\to\infty} \frac{\hat{g}(t)}{\varphi(t)}$.

**Lemma 2.3.** [10] Assume $f, g : [0, \infty) \to [0, \infty)$ are continuous. Then $\hat{f}_0 = f_0$, $\hat{f}_\infty = f_\infty$, $\hat{g}_0 = g_0$ and $\hat{g}_\infty = g_\infty$.

**Lemma 2.4.** Assume $f, g : [0, \infty) \to [0, \infty)$ are continuous, and let $r > 0$. If there exits an $\varepsilon > 0$ such that

$$\hat{f}(r) \le \varphi(\varepsilon)\varphi(r), \ \ \hat{g}(r) \le \varphi(\varepsilon)\varphi(r),$$

then

$$\|\mathbf{T}_\lambda(u, v)\| \le 2\varphi^{-1}(\lambda)\varepsilon\|(u, v)\| \ \text{ for } \ (u, v) \in \partial\Omega_r.$$

*Proof.* From the definition of $T_\lambda$, for $(u, v) \in \partial\Omega_r$, we have

$$\begin{aligned}
\|\mathbf{T}_\lambda(u, v)\| &= \sup_{t\in[0,1]} |A_\lambda(u, v)(t)| + \sup_{t\in[0,1]} |B_\lambda(u, v)(t)| \\
&= \int_0^1 \varphi^{-1}[\frac{1}{s^{N-1}} \int_0^s \tau^{N-1}\lambda f(v(\tau))d\tau]ds \\
&\quad + \int_0^1 \varphi^{-1}[\frac{1}{s^{N-1}} \int_0^s \tau^{N-1}\lambda g(u(\tau))d\tau]ds \\
&\le \int_0^1 \varphi^{-1}[\frac{1}{s^{N-1}} \int_0^s \tau^{N-1}d\tau\lambda \hat{f}(r)]ds \\
&\quad + \int_0^1 \varphi^{-1}[\frac{1}{s^{N-1}} \int_0^s \tau^{N-1}d\tau\lambda \hat{g}(r)]ds \\
&\le 2\varphi^{-1}[\lambda\varphi(\varepsilon)\varphi(r)].
\end{aligned}$$

Then the fact that $\varphi^{-1}(\sigma\varphi(t)) = \varphi^{-1}(\sigma)t, \ t, \sigma \ge 0$ implies that

$$\begin{aligned}
\|\mathbf{T}_\lambda(u, v)\| &\le 2\varphi^{-1}[\lambda\varphi(\varepsilon r)] \\
&= 2\varphi^{-1}(\lambda)\varepsilon\|(u, v)\|.
\end{aligned}$$

□

**Lemma 2.5.** Assume $f, g : [0, \infty) \to [0, \infty)$ are continuous. If $(u, v) \in \partial\Omega_r$, $r > 0$, then

$$\|\mathbf{T}_\lambda(u, v)\| \le 2\varphi^{-1}(\lambda)\varphi^{-1}(\hat{M}_r),$$

where $\hat{M}_r = 1 + \max\{f(v) : 0 \le v \le r\} + \max\{g(u) : 0 \le u \le r\} > 0$.

*Proof.* Since $f(v(t)), g(u(t)) \le \hat{M}_r = \varphi(\varphi^{-1}(\hat{M}_r))$ for t $\in [0, 1]$, , it is easy to see that this lemma can be shown in a similar manner as in Lemma 2.4. □

3. **Proof of Theorem 1.1.** *Proof.* Part (a). Since $f_0 = g_0 = \infty$, there is an $r_1 > 0$ such that

$$f(v) \geq \varphi(\eta)\varphi(v), \quad g(u) \geq \varphi(\eta)\varphi(u), \tag{3.8}$$

for $0 \leq u, v \leq r_1$, where $\eta > 0$ is chosen so that

$$\frac{\eta\varphi^{-1}(\lambda)}{2}\varphi^{-1}(\frac{1}{N4^N}) \geq 1. \tag{3.9}$$

If $(u, v) - \mathbf{T}_\lambda(u, v) = 0$ for some $(u, v) \in \partial U_{r_1}$, we already find the desired solution of (1.1). Therefore we assume that

$$(u, v) - \mathbf{T}_\lambda(u, v) \neq 0 \text{ for all } (u, v) \in \partial U_{r_1}, \tag{3.10}$$

We now claim that

$$(u, v) - \mathbf{T}_\lambda(u, v) \neq t\mathbf{v}, \text{ for all } (u, v) \in \partial\Omega_{r_1} \text{ and } t \geq 0, \tag{3.11}$$

where $\mathbf{v} = (\theta(r), \theta(r))$, and $\theta \in C[0, 1]$ such that $0 \leq \theta(r) \leq 1$ on $[0, 1]$, $\theta(r) \equiv 1$ on $[0, \frac{1}{4}]$ and $\theta(r) \equiv 0$ on $[\frac{1}{2}, 1]$. Thus, $\mathbf{v} \in K \setminus \{0\}$. If there exists $(u^*, v^*) \in \partial\Omega_{r_1}$ and $t_0 \geq 0$ such that $(u^*, v^*) - \mathbf{T}_\lambda(u^*, v^*) = t_0\mathbf{v}$, we shall show this leads to a contradiction. Since (3.10) is true, we have $t_0 > 0$. Since $\mathbf{T}_\lambda(K) \subset K$, we obtain that

$$u^*(r) \geq t_0\theta(r), \text{ for all } r \in [0, 1],$$

and

$$v^*(r) \geq t_0\theta(r) \text{ for all } r \in [0, 1].$$

Let

$$t_u^* = \sup\{t : u^*(r) \geq t\theta(r) \text{ for all } r \in [0, 1]\},$$

and

$$t_v^* = \sup\{t : v^*(r) \geq t\theta(r) \text{ for all } r \in [0, 1]\}.$$

It follows that $t_0 \leq t_u^*, t_v^* < \infty$, and

$$u^*(r) \geq t_u^*\theta(r), \text{ for all } r \in [0, 1],$$

and

$$v^*(r) \geq t_v^*\theta(r) \text{ for all } r \in [0, 1].$$

Without loss of generality assume $t_u^* \leq t_v^*$ (If $t_u^* > t_v^*$, $v^*(r) = B_\lambda(u^*, v^*)(r) + t_0\theta(r)$ will lead to a similar contradiction for $v^*$ and $t_v^*$ ). Now, for $r \in [0, \frac{1}{2}]$, we have

$$\begin{aligned} u^*(r) &= A_\lambda(u^*, v^*)(r) + t_0\theta(r) \\ &= \int_r^1 \varphi^{-1}(\frac{1}{s^{N-1}} \int_0^s \tau^{N-1}\lambda f(v^*(\tau))d\tau)ds + t_0\theta(r). \end{aligned}$$

Note that $v^*(r) \le r_1$ for $r \in [0,1]$. (3.8) implies that, for $r \in [0, \frac{1}{2}]$,

$$\begin{aligned} u^*(r) &\ge \int_{\frac{1}{2}}^{1} \varphi^{-1}\Big(\frac{1}{s^{N-1}} \int_0^s \tau^{N-1}\lambda\varphi(\eta)\varphi(v^*(\tau))d\tau\Big)ds + t_0\theta(r) \\ &\ge \int_{\frac{1}{2}}^{1} \varphi^{-1}\Big(\int_0^s \tau^{N-1}\lambda\varphi(\eta)\varphi(v^*(\tau))d\tau\Big)ds + t_0\theta(r) \\ &\ge \frac{1}{2}\varphi^{-1}\Big(\int_0^{\frac{1}{4}} \tau^{N-1}\lambda\varphi(\eta)\varphi(t_v^*\theta(\tau))d\tau\Big) + t_0\theta(r) \\ &= \frac{1}{2}\varphi^{-1}\Big(\int_0^{\frac{1}{4}} \tau^{N-1}d\tau\varphi(\varphi^{-1}(\lambda))\varphi(\eta)\varphi(t_v^*)\Big) + t_0\theta(r) \\ &= \frac{1}{2}\varphi^{-1}\Big(\frac{1}{N4^N}\varphi(\varphi^{-1}(\lambda)\eta t_v^*)\Big) + t_0\theta(r). \end{aligned}$$

Now, in view of the fact that $\varphi^{-1}(\sigma\varphi(t)) = \varphi^{-1}(\sigma)t,\ t, \sigma \ge 0$, we have, for $r \in [0, \frac{1}{2}]$,

$$\begin{aligned} u^*(r) &\ge t_v^*\frac{\eta\varphi^{-1}(\lambda)}{2}\varphi^{-1}\Big(\frac{1}{N4^N}\Big) + t_0\theta(r). \\ &\ge t_v^* + t_0\theta(r) \\ &\ge (t_v^* + t_0)\theta(r), \end{aligned}$$

and hence

$$u^*(r) \ge (t_u^* + t_0)\theta(r),\ r \in [0,1],$$

which is a contradiction to the definition of $t_u^*$. Thus, in view of Lemma 2.1,

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0.$$

We now determine $\Omega_{r_2}$. Since $f_\infty = g_\infty = 0$, it follows from Lemma 2.3 that $\hat{f}_\infty = \hat{g}_\infty = 0$. Therefore there is an $r_2 > 2r_1$ such that

$$\hat{f}(r_2) \le \varphi(\varepsilon)\varphi(r_2),\ \ \hat{g}(r_2) \le \varphi(\varepsilon)\varphi(r_2),\ ,$$

where the constant $\varepsilon > 0$ satisfies

$$2\varphi^{-1}(\lambda)\varepsilon < 1.$$

Thus, we have by Lemma 2.4 that

$$\|\mathbf{T}_\lambda(u,v)\| \le 2\varphi^{-1}(\lambda)\varepsilon\|(u,v)\| < \|(u,v)\| \quad for \quad (u,v) \in \partial\Omega_{r_2}.$$

By Lemma 2.1,

$$i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1.$$

It follows from the additivity of the fixed point index that $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (1.1).

Part (b). Fix a number $r_2 > 0$. Lemma 2.5 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda(u,v)\| < \|(u,v)\|, \text{ for } (u,v) \in \partial\Omega_{r_2},\ 0 < \lambda < \lambda_0.$$

For each $0 < \lambda < \lambda_0$, it follows from $f_0 = g_0 = \infty$ and the proof of part (a) that there is an $0 < r_1 < r_2$ such that if (3.10) is true, then (3.11) holds. If (3.10) is false, we already find the desired positive solution of (1.1). Therefore we assume that (3.10) is true, and then (3.11) holds. Thus it follows It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0, \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1,$$

and hence, $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$. Consequently, (1.1) has a positive solution for $0 < \lambda < \lambda_0$.

Part (c). Since $f_0 < \infty$ and $f_\infty < \infty$, there exist positive numbers $\varepsilon_1^i$, $\varepsilon_2^i$, $r_1^i$ and $r_2^i$ such that $r_1^i < r_2^i$, $i = 1, 2$

$$f(v) \le \varepsilon_1^1 \varphi(v) \text{ for } 0 \le v \le r_1^1,$$
$$g(u) \le \varepsilon_1^2 \varphi(u) \text{ for } 0 \le u \le r_1^2,$$
$$f(v) \le \varepsilon_2^1 \varphi(v) \text{ for } v \ge r_2^1,$$
$$g(u) \le \varepsilon_2^2 \varphi(u) \text{ for } u \ge r_2^2.$$

Let

$$\varepsilon^1 = \max\{\varepsilon_1^1, \varepsilon_2^1, \max\{\frac{f(v)}{\varphi(v)} : r_1^1 \le v \le r_2^1\} + 1\} > 0$$

and

$$\varepsilon^2 = \max\{\varepsilon_1^2, \varepsilon_2^2, \max\{\frac{g(u)}{\varphi(u)} : r_1^2 \le u \le r_2^2\} + 1\} > 0$$

and $\varepsilon = \max_{i=1,2}\{\varepsilon^i\} > 0$. Thus, we have

$$f(v) \le \varepsilon \varphi(v) \text{ for v} \ge 0,$$

and

$$g(u) \le \varepsilon \varphi(u) \text{ for u} \ge 0,$$

Assume $(u_1, v_1)$ is a positive solution of (2.3). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0$, where

$$\lambda_0 = \varphi(\frac{1}{2\varphi^{-1}(\varepsilon)}).$$

In fact, for $0 < \lambda < \lambda_0$, since $\mathbf{T}_\lambda(u_1, v_1) = (u_1, v_1)$ for $t \in [0, 1]$, we find by Lemma 2.4

$$\begin{aligned} \|(u_1, v_1)\| &= \|\mathbf{T}_\lambda(u_1, v_1)\| \\ &\le 2\varphi^{-1}(\lambda)\varphi^{-1}(\varepsilon)\|(u_1, v_1)\| \\ &< \|(u_1, v_1)\|, \end{aligned}$$

which is a contradiction. □

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