PARTIAL DIFFERENTIAL EQUATIONS WITH ROBIN
BOUNDARY CONDITION IN ONLINE SOCIAL NETWORKS

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Abstract. In recent years, online social networks such as Twitter, have become a major source of information exchange and research on information diffusion in social networks has been accelerated. Partial differential equations are proposed to characterize temporal and spatial patterns of information diffusion over online social networks. The new modeling approach presents a new analytic framework towards quantifying information diffusion through the interplay of structural and topical influences. In this paper we develop a non-autonomous diffusive logistic model with indefinite weight and the Robin boundary condition to describe information diffusion in online social networks. It is validated with a real dataset from an online social network, Digg.com. The simulation shows that the logistic model with the Robin boundary condition is able to more accurately predict the density of influenced users. We study the bifurcation, stability of the diffusive logistic model with heterogeneity in distance. The bifurcation and stability results of the model information describe either information spreading or vanishing in online social networks.

1. Introduction. Online social networking has indisputably become a forefront platform for information exchange in recent years. Social media sites such as Twitter and Facebook have experienced explosive growth and continued popularity. Large amounts of data available to researchers have heightened the interest in studying the information diffusion process for such online networks. Studying the diffusion process can lead to efficiency of distributing relevant information to any given user as well as reducing unwanted information over social media. However, understanding information spreading on online social networks remains a challenge due to the complexity of social interactions and the rapid change of social network platforms.

There is a wealth of research focusing on the measurement and analysis of network structures, user interactions, and traffic characteristics of social media with empirical approaches which utilize data mining and statistical modeling schemes.
There also exists research that has made a substantial effort in using mathematical models to understand and predict information diffusion over a time period in online social networks [3, 10, 13, 15, 24, 28, 45]. [27] discusses dynamical processes on complex networks, dynamical models of network growth and dynamical processes taking place on the networks and reports developments on the structure and function of complex networks.

F. Wang, H. Wang and Xu [40] proposed to use partial differential equations (PDEs) built on intuitive cyber-distance among online users to study both temporal and spatial patterns of information diffusion process in online social networks. In a recent survey paper [11], the PDE model in [40] is cited as one of the three non-graph based modeling predicative models: the epidemiological models based on ordinary differential equations, Linear Influence Model (LIM) and the PDE approach. The PDE-based models for online social networks in [40, 43, 18] are spatial dynamical systems that take into account the influence of the underlying network structure as well as information contents to predict information diffusion over both temporal and spatial dimensions. The PDE-based models in online social networks directly address a number of concerns in modeling information diffusion in online social networks. [24, 39] noticed that nearly all epidemiological models for online social networks only focus on the internal influence, while neglecting the external influence.

The extension of the applications of partial differential equations into information diffusion in online social networks presents a new opportunity and challenge for mathematicians. In the context of spatial ecology and physics, extensive research on eigenvalue analysis, stability, effects of boundary conditions, evolution of dispersal has been done [6, 35, 9]. Despite the tremendous progress in mathematical analysis of reaction-diffusion equations in the last few decades, there remains a host of unsolved mathematical problems in spatial models [6, 21]. In particular, [16, 17, 23] use the dynamical system approach to study eigenvalue, stability and persistence of nonautonomous parabolic partial differential equations.

In this paper we develop a non-autonomous diffusive logistic model with indefinite weight and the Robin boundary condition to describe information diffusion in online social networks. The model is validated with a real dataset from an online social network, Digg.com, and the simulation shows that the logistic model with the Robin boundary condition is able to more accurately predict the density of influenced users than the model with the Neumann boundary condition in [40]. The overall accuracy for the logistic model with the Robin boundary condition is 96.61%; the overall accuracy for Neumann boundary condition is 92.85% [40]. Further, we study stability and bifurcation of nonautonomous parabolic equations with indefinite weight and Robin boundary condition arising from online social networks. We identify the cut-off point for either information spreading or vanishing in online social networks. Mathematically, the result is even sharper than the corresponding result [16] and others.

This paper is organized as follows. Section 2 introduces partial differential equations for modeling information flow in online social network with the Robin boundary condition. We further validate the model with real datasets from Digg. Section 3 presents a number of bifurcation and stability results for the diffusive logistic equation with the Robin boundary condition.

2. PDE model with Robin boundary condition. In [40], F. Wang, H. Wang and Xu use partial differential equations built on intuitive cyber-distance among
online users to study both temporal and spatial patterns of information diffusion process in social media. The information diffusion in online social networks is abstractly divided into two content-based and structure-based processes. The structured-based process results from the underlying network structure and represents information diffusion due to direct links to those who have already been influenced. The content-based process measures information spread among users due to various other activities such as search, and results from the popularity of information content. The content-based process usually is more or less at random. Figure 1 conceptually illustrates the interplay of the two processes in an online social network. The content-based and structure-based processes are analogous to the external and internal influences, respectively, in online social networks in [24]. The two decisive structures (the structure of social networks and the structure of topical affiliations) form the backbone of online social media (Romero et al. [33]). The PDE models for online social networks provide a new analytic framework towards a better understanding of information diffusion mechanisms by studying the interplay of structural and topical influences.

Friendship or following relationship plays a leading role in information diffusion over online social networks. An intuitive approach for defining the distance between two users is to use the number of friendship links in the shortest path from one user to another in the social network graph. We use friendship hops to refer to the number of friendships links or hops. Thus the distance between the initiator and any other user is defined as the length (the number of friendship hop) of the shortest path from the initiator to this user in the social network graph. To be more precisely, let \( U \) denote the user population in an online social network, and \( s \) be the source of information such as a news story that starts to spread in social media. Based on the distance from social network users from this source, the user population \( U \) can be divided into a set of groups, i.e., \( U = \{U_1, U_2, ..., U_i, ..., U_m\} \), where \( m \) is the maximum distance from the users to the source \( s \). The group \( U_x \) consists of users that share the same distance of \( x \) to the source.

We use the \( x \)-axis as the social distance and embed the density \( U_x \) at the location \( x \). Let \( I(x, t) \) denote the density of influenced users at distance \( x \) and time \( t \). Let \( J \) denote the flux of the influenced users at \( x \), at time \( t \). In general, we can assume that information flows from high density to low density and therefore \( J \) can take a simple expression of

\[
J = -c \frac{\partial I}{\partial x}
\]

which results from a principle analogous to Fick’s law ([25]) in Biology or Physics. The minus sign describes the flow is down the gradient. Here \( c \) represents the popularity of information which promotes the spread of the information through non-structure based activities such as search. The content-based action behaves more or less at random. In Twitter, the symbol \# followed by a few characters, called a hashtag, is used to mark keywords or topics in a tweet. With the hashtag symbol anyone can search for the set of tweets that contain a hashtag. It is estimated that Twitter handles 1.6 billion search queries per day [48]. This unstructured phenomenon “jumps” across the network and appears at a seemingly random node [24]. The action results from the popularity of information content rather than the structure of the follower graph of a network. Because of spatial heterogeneity of online users in social media, \( c \) may be dependent on the distance \( x \) from the source in the content-based process in Fig. 1. In general, \( c \) may be a decreasing
function of $x$ since interactions between different groups $U_x$ decrease dramatically as $x$ increases. Therefore, we use an exponential function

$$c = de^{-bx}.$$ 

The structure-based process in Fig. 1 results from local growth due to the underlying network structure and is modeled with a simple nonlinear logistic equation,

$$r(t)I(h(x) - \frac{I}{K}),$$

where $r$ is decay rate of influence of information with respect to time, $h(x)$ the intrinsic growth rate and $K$ the carrying capacity which gives the upper bound of $I$.

Now combining the two processes in Fig. 1, a balance equation in developing the basic diffusion equation in biology [20] gives rise to the following model with the Robin boundary condition.

$$\frac{\partial I}{\partial t} = \frac{\partial (de^{-bx}I_x)}{\partial x} + r(t)I(h(x) - \frac{I}{K})$$

$I(x, 1) = \phi(x), \ l < x < L$

$$\frac{\partial I}{\partial x}(l, t) = 0, \frac{\partial I}{\partial x}(L, t) + \beta I(L, t) = 0, \ t > 1,$$

where

- $I$ represents the density of influenced users with a distance of $x$ at time $t$;
- $de^{-bx}$ represents the popularity of information which promotes the spread of the information through non-structure based activities such as search;
- $r$ represents the intrinsic growth rate of influenced users with the same distance, and measures how fast the information spreads within the users with the same distance;
- $K$ represents the carrying capacity, which is the maximum possible density of influenced users at a given distance;
- $L$ and $l$ represent the lower and upper bounds of the distances between the source $s$ and other social network users;
- $\phi(x)$ is the initial density function, which can be constructed from history data of information spreading;
- $\beta$ indicates that the rate of change of $I$ with respect to $x$ is proportional to the current density $I$.

It is worth noting that there are some differences between information diffusion in online social networks and spatial biological process in mathematical biology. In spatial ecology, the diffusion process often refers to the fact that animals move randomly from one physical location to another. In the context of online social network, online users simply pass on information from one to another and do not necessarily change their network distances within the lifetime of the information.

In [40, 43] it is assumed that $c$ is constant and there is no information exchanging at the ends. As such, the Neumann boundary condition ($I_x(l, t) = I_x(L, t) = 0$) were discussed in [40, 43]. The Neumann boundary condition is reasonable at the left end because there is no news traveling in the left part. Since the shortest friendship hops are used as distance metric, the Neumann boundary condition at the right is motivated by the theory of six degrees of separation in many social networks that most of nodes are six or fewer steps away from any other nodes. Indeed, the average distance on Twitter is 4.67 [47]. Fig. 2 in [40] shows the majority of online social network users in Digg have a distance of 2 to 5 from the initiators. In this figure, for all four stories, the distance 3 users account for more than 40% of all the users from the initiator directly or through other users. As the distance increases from 6 to 8, the number of social networks users reachable from the initiator drops dramatically. However, as we introduce more complex distance metric rather than the shortest network hop, the length of the distance can be much longer. For example, an alternative approach could be a combination of friendship hops and interest distance, for measuring the distance between two users through their shared interests on information or content in social networks. [42] introduced an effective algorithm to identify the shared interests in online social networks. With the shared interest distance, the model reflects how information flows from those who share more common interests to those with less common interests. In this case, there may be substantial information exchanges at the right end. As a result, the Robin boundary condition is more plausible to describe the flow of information at the right side.

![Figure 2](image-url)

**Figure 2.** (a) Predicted (blue, solid) vs. Actual data (red, dotted) of (1). (b) Graph of $h(x)$. (c) Graph of $r(t)$

In order to demonstrate the validity of the model with the Robin boundary condition. We numerically solved the model with Matlab. Figure 2 illustrates the
predicting results for the most popular news with 24,099 votes in Digg.com with the Robin boundary condition. The $x$-axis is the distance measured by friendship hops, while the $y$-axis represents the density of influenced users within each distance. The solid lines denote the actual observations for the density of influenced users for a variety of time periods (i.e., 1-hour, 2-hours, 3-hours, 4-hours and 5-hours), while the dashed lines illustrate the predicted density of influenced users by the model. Figure 2 [b,c] is the graphs of $h(x), r(t)$. The parameters used in the simulations are $d = 2.8863, b = 4, K = 2000, \beta = 0.001$. The overall accuracy for the logistic boundary condition is $96.61\%$; the overall accuracy of the model with Neumann boundary condition for the same dataset is $92.85\%$ [40]. Thus the logistic model with the Robin boundary condition is able to more accurately predict the density of influenced users with different distance over time.


Recently, Lei, Lin and the third author [18] proposed and studied the following free boundary model to describe the spreading of news in online social networks

$$
\begin{cases}
    u_t - du_{xx} = r(t)u(1 - \frac{u}{K}), & t > 0, 0 < x < h(t), \\
    u_x(t, 0) = 0, u(t, h(t)) = 0, & t > 0, \\
    h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
    h(0) = h_0, u(0, x) = u_0(x), & 0 \leq x \leq h_0,
\end{cases}
$$

where $x = h(t)$ is the moving boundary to be determined and represents the spreading front of news (such as movie recommendation) among users. $h'(t) = -\mu u_x(t, h(t))$ is the Stefan condition, where $\mu$ represents the diffusion ability of the information in the new area. Let $r_\infty = \lim_{t \to \infty} r(t) > 0$. It is well known that the Stefan conditions have been used in many areas when phase transitions in matters such as ice passing to water and other biological problems. It was shown in [18] that the free boundary $x = h(t)$ is increasing. Further, it was shown that the information traveling either lasts forever or suspends in finite time. In addition, the impact of the initial condition of news on its spread over online social networks is discussed.

In this section, we study the following model with indefinite weight and the Robin boundary condition

$$
\begin{cases}
    I_t - (a(x)I_x)_x = \lambda r(t)I(h(x) - \frac{I}{K}), & t > 1, l < x < L, \\
    I(1, x) = \phi(x), & l \leq x \leq L, \\
    I_x(t, l) = 0, I_x(t, L) + \beta I(t, L) = 0, & t > 1,
\end{cases}
$$

(2)

where $a(x) \in C^{1+\alpha}([l, L]), 0 < \alpha \leq 1$, is a positive function, $\phi(x) \in C^{2+\alpha}([l, L])$ is nonnegative and not identical to zero, and $r(t)$ is a positive Hölder-continuous function satisfying $\lim_{t \to +\infty} r(t) = r_\infty > 0$, $h$ is a Hölder-continuous function and positive on a subinterval of $[l, L]$. $\phi \leq K h_\infty$, where $h_\infty = \max_{x \in [l, L]} |h(x)|$. The parameter $\lambda$ can be interpreted as a scale or factor of $r(t)$. The Robin boundary condition at $x = L$ reflects the fact that there is an exchange of information at the boundary. For $\beta > 0$, it indicates the flux $-I_x(t, L)$ is positive and therefore information flows to the right.

Note that problem (2) is non-autonomous with respect time $t$. We refer to [16, 17, 22, 31, 38] and their references for further studies. In addition, the fact that $h$ is an indefinite weight also introduces some essential challenges. We shall investigate the bifurcation phenomena of (2) and find the bifurcation point of either information spreading or vanishing in online social networks.
3.1. Preliminaries. Firstly, we consider the following eigenvalue problem
\begin{equation}
\begin{aligned}
- (a(x) u')' &= \lambda h(x) u, \quad l < x < L, \\
 u'(l) &= 0, \quad u'(L) + \beta u(L) = 0,
\end{aligned}
\end{equation}
where \( \lambda \) is a parameter. \( \lambda_1 \) is called an eigenvalue of (3) if there exists a nontrivial solution of (3) with \( \lambda = \lambda_1 \).

**Lemma 1.** Problem (3) admits a positive principal eigenvalue \( \lambda_1^+ \), and it is a simple eigenvalue. Moreover, any eigenfunction associated to \( \lambda_1^+ \) is positive on \([l, L]\).

**Proof.** In view of (A2), we can see that \( h(x) \) changes sign and is positive in some subdomain of \([l, L]\). From Theorem 2.4 of [6], we know that (3) possesses a positive principal eigenvalue \( \lambda_1^+ \) determined by
\[
\frac{1}{\lambda_1^+} = \max_{u \in W^{1,2}(l, L), u \neq 0} \left[ \frac{\int_l^L h(x) u^2 \, dx}{\int_l^L a(x) |u'|^2 \, dx + \beta a(L) u^2(L)} \right].
\]
Moreover, \( \lambda_1^+ \) is the only positive eigenvalue admitting a positive eigenfunction, and it is a simple eigenvalue. Let \( \phi_1 \) be any (weak) eigenfunction corresponding to \( \lambda_1^+ \). By standard elliptic regularity theory, we know that \( \phi_1 \in C^2[l, L] \).

Without loss of generality, we assume that \( \phi_1 > 0 \) in \((l, L)\). We claim that \( \phi_1(l) > 0 \) and \( \phi_1(L) > 0 \). We only prove the case \( \phi_1(l) > 0 \) because of the case \( \phi_1(L) > 0 \) being completely analogous. Suppose on contrary that \( \phi_1(l) = 0 \). We note that
\[
\phi_1(x) = \int_l^x \phi_1'(\tau) \, d\tau \quad \text{and} \quad \phi_1''(x) = \int_l^x \phi_1''(\tau) \, d\tau.
\]

Then by above relations and the equations, one can find that
\[
|\phi_1'(x)| = \left| \int_l^x \phi_1''(\tau) \, d\tau \right| \leq \lambda_1^+ h(x) \int_l^x \frac{\phi_1^1}{a(x)} \, d\tau + c \int_l^x \left| \phi_1'(\tau) \right| \, d\tau \leq c_1 \int_l^x \left| \phi_1'(\tau) \right| \, d\tau,
\]
where \( c, c_1 \) are positive constants. By the Gronwall inequality [8], we get that \( \phi_1 \equiv 0 \) on \([l, L]\). This is a contradiction. \( \square \)

**Remark 2.** Note that the complete spectral picture has been obtained in [12]. However, Lemma 1 is better than the corresponding one in [12] because any eigenfunction associated to \( \lambda_1^+ \) is positive on \([l, L]\). Moreover, by an argument similar to Lemma 1, we can show that any nonnegative solution of (5) is positive on \([l, L]\).

Let \( X = \{ u \in C^1[l, L] : u'(l) = 0, \ u'(L) + \beta u(L) = 0 \} \) with the standard norm \( \| \cdot \| \). Consider
\begin{equation}
\begin{aligned}
- (a(x) u')' &= \lambda \left( h(x) - \frac{u}{L} \right), \quad l < x < L, \\
 u'(l) &= 0, \quad u'(L) + \beta u(L) = 0,
\end{aligned}
\end{equation}

**Lemma 3.** Problem (4) has an unbounded branch \( C \) in \( \mathbb{R} \times X \) of positive solutions bifurcating from \((\lambda_1^+, 0)\). Moreover,
(i) \( C \) is bounded from above in \( u \), bounded away from zero in \( \lambda \), and extends to \(+\infty \) in \( \lambda \);
(ii) \( C \) is contained in the set of \((\lambda, u)\) with \( \lambda \geq \lambda_1^+ \);
(iii) For each \( \lambda > \lambda_1^+ \) the positive solution with \((\lambda, u) \in C \) is the unique positive solution and \( C \) is a smooth arc; i.e., \( u(\lambda) \) depends differentiably on \( \lambda \) for \( \lambda > \lambda_1^+ \).
Proof. Let \( g(x, s) = h(x) - s/K \). Then we have \( g(x, s) < 0 \) for \( s > h_\infty K \), \( g(x, s) < g(x, 0) \) for \( s > 0 \) and \( g_u(x, s) < 0 \) for all \( s \geq 0 \). Applying Corollary 3.14 of [6] to (4), we can obtain (i)–(iii).

Clearly, the steady state solution equation of (2) is

\[
\begin{cases}
- (a(x)u')' = \lambda r_\infty u \left( h(x) - \frac{w}{K} \right), & l < x < L, \\
u'(l) = 0, & u'(L) + \beta u(L) = 0,
\end{cases}
\]

where \( u(x) = \lim_{t \to +\infty} I(t, x) \). The results of Lemma 3 imply that there exists a unique positive solution of (5) if \( \lambda > \lambda_1^+ / r_\infty \); and there is only a trivial solution of (5) if \( \lambda \leq \lambda_1^+ / r_\infty \).

Next we give the following definition of upper and lower solutions of (2).

**Definition 4.** A function \( I \in C^{2,1}((l, L) \times (1, +\infty)) \cap C^{1,0}([l, L] \times [1, +\infty)) \) is called an upper solution of (2) if it satisfies

\[
\begin{cases}
I_t - (a(x)I')' \geq f(t, x, I, \lambda), & l < x < L, \ t > 1 \\
I(1, x) \geq \phi(x), & l \leq x \leq L, \\
I_x(t, l) \geq 0, & I_x(t, L) + \beta I(t, L) \geq 0, \ t > 1,
\end{cases}
\]

where \( f(t, x, I, \lambda) = \lambda r_\infty (h(x) - \frac{w}{K}) \). Similarly, \( I \) is called a lower solution of (2) if it satisfies all the reversed inequalities in (6).

**Lemma 5.** Let \( I(t, x) \) be a solution of (2), \( \overline{I}(t, x) \) and \( \underline{I}(t, x) \) are upper and lower solutions of (2) respectively, then \( \overline{I}(t, x) \leq I(t, x) \leq \underline{I}(t, x) \) in \([l, L] \times [1, +\infty)\).

**Proof.** Let \( w = \overline{I} - \underline{I} \). Then by Definition 2.1 and the mean value theorem,

\[
\begin{cases}
w_t - (a(x)w')' \geq f(t, x, \overline{I}, \lambda)w, & l < x < L, \ t > 1 \\
w(1, x) \geq 0, & l \leq x \leq L, \\
w_x(t, l) \geq 0, & w_x(t, L) + \beta w(t, L) \geq 0, \ t > 1,
\end{cases}
\]

where \( \overline{I} \) is an intermediate value between \( \overline{I} \) and \( \underline{I} \). By Lemma 2.2.1 of [29], we have \( \overline{I} \geq \underline{I} \). Since \( I \) may be considered as a lower solution or an upper solution the relation \( \overline{I}(t, x) \leq I(t, x) \leq \underline{I}(t, x) \) follows immediately.

Finally, we prove two lemmata which will be used later.

**Lemma 6.** For any \( \epsilon > 0 \), let \( u_\epsilon(x) \) be any positive solution of

\[
\begin{cases}
- (a(x)u')' = \lambda r_\infty (h(x) + \epsilon) v - \lambda (r_\infty - \epsilon) \frac{w^2}{K}, & l < x < L, \\
v'(l) = 0, & v'(L) + \beta v(L) = 0,
\end{cases}
\]

Then \( \lim_{\epsilon \to 0^+} u_\epsilon(x) = u(x) \), where \( u(x) \) is the unique positive of (7) with \( \epsilon = 0 \).

**Proof.** Consider

\[
\begin{cases}
- (a(x)v')' = \lambda r_\infty h(x)v - \lambda r_\infty \frac{w^2}{K}, & l < x < L, \\
v'(l) = 0, & v'(L) + \beta v(L) = 0.
\end{cases}
\]

Clearly, 0 is a lower solution of (8). It is easy to show that \( u_\epsilon \) is an upper solution of (8). So we have that \( 0 \leq u(x) \leq u_\epsilon(x) \) for any \( x \in [l, L] \). Let \( w = \lim_{\epsilon \to 0^+} u_\epsilon(x) \). From (7), we get

\[
\begin{cases}
- (a(x)w')' = \lambda r_\infty h(x)w - \lambda r_\infty \frac{w^2}{K}, & l < x < L, \\
w'(l) = 0, & w'(L) + \beta w(L) = 0.
\end{cases}
\]

It follows that \( w \equiv u \).

Similarly to the proof of Lemma 6, we can obtain the following result.
Lemma 7. For any $\epsilon > 0$, let $u_\epsilon(x)$ be any positive solution of
\[
\begin{cases}
-(a(x)u_\epsilon')' = \lambda r_\infty (h(x) - \epsilon) v - \lambda (r_\infty + \epsilon) \frac{u_\epsilon^2}{v^2}, & l < x < L, \\
v'(l) = 0, \quad v'(L) + \beta v(L) = 0.
\end{cases}
\] (9)
Then $\limsup_{\epsilon \to 0^+} u_\epsilon(x) = u(x)$, where $u(x)$ is the unique solution of (9) with $\epsilon = 0$.

3.2. Spreading and vanishing. In this subsection, we shall get some asymptotically stable results about the solutions of (2). Firstly, consider the following auxiliary problem
\[
\begin{cases}
I_t = (a(x)I')' + I (bh(x) - cI), & l < x < L, \quad t > 1 \\
I(1, x) = \phi(x), & l \leq x \leq L, \\
I_x(t, l) = 0, \quad I_x(t, L) + \beta I(t, L) = 0, & t > 1,
\end{cases}
\] (10)
where $b$, $c$ are positive constants, $\phi(x) \leq bh_\infty/c$ for any $x \in [l, L]$. Consider the following eigenvalue problem
\[
\begin{cases}
-(a(x)u')' = \lambda u + bh(x)u, & l < x < L, \\
u'(l) = 0, \quad u'(L) + \beta u(L) = 0.
\end{cases}
\] (11)
Let $\Lambda(bh)$ be the first eigenvalue of (11). It is well known that $\Lambda(bh)$ is a simple, principal eigenvalue. We write $\varphi_1$ for the positive eigenfunction associated to $\Lambda(bh)$, normalized such that $\|\varphi_1\|_\infty = 1$. Moreover, we also have the following lemma.

Lemma 8. (11) has a positive solution if and only if $\lambda = \Lambda(bh)$.

Proof. Let $v$ be any eigenfunction associated to an eigenvalue $\mu \geq \Lambda(bh)$. We need to show that $v$ changes sign. Assume by contradiction that $v \geq 0$, the case $v \leq 0$ being completely analogous. By the Gronwall inequality, we can easily show that $v > 0$ on $[l, L]$. By the following Picone’s identity (see [2, 14, 30, 36])
\[
|\varphi_1'|^2 - \left(\frac{\varphi_1^2}{v}\right)' v' = |\varphi_1'|^2 + \frac{\varphi_1^2}{v^2} |v'|^2 - 2 \frac{\varphi_1}{v} \varphi_1' v' \geq 0,
\]
we can easily get the following identity
\[
a |\varphi_1'|^2 - \left(\frac{\varphi_1^2}{v}\right)' a v' = a |\varphi_1'|^2 + \frac{a \varphi_1^2}{v^2} |v'|^2 - 2 \frac{a \varphi_1}{v} \varphi_1' v' \geq 0.
\]
By an easy calculation and using above identity, we obtain that
\[
0 \leq \int_l^L (\Lambda(bh) - \mu) \varphi_1^2 dx < 0,
\]
which is impossible. Hence we have proved that $v$ must change sign.

Using Lemma 8, we can get the following result.

Lemma 9. If $h_2$ is constant, then
\[
\Lambda(bh_1 + h_2) = \Lambda(bh_1) - h_2.
\]

Proof. Let $\varphi_2$ be any positive eigenfunction associated to $\Lambda(bh_1 + h_2)$. Then we have
\[
-(a(x)\varphi_2')' = \Lambda(bh_1 + h_2) \varphi_2 + (bh_1 + h_2) \varphi_2 = (\Lambda(bh_1 + h_2) + h_2) \varphi_2 + bh_1 \varphi_2.
\]
Lemma 8 implies that
\[
\Lambda(bh_1 + h_2) + h_2 = \Lambda(bh_1),
\]
i.e., $\Lambda(bh_1 + h_2) = \Lambda(bh_1) - h_2$. 

Remark 10. If \( a(x) \equiv 1 \), Lemma 9 can obtain from relation (3.5) of [16]. However, the authors of [16] did not give the proof of relation (3.5). For reader’s convenience, we give its proof here. Note that our proof can be easily applied to the corresponding high dimensional problem.

Lemma 11. Let \( \Lambda(bh) \) be the first eigenvalue of (11). Then:

(i) There exists a unique positive solution of (10) denoted by \( I^*(t,x) \). Moreover, \( 0 < I^*(t,x) \leq bh_\infty/c \) for any \( x \in [l,L] \) and \( t > 1 \).

(ii) If \( \Lambda(bh) > 0 \), then (10) admits only one nonnegative steady state solution \( u = 0 \), which is globally asymptotically stable, that is \( \lim_{t \to +\infty} I^*(t,x) = 0 \) in \( C^1[l,L] \). Moreover the convergence is exponentially fast and uniform for bounded sets of initial data \( \phi \).

(iii) If \( \Lambda(bh) < 0 \), then (10) has only one positive steady state solution \( u = u^*(x) \), which is globally asymptotically stable, that is \( \lim_{t \to +\infty} I^*(t,x) = u^*(x) \) in \( C^1[l,L] \). Moreover the convergence is exponentially fast and uniform for bounded sets of initial data \( \phi \).

Proof. (i) The global existence and uniqueness of solutions of problem (10) are standard results (also see Theorem 2.1 and 2.5 of [32]). It is easy to verify that 0 is lower solution of (10) and \( C_* := bh_\infty/c \) is an upper solution of (10). Lemma 5 implies that \( 0 \leq I^*(x,t) \leq C_* \) for any \( x \in [l,L] \) and \( t \geq 1 \). The strong maximum principle implies \( 0 < I^*(x,t) \leq C_* \) for any \( x \in [l,L] \) and \( t > 1 \).

(ii) Set \( Y = \{ u \in C^1[l,L] : 0 \leq u \leq C_* \} \). Similar to [5], we can view (10) as a dynamical system on \( Y \) with

\[
V(I) = \int_l^L \left( \frac{1}{2}a(x)|I'|^2 - bh(x)I^2 + \frac{\beta a(L)I^2}{2} \right) dx + \frac{1}{2} \beta a(L) I^2(t,L).
\]

as a Lyapunov function. It is easy to see that \( V \) is bounded on bounded subsets of \( Y \). If \( I(t,x) \) is an orbit of (10) starting at \( I(1,x) = \phi(x) \), we have

\[
\frac{dV(I)}{dt} = -\int_l^L I_t^2 dx \leq 0.
\]

Thus, \( V \) decreases monotonically along the orbit. From (10), we can see that \( I \in C^{2,1}([l,L] \times [1,+,\infty]) \). So the orbits \( I \) is boundedness and precompactness. It follows that \( V(I(t,x)) \) is bounded blow and there exists \( V_1 = \lim_{t \to -\infty} V(I(t,x)) \). If \( u \) is in the \( \omega \)-limit set of \( \phi(x) \) then \( V(u) = V_1 \). Let \( w^*(t,x) \) be the orbit starting at point \( w_1(x) \) in the \( \omega \)-limit set of \( \phi \). Since \( \omega \)-limit set is invariant, the entire orbit \( w^*(t,x) \) must belong to it, so \( V(w^*(t,x)) = V_1 \). While, this is possible only \( w_1^*(t,x) = 0 \). So \( w_1(x) \) must be an equilibrium of (10). By the arbitrariness of \( w_1(x) \), we have that the \( \omega \)-limit set for (10) consists of equilibrium points.

From now on, for given \( h \), we suppress the dependence on \( h \) in \( \Lambda(bh) \). Now we study the graph of \( \Lambda(b) \) on \( [0,+,\infty) \times \mathbb{R} \) by a similar method of [1]. Let

\[
S_b = \left\{ \int_l^L a|v|^2 dx + \beta a(L)v(L)^2 - b \int_l^L hv^2 dx : v \in W^{1,2}(l,L), \int_l^L v^2 dx = 1 \right\}.
\]

It is shown in [6] by variational arguments that \( \Lambda(b) = \inf S_b \). Clearly, \( b \) is a principal eigenvalue of (3) if and only if \( \Lambda(b) = 0 \). For fixed \( v \in W^{1,2}(l,L) \), \( b \to \int_l^L a|v|^2 dx + \beta a(L)v(L)^2 \) is an affine and so concave function. As the infimum of any collection of concave functions is concave, it follows that \( b \to \Lambda(b) \) is a concave function. Also, by considering test functions \( v_1 \in W^{1,2}(l,L) \) such that
\[ \int_0^L hv_t^2 > 0, \] it is easy to see that \( \Lambda(b) \to -\infty \) as \( b \to +\infty \). Clearly, we have \( \Lambda(0) > 0 \). Thus, \( b \to \Lambda(b) \) is an increasing function until it attains its maximum, and is a decreasing function thereafter; or is a decreasing function. From these facts, we have that \( \Lambda(\lambda_1^+) = 0, \Lambda(b) > 0 \iff b \in (0, \lambda_1^+) \) and \( \Lambda(b) < 0 \iff b \in (\lambda_1^+, +\infty) \).

By Lemma 3, we get that (10) admits only one nonnegative steady state solution \( u = 0 \) if \( \Lambda(b) > 0 \). So for orbit \( I^*(t, x) \) we have \( I^*(t, x) \to 0 \) (as \( t \to +\infty \)) in \( Y \) uniform for bounded sets of initial data.

Let \( \mathcal{T} = ce^{-\Lambda(b)t}\psi_1 \). Then we have that

\[
\mathcal{T}_t - (a(x)\mathcal{T})' - \mathcal{T} (bh(x) - c\mathcal{T}) = -\Lambda(b)\mathcal{T} - \left[ (a(x)\mathcal{T})' + bh(x)\mathcal{T} \right] + c\mathcal{T}^2 > 0. 
\]

By the Gronwall inequality, we can easily show that \( \psi_1 > 0 \) on \([l, L]\). Choose \( c \) so large that \( \mathcal{T}(1, x) > \phi(x) \) for any \( x \in [l, L] \). Then \( \mathcal{T} \) is an upper solution of (10). Then \( \mathcal{T}(t, x) > I^*(t, x) \) for all \( t > 1 \), and since \( \Lambda(b) < 0 \) we have \( \mathcal{T}(t, x) \to 0 \) exponentially as \( t \to +\infty \), so \( I^*(t, x) \) must decay toward zero exponentially as well.

(iii) From (i) and Lemma 3, we know that (10) has only one positive steady state solution \( u = u^*(x) \) if \( \Lambda(b) < 0 \). Since the \( \omega \)-limit set for (10) consists of equilibrium points, \( I^*(t, x) \to u^*(x) \) as \( t \to +\infty \) in \( Y \) uniform for bounded sets of initial data.

Next, we prove the convergence is exponentially fast.

Let \( \mathcal{T} = u^* (1 - e^{-\rho t}) \) for \( t > 1 \), where \( \rho \) is some positive constant which are determined later. Then we have that

\[
\mathcal{T}_t - (a(x)\mathcal{T})' - \mathcal{T} (bh(x) - c\mathcal{T})
= \rho e^{-\rho t} u^* - \left[ (a(x)\mathcal{T})' + bh(x)\mathcal{T} \right] + c\mathcal{T}^2
= \rho e^{-\rho t} u^* - (1 - e^{-\rho t}) c (u^*)^2 + c (1 - e^{-\rho t})^2 (u^*)^2
= e^{-\rho t} u^* (\rho - cu^* + ce^{-\rho t} u^*). 
\]

We take \( \rho \leq cm/2 \), where \( m = \min_{x \in [l, L]} u^*(x) \). Let \( M = \max_{x \in [l, L]} u^*(x) \). Then there exists \( t_0 > 1 \) such that

\[ e^{-\rho t} M \leq \frac{m}{2} \text{ for all } t \geq t_0. \]

So we have

\[ \mathcal{T}_t - (a(x)\mathcal{T})' - \mathcal{T} (bh(x) - c\mathcal{T}) \leq 0 \text{ for all } t \geq t_0. \]

The strong maximum principle implies \( I^*(t, x) > 0 \) on \([l, L] \) for \( t \geq t_0 \). So we can take \( \rho \) small enough such that

\[ \mathcal{T}(t_0, x) = u^* (1 - e^{-\rho t_0}) \leq I^*(t_0, x). \]

Thus \( \mathcal{T} \) is a lower solution of the following problem

\[
\begin{align*}
W_t &= (a(x)W)' + W (bh(x) - cW), \quad l < x < L, \quad t > t_0, \\
W (t_0, x) &= I^*(t_0, x), \quad l \leq x \leq L, \\
W_x (t, l) &= 0, \quad W_x (t, L) + \beta I(t, L) = 0, \quad t > t_0.
\end{align*}
\]

So we have \( \mathcal{T}(t, x) \leq I^*(t, x) \) on \([l, L] \) for \( t \geq t_0 \).
Let \( \hat{I} = u^* (1 + \vartheta e^{-\rho t}) \), where \( \vartheta > 0 \) is a constant. Then we have that

\[
\begin{align*}
\hat{I}_t - (a(x)\hat{P})' - \hat{I} (bh(x) - c\hat{I}) &= -\rho \vartheta e^{-\rho t} u^* - \left( (a(x)\hat{P})' + bh(x)\hat{I}) + c\hat{I}\right)^2 \\
&= -\rho \vartheta e^{-\rho t} u^* - (1 + \vartheta e^{-\rho t}) c (u^*)^2 + c (1 + \vartheta e^{-\rho t})^2 (u^*)^2 \\
&= \vartheta e^{-\rho t} u^* (cm - \rho + c\vartheta e^{-\rho t} u^*) \\
&\geq \vartheta e^{-\rho t} u^* (cm - \rho) \\
&\geq 0.
\end{align*}
\]

Choose \( \vartheta \) large enough such that \( \hat{I}(1,x) > \phi(x) \). So \( \hat{I} \) is an upper solution of (10). It follows that \( |I^* - u^*| \leq \xi e^{-\rho t} \) for \( t \geq t_0 \), where \( \xi = \max \{1, \vartheta \} \). Therefore, the convergence of \( I^*(t,x) \to u^*(x) \) is exponentially fast.

**Remark 12.** If \( a(x) \equiv 1 \), Lemma 11 is just the corollary of Theorem 3.3 of [16]. However, our proof is more simple and direct even in the case of \( a \equiv 1 \).

By Lemma 11, we can easily get the following results.

**Proposition 13.** Let \( \lambda_1^+ \) be the positive principal eigenvalue of (9).

(i) If \( b < \lambda_1^+ \), then (10) admits only one nonnegative steady state solution \( u = 0 \), which is globally asymptotically stable, that is \( \lim_{t \to +\infty} I^*(t,x) = 0 \) uniformly on \([l,L]\).

(ii) If \( b > \lambda_1^+ \), then (10) has only one positive steady state solution \( u = u^*(x) \), which is globally asymptotically stable, that is \( \lim_{t \to +\infty} I^*(t,x) = u^*(x) \) uniformly on \([l,L]\).

The following two theorems are our main results.

**Theorem 14.** If \( \lambda < \lambda_1^+ / r_\infty \), then (2) no positive equilibria and all solutions decay to zero uniformly on \([l,L]\) as \( t \to +\infty \).

**Proof.** Clearly, 0 is a lower solution of (2). Now, consider the following problem

\[
\begin{align*}
I_t - (a(x)I')' = \lambda r(t)I (h(x) - \frac{1}{K}), & \quad l < x < L, \ t > 1 \\
I(1,x) = M \phi_1(x), & \quad l < x < L, \\
I_x(t,l) = 0, I_x(t,L) + \beta I(t,L) = 0, & \quad t > 1,
\end{align*}
\]

where \( \phi_1 > 0 \) is the eigenfunction corresponding to \( \lambda_1^+ \) and \( M > 0 \) is a constant. We choose \( M \) so large that \( M \phi_1(x) \geq \phi(x) \). For any solution \( \tilde{I}(t,x) \) of (14), we can see that \( \tilde{I} \) is an upper solution of (2). Let \( I \) be any solution of (2). It follows from Lemma 5 that

\[
0 \leq I(t,x) \leq \tilde{I}(t,x) \text{ for any } x \in [l,L] \text{ and } t \geq 1.
\]

Since \( \lim_{t \to +\infty} r(t) = r_\infty \), for any \( \epsilon > 0 \), there exists a \( \tilde{T}_0 > 1 \) such that \( r_\infty - \epsilon \leq r(t) \leq r_\infty + \epsilon \) for \( t \geq \tilde{T}_0 \). Since \( \lim_{t \to +\infty} r(t)h(x) = r_\infty h(x) \), for above \( \epsilon \), there exists a \( \tilde{T}_0 > 1 \) such that \( r_\infty h(x) - \epsilon r_\infty \leq r(t)h(x) \leq r_\infty h(x) + \epsilon r_\infty \) for \( t \geq \tilde{T}_0 \) and \( x \in [l,L] \). Let \( T_0 = \max \left\{ \tilde{T}_0, \tilde{T}_0 \right\} \). Then \( \tilde{I}(t,x) \) satisfies that

\[
\tilde{I}_t \leq \left( a(x)\tilde{P} \right)' + \lambda r_\infty (h(x) + \epsilon) \tilde{I} - \lambda (r_\infty - \epsilon) \frac{\tilde{T}^2}{K}, \quad l < x < L, \ t > T_0.
\]
Now consider the following problem
\begin{equation}
I_1 - (a(x) I')' = \lambda x (h(x) + e) I - \lambda (r_\infty - e) \frac{I^2}{K}, \quad l < x < L, \quad t > T_0
\end{equation}
\begin{align*}
I(T_0, x) &= \bar{I}(T_0, x), \quad l \leq x \leq L, \\
I_x(t, l) &= 0, \quad I_x(t, L) + \beta I(t, L) = 0, \quad t > T_0.
\end{align*}

For the unique positive solution \( \bar{T}_x(t, x) \) of (15), by the comparison principle again, we can see that
\[
\bar{T}(t, x) \leq \bar{T}_x(t, x) \quad \text{for any} \quad x \in [l, L] \quad \text{and} \quad t \geq T_0.
\]

On the other hand, consider the following eigenvalue problem
\begin{equation}
\begin{cases}
-a(x)u' - \lambda x (h(x) + e) u = \mu u + \epsilon \lambda x u, & l < x < L, \\
u'(l) = 0, \quad u'(L) + \beta u(L) = 0.
\end{cases}
\end{equation}

By Lemma 9, we have that \( \Lambda (\lambda x (h + e)) = \Lambda (\lambda x) - \epsilon \lambda x \). The fact of \( \lambda < \lambda_1^+ / r_\infty \) implies \( \lambda x \in (0, \lambda_1^+) \). So we have that \( \Lambda (\lambda x) > 0 \). Choose \( \epsilon > 0 \) sufficiently small such that \( \lambda = \lambda (\lambda x (h + e)) > 0 \). By Lemma 11, we have \( \bar{T}_x(t, x) \to 0 \) uniformly for \( x \in [l, L] \) as \( t \to +\infty \). Therefore, \( I(t, x) \to 0 \) uniformly for \( x \in [l, L] \) as \( t \to +\infty \).

**Theorem 15.** If \( \lambda > \lambda_1^+ / r_\infty \), then (2) has a unique positive steady state solution \( u^* \), and all solutions to (2) satisfies \( I(t, x) \to u^*(x) \) uniformly on \( [l, L] \) as \( t \to +\infty \).

**Proof.** In view of the proof of Theorem 14, \( \lambda > \lambda_1^+ / r_\infty \) implies that \( \Lambda (\lambda x) < 0 \). It follows that \( \Lambda (\lambda x (h + e)) < 0 \). Lemma 11 implies that \( \bar{T}_x(t, x) \to \pi_\infty(x) \) uniformly for \( x \in [l, L] \) as \( t \to +\infty \), where \( \pi_\infty(x) \) is the unique positive solution of
\begin{equation}
\begin{cases}
-a(x)u' = \lambda x (h(x) + e) u - \lambda (r_\infty - e) \frac{u^2}{K}, & l < x < L, \\
u'(l) = 0, \quad u'(L) + \beta u(L) = 0.
\end{cases}
\end{equation}

So we have
\[
\limsup_{t \to +\infty} I(t, x) \leq \pi_\infty(x) \quad \text{for any} \quad x \in [l, L].
\]

By Lemma 6, we get that
\[
\limsup_{t \to +\infty} I(t, x) \leq u^*(x) \quad \text{for any} \quad x \in [l, L].
\]

Since \( \lim_{t \to +\infty} r(t) = r_\infty \), for any \( \epsilon > 0 \), there exists a \( \bar{T}_1 > 1 \), such that \( r_\infty - \epsilon \leq r(t) \leq r_\infty + \epsilon \) for \( t \geq \bar{T}_1 \). Since \( \lim_{t \to +\infty} r(t) h(t) = r_\infty h(x) \), for above \( \epsilon \), there exists a \( \bar{T}_1 > 1 \), such that \( r_\infty h(x) - \epsilon d \leq r(t) h(t) \leq r_\infty h(x) + \epsilon d \) for \( t \geq \bar{T}_1 \) and \( x \in [l, L] \). Let \( T_1 = \max \{ \bar{T}_1, \bar{T}_1 \} \). Now let \( I(t, x) \) be the solution of
\begin{equation}
\begin{cases}
W_t - (a(x) W')' = \lambda (r(t) h(t) - \frac{W}{K}), & l < x < L, \quad t > T_1 \\
W(T_1, x) = \delta I(T_1, x), & l \leq x \leq L, \\
W_x(t, l) = 0, \quad W_x(t, L) + \beta W(t, L) = 0, & t > T_1,
\end{cases}
\end{equation}

where \( \delta \in (0, 1) \) is a positive constant, \( I(t, x) \) is a solution of (2). Then we can easily show \( I(t, x) \leq I(t, x) \) for \( t = T_1 \). Hence \( I(t, x) \) is a lower solution of (2) in \([l, L] \times [T_1, +\infty)\).

Clearly, we have that
\[
L_x \geq (a(x) L')' + (r_\infty h(x) - \epsilon d) L - \lambda (r_\infty + \epsilon) \frac{L^2}{K}, \quad l < x < L, \quad t > T_1.
\]
Now consider the problem
\[
\begin{align*}
W_t - (a(x)W')' &= \lambda (r_\infty h(x) - \epsilon d) W - \lambda (r_\infty + \epsilon) \frac{W^2}{K}, \quad l < x < L, \quad t > T_1 \\
W(T_1, x) &= \delta I(T_1, x), \quad l \leq x \leq L, \\
W_x(t, l) &= 0, \quad W_x(t, L) + \beta W(t, L) = 0, \quad t > T_1.
\end{align*}
\]
(16)

For the unique positive solution \( \hat{I}_\epsilon(t, x) \) of (16), by the comparison principle, we have that
\[ I(t, x) \geq \hat{I}_\epsilon(t, x) \quad \text{for any } x \in [l, L] \text{ and } t \geq T_1. \]

On the other hand, consider the following eigenvalue problem
\[
\begin{align*}
- (a(x)u')' - \lambda r_\infty h(x) u &= \mu u - \epsilon r_\infty u, \quad l < x < L, \\
u'(l) &= 0, \quad u'(L) + \beta u(L) = 0.
\end{align*}
\]
By Lemma 9, we get that \( \Lambda (\lambda r_\infty (h - \epsilon)) = \Lambda (\lambda r_\infty) + \epsilon r_\infty \). The fact of \( \lambda > \lambda_1^+ / r_\infty \) implies \( \lambda r_\infty > \lambda_1^+ \). So we have that \( \Lambda (\lambda r_\infty) < 0 \). Choose \( \epsilon > 0 \) sufficiently small such that \( \Lambda (\lambda r_\infty (h - \epsilon)) < 0 \). By Lemma 11, we have \( \hat{I}_\epsilon(t, x) \rightarrow \hat{u}_\epsilon(x) \) uniformly for \( x \in [l, L] \) as \( t \rightarrow +\infty \), where \( \hat{u}_\epsilon(x) \) is the unique positive solution of
\[
\begin{align*}
- (a(x)u')' &= \lambda r_\infty (h(x) - \epsilon) u - \lambda (r_\infty + \epsilon) \frac{u^2}{K}, \quad l < x < L, \\
u'(l) &= 0, \quad u'(L) + \beta u(L) = 0.
\end{align*}
\]
Therefore, we have that
\[ \liminf_{t \rightarrow +\infty} I(t, x) \geq \hat{u}_\epsilon(x) \quad \text{for } x \in [l, L]. \]
By Lemma 7, we get that
\[ \liminf_{t \rightarrow +\infty} I(t, x) \geq u^*(x) \quad \text{for any } x \in [l, L]. \]
Therefore, we have \( I(t, x) \rightarrow u^*(x) \) uniformly on \([l, L]\) as \( t \rightarrow +\infty \).

**Remark 16.** The results of Theorem 14 and 15 show that if \( \lambda < \lambda_1^+ / r_\infty \), the information vanishes in finite time; if \( \lambda > \lambda_1^+ / r_\infty \), the information diffusion lasts forever.

**Remark 17.** While [16, 17] deal with more general equations, Theorem 14 and 15 are sharper than the corresponding results in [16, 17] as we are able to identify the cut-off point \( (\lambda_1^+ / r_\infty) \) for which the solutions of (2) go to either zero or a positive static state.

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