Global bifurcation and positive solution for a class of fully nonlinear problems

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\textbf{ABSTRACT}

In this paper, we study global bifurcation phenomena for the following Kirchhoff type problem
\[
\begin{aligned}
-\int_{\Omega} M(|\nabla u(x)|^2) \Delta u &= \lambda f(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \(M\) is a continuous function. Under some natural hypotheses, we show that \((\lambda_1(a)M(0), 0)\) is a bifurcation point and there is a global continuum \(C\) emanating from \((\lambda_1(a)M(0), 0)\), where \(\lambda_1(a)\) denotes the first eigenvalue of the above problem with \(f(x, s) = a(x)s\). As an application of the above result, we study the existence of positive solution for this problem with asymptotically linear nonlinearity.

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1. Introduction

Consider the following Kirchhoff type problem
\[
\begin{aligned}
-\int_{\Omega} M(|\nabla u(x)|^2) \Delta u &= \lambda a(x)u(x) + g(x, u, \lambda) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \(\Omega \subset \mathbb{R}^N\) is a bounded domain with smooth boundary \(\partial \Omega\), \(M\) is a continuous function on \(\mathbb{R}^+\), \(a \in L^\infty(\Omega)\) with \(a \neq 0\), \(\lambda > 0\) is a parameter, \(g : \Omega \times \mathbb{R}^2 \to \mathbb{R}\) satisfies the carathéodory condition in the first two variables and
\[
\lim_{s \to 0} \frac{g(x, s, \lambda)}{s} = 0
\]
uniformly for a.e. \(x \in \Omega\) and \(\lambda\) on bounded sets. Moreover, we also assume that \(g\) satisfies the growth restriction

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(G) There exist $c > 0$ and $p \in (1, 2^*)$ such that
$$|g(x, s, \lambda)| \leq c (1 + |s|^{p-1})$$
for a.e. $x \in \Omega$ and $\lambda$ on bounded sets, where
$$2^* = \begin{cases} \frac{2N}{N-2}, & \text{if } N > 2, \\ +\infty, & \text{if } N \leq 2. \end{cases}$$

The problem (1.1) is nonlocal as the appearance of the term $\int_{\Omega} |
abla u(x)|^2 \, dx$ which implies that it is not a pointwise identity. This causes some mathematical difficulties which make the study of problem (1.1) particularly interesting. Moreover, problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string [1]. After the famous paper by Lions [2], this type of problems has been the subject of numerous studies, and some important and interesting results have been obtained, for example, see [3–6]. Recently, there are many mathematicians studying this kind of problems by variational method, see [7–13] and the references therein. We refer to [14–20] for Kirchhoff models with critical exponents. For evolution problems, we refer to [21–23] and the references therein.

To the best of our knowledge, there are few papers that studied Kirchhoff type problems using the bifurcation theory, see for example [24,25]. The first aim of this paper is to study global bifurcation phenomena for problem (1.1). Let $\lambda_1(a)$ denote the first eigenvalue of the following problem
$$\begin{cases} -\Delta u = a(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

It is well known that $\lambda_1(a)$ is simple, isolated and is the unique principle eigenvalue of problem (1.3). Now, we make the following assumptions on $M$.

(M$_0$) $M$ is a continuous function on $\mathbb{R}^+$ such that for some $m_0 > 0$, we have $M(t) \geq m_0$, for all $t \in \mathbb{R}^+$;

(M$_1$) there exists $m_1 > 0$, such that $\lim_{t \to +\infty} M(t) = m_1$.

The hypothesis (M$_0$) shows that our problem is non-degenerate. In [14,16] the so-called “degenerate” case is covered (see also [22,23,20]), that is the main Kirchhoff non-negative function $M$ could be zero at 0.

Our first main result is the following theorem.

**Theorem 1.1.** Assume that (1.2), (G) and (M$_0$) hold. Then $(\lambda_1(a)M(0), 0)$ is a bifurcation point of problem (1.1) and the associated bifurcation continuum $\mathcal{C}$ in $\mathbb{R} \times H^1_0(\Omega)$, whose closure contains $(\lambda_1(a)M(0), 0)$, is either unbounded or contains a pair $(\mu M(0), 0)$, where $\mu$ is another eigenvalue of problem (1.3).

On the basis of Theorem 1.1, the second aim of this paper is to determine the interval of $\lambda$, for which there exists a positive solution for the following Kirchhoff type problem
$$\begin{cases} -M \left( \int_{\Omega} |
abla u(x)|^2 \, dx \right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $f \in C(\Omega \times \mathbb{R}^+ \times \mathbb{R})$ satisfies that

(f$_1$) $f : \Omega \times [0, \infty) \to \mathbb{R}^+$ such that $f(x, s)s > 0$ for $x \in \Omega$ and any $s > 0$;

(f$_2$) $\lim_{s \to 0^+} \frac{f(x, s)}{s} = a(x)$, $\lim_{s \to +\infty} \frac{f(x, s)}{s} = c(x) \neq 0$ uniformly in $x \in \Omega$, where $a(x), c(x)$ such that they are strict positive on some subset of positive measure in $\Omega$ and $\lambda_1(c)m_1 \neq \lambda_1(a)M(0)$.

The following theorem is our second main result.

**Theorem 1.2.** Suppose that (M$_0$), (M$_1$) and (f$_1$), (f$_2$) hold, then for
$$\lambda \in (\min \{\lambda_1(c)m_1, \lambda_1(a)M(0)\}, \max \{\lambda_1(c)m_1, \lambda_1(a)M(0)\}),$$
problem (1.4) possesses at least one positive solution.

**Remark 1.3.** Note that the corresponding existence result of [7] is a corollary of Theorem 1.2. In fact, by the monotonicity of eigenvalue with respect to weight, we get $1 \in (\lambda_1(a)M(0), \lambda_1(c)m_1)$ under the assumptions of Theorem 1 in [7]. So problem (1.4) with $\lambda = 1$ possesses at least one positive solution. Clearly, our assumptions are weaker than corresponding ones of [7]. Therefore, we improve and extend the corresponding result of [7].

The rest of this paper is organized as follows. Sections 2 and 3 present the proofs of Theorems 1.1 and 1.2, respectively.
2. Global bifurcation

Firstly, we study the global bifurcation phenomena for the following fully nonlinear abstract operator equation

\[ u = F(\lambda, u), \quad u \in X \]  

(2.1)

where \( \lambda \) varies in \( \mathbb{R} \), \( X \) is a real Banach space with norm \( \| \cdot \| \), \( F : \mathbb{R} \times X \rightarrow X \) is completely continuous. Moreover, we assume that there exists a linear, completely continuous operator \( L \) such that

\[ F(\lambda, u) - \lambda Lu = o(\|u\|) \]

for \( u \in X \) near 0 uniformly on bounded \( \lambda \) intervals.

Let \( r \) (\( L \)) denote the set of real characteristic values of \( L \), and \( \mathcal{S} \) be the closure of the set of nontrivial solution pairs of Eq. (2.1). Using Theorem 1.3 of [26], we can easily get the following lemma.

**Lemma 2.1.** If \( \mu \in r (L) \setminus \{0\} \) has odd algebraic multiplicity, then \( \mathcal{S} \) possesses a component \( \mathcal{C}_\mu \) such that \( (\mu, 0) \in \mathcal{C}_\mu \) and \( \mathcal{C}_\mu \) either

(i) meets infinity in \( \mathbb{R} \times X \), or

(ii) meets \( (\tilde{\mu}, 0) \), where \( \tilde{\mu} \) is another characteristic value of \( L \).

Clearly, problem (1.1) can be equivalently written as

\[ u = (-\Delta)^{-1} \left( \frac{1}{M(\int_{\Omega} |\nabla u|^2 \, dx)} (\lambda au + H(\lambda, u)) \right) \]

where \( H(\lambda, \cdot) \) denotes the usual Nemitsky operator associated with \( g \). We write \( X := H^1_0(\Omega) \) with the norm \( \|u\| = (\int_{\Omega} |\nabla u|^2 \, dx)^{1/2} \).

**Proof of Theorem 1.1.** Let

\[ Lu = \frac{(-\Delta)^{-1}(au)}{M(0)}, \quad \tilde{H}(\lambda, u) = \frac{(-\Delta)^{-1}(H(\lambda, u))}{M(\|u\|^2)} + \frac{\lambda (M(0) - M(\|u\|^2))}{M(0)M(\|u\|^2)} (-\Delta)^{-1}(au). \]

Clearly, \( L : X \rightarrow X \) is linear completely continuous. From condition (G), \( (M_0) \) and noting \( 2 < 2^* \), we can see that \( \tilde{H} : \mathbb{R} \times X \rightarrow X \) is completely continuous. Moreover, it is easy to see that \( \lambda_1(a)M(0) \) is a simple characteristic value of \( L \). Then

\[ (-\Delta)^{-1} \left( \frac{1}{M(\|u\|^2)} (\lambda u + H(\lambda, u)) \right) = \lambda Lu + \tilde{H}(\lambda, u). \]

Next, we show that \( \tilde{H} = o(\|u\|) \) at \( u = 0 \) uniformly on bounded \( \lambda \) intervals. It is sufficient to show that

\[ \lim_{\|u\| \rightarrow 0} \frac{H(x, u)}{\|u\|} = 0 \quad \text{in} \quad L^p(\Omega). \]

Without loss of generality, we may assume that \( p > 2 \). Otherwise, we can consider \( \tilde{p} = cp, c > 1 \) such that \( \tilde{p} \in (2, 2^*) \). From \( p < 2^* \), we can see that

\[ \frac{p'(p - 2)}{2^*} < \frac{2^* - p'}{2^*}. \]

So we can choose a real number \( r > 1 \) such that

\[ \frac{p'(p - 2)}{2^*} \leq \frac{1}{r} \leq \frac{2^* - p'}{2^*}. \]

It follows that

\[ p'r(p - 2) \leq 2^* \quad \text{and} \quad p'r' \leq 2^*. \]

For any \( \varepsilon > 0 \), in view of (1.2) and (G), we can choose positive numbers \( \delta = \delta(\varepsilon) \) and \( M = M(\delta) \) such that for a.e. \( x \in \Omega \), the following relations hold:

\[ \left| \frac{g(x, s, \lambda)}{s} \right| \leq \varepsilon \quad \text{for} \quad 0 < |s| \leq \delta, \]

\[ \left| \frac{g(x, s, \lambda)}{s} \right| \leq M|s|^{p - 2} \quad \text{for} \quad |s| > \delta. \]
Then we can obtain that
\[
\int_\Omega \left| \frac{H(\lambda, u)}{u} \right|^\beta dx \leq \varepsilon |\Omega| + M^\beta \int_\Omega |u|^{\beta(p-2)} dx.
\]
From this inequality, (2.2) and \(u \to 0\) in \(X\), we get that
\[
\left| \frac{H(\lambda, u)}{u} \right| \to 0 \quad \text{in} \quad L^1(\Omega).
\]
Let \(v = u/\|u\|\). By the boundedness of \(v\) in \(X\), (2.2) and the continuous embedding of \(X \hookrightarrow L^2(\Omega)\), we have that
\[
\int_\Omega |v|^{p'} dx \leq c
\]
for some constant \(c > 0\). Then from (2.3), (2.4) and Hölder’s inequality, we obtain that
\[
\int_\Omega \left| \frac{H(\lambda, u)}{u} \right|^\beta dx = \int_\Omega \left| \frac{H(\lambda, u)}{u} \right|^{\beta'} dx
\leq \left( \int_\Omega \left| \frac{H(\lambda, u)}{u} \right|^{p'} dx \right)^{1/r} \left( \int_\Omega |v|^{p'} dx \right)^{1/\beta'}
\to 0.
\]

Now, from Lemma 2.1, we get the existence of a global branch of the set of nontrivial solution of problem (1.1) emanating from \((\lambda_1(a)M(0), 0)\).

### 3. Positive solution

In this section, based on the Theorem 1.1, we study the existence of positive solution for problem (1.4).

**Lemma 3.1.** Assume that (M_0) and \((f_1) - (f_2)\) hold. Then \((\lambda_1(a)M(0), 0)\) is a bifurcation point of problem (1.4) and the associated bifurcation branch \(C\) in \(\mathbb{R} \times X\) whose closure contains \((\lambda_1(a)M(0), 0)\), is either unbounded or contains a pair \((\mu M(0), 0)\) in which \(\mu\) is another eigenvalue of problem (1.3).

**Proof.** Let \(\vartheta : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+\) be a continuous function such that
\[
f(x, u) = a(x)u + \vartheta(x, u)
\]
with
\[
\lim_{s \to 0^+} \frac{\vartheta(x, s)}{s} = 0 \quad \text{and} \quad \lim_{s \to +\infty} \frac{\vartheta(x, s)}{s} = c(x) - a(x) \quad \text{uniformly in} \ \Omega.
\]
From (3.1), we can see that \(\lambda \vartheta(x, u)\) satisfies the hypotheses of (1.2) and (G). Now, Theorem 1.1 can be applied to get the desired results.

Next, we shall prove that the first choice of the alternatives of Lemma 3.1 is valid. Let \(P\) denote the set of functions in \(X\) which are positive in \(\Omega\). Let \(K = \mathbb{R} \times P\) under the product topology.

**Lemma 3.2.** We have \(C \subseteq (K \cup \{(\lambda_1(a)M(0), 0)\})\) and the last alternative of Lemma 3.1 is impossible.

**Proof.** Clearly, we have \(u \geq 0\) for any nontrivial solution \((\lambda, u) \in C\) because \(f \geq 0\). By the strong maximum principle [27], we know that \(u > 0\) in the whole domain for any nontrivial solution \((\lambda, u) \in C\). So we have \(C \subseteq (K \cup (\mathbb{R} \times \{0\}))\). Suppose on the contrary, if there exists \((\lambda_m, u_m) \to (\mu M(0), 0)\) when \(m \to +\infty\) with \((\lambda_m, u_m) \in C, u_m \not\equiv 0\) and \(\mu \neq \lambda_1(a)\). Let \(v_m := u_m/\|u_m\|\), then \(v_m\) satisfies
\[
v_m = (-\Delta)^{-1} \left( \frac{\lambda_m a(x) v_m(x)}{M(\|u_m\|)} + \frac{\lambda_m \vartheta(x, u_m(x))}{M(\|u_m\|) \|u_m\|} \right).
\]
By an argument similar to that of Theorem 1.1, we obtain that for some convenient subsequence, \(v_m \to v_0 \neq 0\) as \(m \to +\infty\). It is easy to see that \(v_0\) verifies the equation
\[
\begin{align*}
-\Delta v_0 &= \mu a(x) v_0 \quad \text{in} \ \Omega, \\
v_0 &= 0 \quad \text{on} \ \partial \Omega.
\end{align*}
\]
So it follows from [28] that \(v_0\) must change its sign. This is a contradiction.
Proof of Theorem 1.2. We only need to show $E$ links $(\lambda_1(a)M(0), 0)$ to $(\lambda_1(c)M_1, +\infty)$ in $\mathbb{R} \times X$. Let $(\lambda_n, u_n) \in E$ with $u_n \not\equiv 0$ satisfies

$$\lambda_n + \|u_n\| \to +\infty.$$  

We note that $\lambda_n > 0$ for all $n \in \mathbb{N}$, since $(0, 0)$ is the only solution of (1.4) for $\lambda = 0$ and $E \cap (\{0\} \times X) = \emptyset$.

We claim that there exists a constant $M > 0$ such that

$$\lambda_n \in (0, M]$$

for $n \in \mathbb{N}$ large enough. On the contrary, we suppose that $\lim_{n \to \infty} \lambda_n \to +\infty$. Since $(\lambda_n, u_n) \in E$, it follows that

$$-\Delta u_n(x) = \frac{\lambda_n}{M(\|u_n\|)} f(x, u_n) - u_n \quad \text{in} \quad \Omega.$$  

It follows from $(M_0)$--$(M_1)$ that there exists $C > 0$ such that $\frac{1}{M(\|u_n\|)} \geq 1/C$. It follows from $(f_2)$ that there exists some open subset $\Omega_0$ of positive measure in $\Omega$ such that $a, c$ are positive on $\Omega_0$. Furthermore, there exist two positive constants $\delta_1, \delta_2$ such that $\delta_1 < \delta_2$,

$$f(x, s) > \frac{a(x)}{2} \quad \text{for} \quad s \in (0, \delta_1), x \in \Omega_0$$

and

$$f(x, s) > \frac{c(x)}{2} \quad \text{for} \quad s \in (\delta_2, +\infty), x \in \Omega_0.$$

Let

$$\sigma_1 = \min_{\Omega_0 \times [\delta_1, \delta_2]} \frac{f(x, s)}{s}.$$  

Then $(f_1)$ shows that $\sigma_1 > 0$. Clearly, we have $a, c \in C (\Omega_0)$. Let

$$\sigma = \min \left\{ \delta_1, \min_{\Omega_0} \frac{a(x)}{2}, \min_{\Omega_0} \frac{c(x)}{2} \right\}.$$  

Obviously, one has $\sigma > 0$. Then we can see that $\frac{f(x, u_n)}{u_n} \geq \sigma$ all $x \in \Omega_0$, $n \in \mathbb{N}$. Set

$$\widetilde{u}_n = \begin{cases} u_n & \text{if} \ x \in \Omega_0, \\ 0 & \text{if} \ x \in \Omega \setminus \Omega_0. \end{cases}$$

Then we have that

$$\begin{cases} -\Delta \widetilde{u}_n(x) \geq \frac{\sigma}{C} \lambda_n \widetilde{u}_n \quad \text{in} \quad \Omega_0, \\ \widetilde{u}_n = 0 \quad \text{on} \quad \partial \Omega. \end{cases} \quad (3.2)$$

Multiplying the first equation of problem (3.2) by a positive eigenfunction $\phi_1$ associated to $\lambda_1(1)$, we get that

$$\lambda_1 \geq \frac{\sigma}{C} \lambda_n,$$

an absurdum.

Therefore, we get that $\|u_n\| \to \infty$. Let $\xi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that

$$f(x, u) = c(x)u + \xi(x, u)$$

with

$$\lim_{s \to +\infty} \frac{\xi(x, s)}{s} = 0 \quad \text{and} \quad \lim_{s \to 0^+} \frac{\xi(x, s)}{s} = a(x)$$

uniformly in $\Omega$.  

(3.3)

We divide the equation

$$\begin{cases} -\Delta u_n(x) = \frac{\lambda_n c(x) u_n(x)}{M(\|u_n\|)} + \frac{\lambda_n \xi(x, u_n(x))}{M(\|u_n\|)} \quad \text{in} \quad \Omega, \\ u_n = 0 \quad \text{on} \quad \partial \Omega \end{cases}$$

by $\|u_n\|$ and set $\overline{u}_n = u_n/\|u_n\|$. Since $\overline{u}_n$ is bounded in $X$, after taking a subsequence if necessary, we have that $\overline{u}_n \to \overline{u}$ for some $\overline{u} \in X$ and $\overline{u}_n \to \overline{u}$ in $L^2(\Omega)$. 

It follows from (3.3) that for any \( \epsilon > 0 \), there exists a constant \( C \) such that
\[
|\xi(x, u_n)| \leq C + \epsilon |u_n|.
\]
By (3.4), we can easily show that
\[
\lim_{n \to +\infty} \frac{\xi(x, u_n(x))}{\|u_n\|} = 0 \quad \text{in} \quad L^2(\Omega).
\]
By the compactness of \((-\Delta)^{-1} : L^2(\Omega) \to X\), we obtain
\[
\begin{cases}
-\Delta \overline{u}(x) = \frac{\overline{\lambda}}{m_1} c(x) \overline{\Pi}(x) & \text{in} \quad \Omega, \\
\overline{u} = 0 & \text{on} \quad \partial \Omega,
\end{cases}
\]
where \( \overline{\lambda} = \lim_{n \to +\infty} \lambda_n \), again choosing a subsequence and relabeling it if necessary.

It is clear that \( \overline{u} \in \overline{C} \subseteq C \) since \( C \) is closed in \( \mathbb{R} \times X \). So \( \overline{\lambda} = \lambda_1(c)m_1 \). Therefore, \( C \) links \( (\lambda_1(a)M(0), 0) \) to \( (\lambda_1(c)m_1, \infty) \) in \( \mathbb{R} \times X \).

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\section*{References}