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CONVEX SOLUTIONS OF BOUNDARY VALUE PROBLEMS ARISING FROM MONGE-AMPÈRE EQUATIONS

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Abstract. In this paper we study an eigenvalue boundary value problem which arises when seeking radial convex solutions of the Monge-Ampère equations. We shall establish several criteria for the existence, multiplicity and nonexistence of strictly convex solutions for the boundary value problem with or without an eigenvalue parameter.

1. **Introduction.** In this paper we consider the existence, multiplicity and nonexistence of strictly convex solutions for the boundary value problem:

$$\begin{cases} \Big(\big(u'(r)\big)^n\Big)' = \lambda n r^{n-1} f(-u(r)) \text{ in } \ 0 < r < 1, \\ u'(0) = 0, \ \ u(1) = 0, \end{cases} \tag{1.1}$$

where $n \geq 1$ is the dimension of the space, $f(u) \geq 0$ for $u \geq 0$, and f is not identical to zero. By a solution of (1.1) we understand it is a function which belongs to $C^2[0,1]$ and satisfies (1.1). A strictly convex solution of (1.1) is negative on [0,1). Such a problem arises in the study of the existence of convex radially symmetric solutions for the following Dirichlet problem of the Monge-Ampère equations in $\mathbb{R}^n$:

$$\begin{cases} \det(D^2 u) = \lambda f(-u) \ \text{ in } \ B \\ u = 0 \ \text{ on } \ \partial B, \end{cases} \tag{1.2}$$

where $B = \{x \in \mathbb{R}^n : |x| < 1\}$ is the unit ball in $\mathbb{R}^n$ and $D^2 u = (\frac{\partial^2 u}{\partial x_i \partial x_j})$ is the Hessian of u.

To see how Equation 1.2 reduces to Equation 1.1 for a radially symmetric solution $u = u(x)$, consider $u = v(|x|)$ for some function $v(r)$ defined for $r > 0$. We claim that

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$$det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = detA. \tag{1.3}$$

where

$$A = \begin{pmatrix} v''(r) & 0 & 0 & \dots & 0 \\ 0 & \frac{v'(r)}{r} & 0 & \dots & 0 \\ 0 & 0 & \frac{v'(r)}{r} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \frac{v'(r)}{r} \end{pmatrix}$$

To verify this claim, first of all it can be easily checked that we actually have $\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = A$ at the point $x^0 = (r, 0, \dots, 0)$, where $r = |x|$. Now for a general point $x \in \mathbb{R}^n$, there is a rotational matrix B such that $Bx = x^0$. Since $u = u(x) = v(|Bx|) = v(|x|)$, by viewing $(\frac{\partial u}{\partial x_i})$ as a linear operator and $\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ as a bilinear operator we deduce that

$$\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = B^2 \left(\frac{\partial^2 v(|y|)}{\partial y_i \partial y_j}\right)|_{y=x^0} = B^2 A.$$

Hence we have (1.3) since $det(B) = 1$. Consequently, we have

$$det(D^2 u) = v'' \frac{(v')^{n-1}}{r^{n-1}}.$$

Thus, a radially symmetric solution of (1.2) can be viewed as a solution of (1.1). Note that the equality 1.3 was used in [1].

As noticed by Lions [8], the particular function $f(u) = u^n$ acts like a "linear" term to the fully nonlinear operation $det(D^2 u)$. In fact, in [8] Lions proved the existence of a unique eigenvalue λ_1 for Equation 1.2 with $f(u) = u^n$. Specifically, $\lambda_1 > 0$ and the corresponding eigenfunction ψ_1 is negative convex, and that any other eigenfunction would be a positive constant multiple of ψ_1. Furthermore, λ_1 acts like a bifurcation point for 1.2 with general functions f, which is reminiscent of the well-known properties of the first eigenvalues of linear second-order elliptic operators or more generally of positive operators as given by the famous Krein-Rutman theorem. For this reason, the so-called sublinear and superlinear functions, $f(u)$, are defined in relation to u^n, as will be done in this paper.

Kutev [9] obtained the existence of a unique nontrivial convex radially symmetric solution of (1.2) with $f(u) = (u)^p$ for any p with $0 < p \neq n$ by reducing (1.2) to (1.1). We refer to [3, 9] and references therein for further discussions regarding convex radially symmetric solutions of (1.2).

The results we are going to present reveal how the behavior of the function f at zero and infinity (in particular against u^n) and its number of zeros, have a profound effect on the number of nontrivial solutions of the corresponding boundary value problem. We would like to point out that we do not assume $f(u) > 0$ for all $u > 0$, throughout the paper.

2. **Preliminaries.** Upon the transformation $v = -u$, (1.1) can be written as

$$\begin{cases} \Big(\big(-v'(r)\big)^n\Big)' = \lambda n r^{n-1} f(v(r)) \ \text{ in } \ 0 < r < 1, \\ v'(0) = 0, \ v(1) = 0. \end{cases} \tag{2.4}$$

Therefore, throughout the paper we shall study positive concave classical solutions of (2.4). The following well-known result of the fixed point index, due to Krasnoselskii, is the base of our approaches. See, for instance, ([2, 6, 7]).

Lemma 2.1. *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{v \in K : \|x\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{v \in K : \|x\| = r\}$.*

(i) If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.
(ii) If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

In order to apply Lemma 2.1 to (2.4), let X be the Banach space $C[0,1]$ with $\|v\| = \sup\limits_{t \in [0,1]} |v(t)|$.

Define a set $K \subseteq X$ by

$$\begin{aligned} K = \{v \in X : \ & v(t) \text{ is decreasing in } t, v(1) = 0, \\ & \text{and } v(t) \geq \|v\|(1-t) \text{ for all } t \in [0,1]\}. \end{aligned}$$

It can be easily verified that K is indeed a cone in X.

For any $r > 0$, define Ω_r by

$$\Omega_r = \{v \in K : \|v\| < r\}.$$

Note that $\partial \Omega_r = \{v \in K : \|v\| = r\}$.

To study (2.4), consider the map $T_\lambda : K \to X$, defined by

$$(T_\lambda)v(r) = \int_r^1 \big(\lambda \int_0^s n\tau^{n-1} f(v(\tau)) d\tau\big)^{\frac{1}{n}} ds, \ \ 0 \leq r \leq 1. \tag{2.5}$$

We point out that (1.1) is equivalent to the fixed point equation

$$(T_\lambda)v = v,$$

in K. In fact, if $v \in K$ is a positive fixed point of T_λ, then $-v$ is a convex solution of (1.1). It is further clear that so long as f does not vanish on any entire interval, we should have $v'' < 0$ for $r \in (0,1)$ and hence $-v$ must be a strictly convex solution of (1.1). Conversely, if u is a strictly convex solution of (1.1), then $-u$ is a positive fixed point of T_λ in K.

The following lemma is a simple consequence of the concavity of a function v.

Lemma 2.2. *For any function $v \in C[0,1]$ with $v(t) \geq 0$ and $v'(t)$ decreasing in $[0,1]$, we have*

$$v(t) \geq \min\{t, 1-t\}||v||, \quad t \in [0,1].$$

In particular, for any pair $0 < \alpha < \beta < 1$ we have,

$$\min_{\alpha \leq t \leq \beta} v(t) \geq \min\{\alpha, 1-\beta\}\,||v||.$$

For instance, we have

$$\min_{\alpha \le t \le 1-\alpha} v(t) \ge \alpha ||v||.$$

Furthermore, if $v(0) = \|v\|$*, then we have*

$$v(t) \ge \|v\|(1-t) \text{ for all } t \in [0,1]. \tag{2.6}$$

Proof. Since $v'(t)$ is decreasing, we have for $0 \le t_0 < t < t_1 \le 1$,

$$v(t) - v(t_0) = \int_{t_0}^{t} v'(s)ds \ge (t-t_0)v'(t)$$

and

$$v(t_1) - v(t) = \int_{t}^{t_1} v'(s)ds \le (t_1-t)v'(t),$$

from which, we have

$$v(t) \ge \frac{(t_1-t)v(t_0) + (t-t_0)v(t_1)}{t_1-t_0}.$$

Choose some $\sigma \in [0,1]$ such that $v(\sigma) = ||v||$ and consider $[t_0, t_1]$ as either of $[0,\sigma]$ and $[\sigma, 1]$, we have

$$v(t) \ge t||v|| \quad \text{for} \quad t \in [0,\sigma],$$

and

$$v(t) \ge (1-t)||v|| \quad \text{for} \quad t \in [\sigma, 1].$$

Clearly, this last inequality implies (2.6), and the last two inequalities imply

$$v(t) \ge \min\{t, 1-t\}||v||, \quad t \in [0,1].$$

□

Lemma 2.3. $T_\lambda(K) \subset K$ *and the map* $T_\lambda : K \to K$ *is completely continuous.*

Proof. The inequality (2.6) and the definition of T_λ imply that $T_\lambda(K) \subset K$. The complete continuity of the integral operator T_λ is well known. □

3. **Uniqueness results.** In this section we are going to prove a general result of uniqueness and approximation by iterations of the solution. Results to be proved in this section are true for any positive parameter λ. So, we may assume $\lambda = 1$ for simplicity and therefor consider

$$\begin{cases} \Big(\big(-v'(r)\big)^n\Big)' = nr^{n-1}f(v(r)) \text{ in } 0 < r < 1, \\ \\ v'(0) = 0, \ \ v(1) = 0, \end{cases} \tag{3.7}$$

and the corresponding operator

$$(Tv)(r) = \int_r^1 \Big(\int_0^s n\tau^{n-1} f(v(\tau))d\tau\Big)^{\frac{1}{n}} ds, \ \ 0 \le r \le 1, \tag{3.8}$$

defined on the cone K and also on the cone K_1, where

$$K_1 = \{v \in X : v(t) \ge 0\},$$

Now we can state the following theorem.

Theorem 3.1. *Let $f : [0, \infty) \to [0, \infty)$ be continuous and increasing, such that for any $u > 0$ and $t \in (0, 1)$ there always exists some $\eta > 0$ such that*

$$f(tu) \geq [(1 + \eta)t]^n f(u). \tag{3.9}$$

Then (3.7) can have at most one positive solution. Furthermore, if (3.7) has a positive solution v^*, then for any $v_0 \in K_1$ with $v_0 > 0$, the iteration v_n, defined by

$$v_{n+1} = Tv_n,$$

converges to v^*.

Corollary 3.2. *For $f(u) = \sum_{i=1}^{i=m} a_i u^{p_i}$, with $a_i > 0$ and $0 < p_i < n$, the IVP (3.7) has a unique positive solution v^* which can be approximated via iterations with any positive initial point from K_1.*

To compare with the results of Kutev's in [9], the results in this section cover a broad class of functions of f versus his case when $f(u) = u^p$ with $0 < p < n$.

Remark: By slightly modifying the proof of the Theorem 3.1, we can show that condition 3.9 implies that $f(u) > 0$ when $u > 0$, and can be replaced by the following simpler condition:

$$f(tu) > t^n f(u).$$

The theorem is proved via a sequence of lemmas. But we need to have a definition first.

Definition 3.1. *Let P be a cone from a Banach space Y. With some $u_o \in P$ positive, $A : P \to P$ is called u_o-sublinear if*

a. *for any $x > 0$, there exist $\alpha, \beta > 0$ such that*

$$\alpha u_0 \leq Ax \leq \beta u_0;$$

and

b. *for any $\alpha u_0 \leq x \leq \beta u_0$ and $t \in (0, 1)$, there always exists some $\eta > 0$ such that*

$$A(tx) \geq (1 + \eta)tAx.$$

Lemma 3.3. *An increasing and u_0-subliner operator A can have at most one positive fixed point.*

Proof. Let $u, v > 0$ be two positive fixed points. Then $u \geq cv$ for some $c > 0$. Set $c^* = \sup\{t |\ u \geq tv\}$. We claim that $c^* \geq 1$. If not, then there exists some $\eta > 0$ such that $A(c^*v) \geq (1 + \eta)c^*Av$. Thus,

$$\begin{aligned} u &\geq A(c^*v) \\ &\geq (1 + \eta)c^*Av, \end{aligned}$$

which is a contraction since $(1 + \eta)c^* > c^*$. So, $u \geq v$ and similarly we can show that $v \geq u$. □

Lemma 3.4. *Let f be from Theorem 3.1, and $u_0 = 1 - t$. Then the operator T, defined by 3.8, is u_0-sublinear.*

Proof. First, we show that for any $u > 0$ from K_1, there exist $\alpha, \beta > 0$ such that

$$\alpha u_0 \leq Tu \leq \beta u_0. \tag{3.10}$$

Let $M = \max_{t\in[0,1]}\{f(u(t))\}$. Then,

$$(Tu)(t) \leq M^{\frac{1}{n}} \int_t^1 s\ ds \leq \frac{1}{2} M^{\frac{1}{n}}(1-t^2) \leq M^{\frac{1}{n}}(1-t).$$

So, we may take $\beta = M^{\frac{1}{n}}$.

Clearly, we may take $\alpha = \|Tu\| = (Tu)(1)$ since $T(K_1) \subseteq K$. To stage a direct proof, notice that $(Tu)(t)$ is strictly decreasing in t and vanishes at $t = 0$. Choose any $c \in (0, 1)$ and set

$$m = \Big(\int_0^c nt^{n-1} f(u(t))\ dt \Big)^{\frac{1}{n}}.$$

Then for all $t \in [c, 1]$, we have

$$(Tu)(t) \geq \int_t^1 m\ ds \geq m(1-t).$$

Since $(Tu)(t) \geq (Tu)(c) = m(1 - c)$ for all $t \in [0, c]$, we have

$$(Tu)(t) \geq m(1-c)(1-t)$$

for all $t \in [0, 1]$. So, we may choose $\alpha = m(1 - c)$ and (3.10) is proved.

Now we need to show that for any $\alpha u_0 \leq u \leq \beta u_0$ and $\xi \in (0, 1)$, there always exists some $\eta > 0$ such that

$$T(\xi u) \geq (1+\eta)\xi Tu.$$

To this end, we note that due to the conditions satisfied by f, there exists an $\eta > 0$ such that

$$f(\xi u) \geq [(1+\eta)\xi]^n f(u).$$

Thus we have

$$(T(\xi u))(t) \geq \big[(1+\eta)\xi\big](Tu)(t).$$

Therefore, the proof is complete. □

Lemma 3.5. *If an increasing u_0-sublinear operator A from a normal cone E into itself has a positive fixed x^*, then for any $x_0 > 0$ the iterations*

$$x_{n+1} = Ax_n$$

converge to x^*. Namely,

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

Proof. For any $t_1 \in (0, 1)$, set $v_0 = t_1 x^*$ and $v_{n+1} = Av_n$. Then,

$$t_1 x^* = v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq x^*.$$

Set $\rho_n = \sup\{t | tx^* \leq v_n\}$, then

$$0 < t_1 = \rho_0 \leq \rho_1 \leq \rho_2 \leq \cdots \leq \rho_n \leq \cdots \leq 1,$$

and $\rho_n x^* \le v_n$. We claim that

$$\lim_{n\to\infty} \rho_n = 1.$$

If not, we have $\lim_{n\to\infty} \rho_n = \gamma < 1$. Then there is $\eta > 0$ such that

$$A(\gamma x^*) \ge (1+\eta)\gamma A x^* = (1+\eta)\gamma x^*.$$

So, for $0 < t \le \gamma$ we have

$$A(tx^*) = A\left(\frac{t}{\gamma}\gamma x^*\right) \ge \frac{t}{\gamma} A(\gamma x^*) \ge (1+\eta)tx^*.$$

In particular, $A(\rho_n x^*) \ge (1+\eta)\rho_n x^*$. Thus,

$$v_{n+1} = Av_n \ge A(\rho_n x^*) \ge (1+\eta)\rho_n x^*,$$

hence $\rho_{n+1} \ge (1+\eta)\rho_n$ and

$$\rho_n \ge (1+\eta)^n \rho_0,$$

a contradiction. Therefore, the claim is proved.

Now we take any $t_2 > 1$ and set $w_0 = t_2 x^*$ and $w_{n+1} = Aw_n$. We have

$$t_2 x^* = w_0 \ge w_1 \ge w_2 \ge \cdots \ge w_n \ge \cdots \ge x^*.$$

Define $\xi_n = \inf\{t | tx^* \ge w_n\}$. We have

$$t_2 = \xi_0 \ge \xi_1 \ge \xi_2 \ge \cdots \ge \xi_n \ge 1,$$

and $\xi_n x^* \ge w_n$. We claim that

$$\lim_{n\to\infty} \xi_n = 1.$$

If not, let $\lim_{n\to\infty} \xi_n = \gamma_0 > 1$. Then there exists some $\eta_0 > 0$ such that

$$Ax^* = A\left(\frac{1}{\gamma_0}\gamma_0 x^*\right) \ge \frac{1+\eta_0}{\gamma_0} A(\gamma_0 x^*),$$

namely,

$$A(\gamma_0 x^*) \le \frac{\gamma_0}{1+\eta_0} Ax^* = \frac{\gamma_0 x^*}{1+\eta_0}.$$

For $t \ge \gamma_0$ we have

$$A(\gamma_0 x^*) = A\left(\frac{\gamma_0}{t} tx^*\right) \ge \frac{\gamma_0}{t} A(tx^*).$$

Thus,

$$A(tx^*) \le \frac{t}{\gamma_0} A(\gamma_0 x^*) \le \frac{tx^*}{1+\eta_0}.$$

In particular, $A(\xi_n x^*) \le \frac{\xi_n x^*}{1+\eta_0}$. Then we have

$$w_{n+1} = Aw_n \le A(\xi_n x^*) \le \frac{\xi_n x^*}{1+\eta_0},$$

and $\xi_{n+1} \le \frac{\xi_n}{1+\eta_0}$. Thus,

$$\xi_n \le \frac{\xi_0}{(1+\eta_0)^n},$$

a contradiction. The second claim is prove.

Due to the u_0-sublinear nature of A, we have

$$\alpha_0 u_0 \le x^* = Ax^* \le \beta_0 u_0$$

and

$$\alpha_1 u_0 \le x_1 = Ax_0 \le \beta_1 u_0,$$

for a group of positive constants involved. Therefore,

$$\frac{\alpha_1}{\beta_0}x^* \le x_1 \le \frac{\beta_1}{\alpha_0}x^*.$$

Take $0 < t_1 < \min\left\{1, \frac{\alpha_1}{\beta_0}\right\}$ and $t_2 > \max\left\{1, \frac{\beta_1}{\alpha_0}\right\}$. Then

$$v_0 = t_1 x^* \le x_1 \le t_2 x^* = w_0.$$

With the above chosen t_1 and t_2, we may repeat the prior defined process to have

$$\rho_n x^* \le v_n \le x_{n+1} \le w_n \le \xi_n x^*.$$

Thus,

$$\begin{aligned}(\rho_n - 1)\beta_0 u_0 &\le (\rho_n - 1)x^* \\ &\le x_{n+1} - x^* \\ &\le (\xi_n - 1)x^* \\ &\le (\xi_n - 1)\beta_0 u_0.\end{aligned}$$

From these inequalities we arrive at

$$0 \le (x_{n+1} - x^*) + (1 - \rho_n)\beta_0 u_0 \le (\xi_n - 1)\beta_0 u_0 + (1 - \rho_n)\beta_0 u_0$$

Now, since E is a normal cone, namely, there is a constant N such that for any $0 \le x \le y$ we have

$$\|x\| \le N\|y\|.$$

We thus have

$$\begin{aligned}\|x_{n+1} - x^*\| &\le \|(x_{n+1} - x^*) + (1 - \rho_n)\beta_0 u_0\| + \|(1 - \rho_n)\beta_0 u_0\| \\ &\le N\|(\xi_n - 1)\beta_0 u_0\| + (N + 1)\|(1 - \rho_n)\beta_0 u_0\|\end{aligned}$$

which implies readily

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

□

Remark. Note that the method and procedure used in the proof of Lemma (3.5) is known in the literature. See, for instance, [6]

Now with the above lemmas in place, Theorem (3.1) follows readily. Corollary (3.2) also follows since the polynomial type function f satisfies all the conditions postulated in Theorem (3.1) and furthermore, with the said f the IVP problem does have a positive solution which can be proved by the standard fixed point index argument as to be explained in Section 5.

4. **Multiple solutions.** In the previous session we established some uniqueness results for functions f which, in practice, may not vanish except for, possibly, at $u = 0$. In the present session we are going to examine how the number of zeros of f may have a huge impact on the number of solutions of the BVP (1.1) or (3.7).

Unlike in the previous session, in this session we consider the BVP with a parameter. So, we now focus on BVP (1.1). Here is the main result of the section.

Theorem 4.1. *Assume that $f : [0, \infty) \to [0, \infty)$ is continuous and there exist two sequences of positive numbers a_i and b_i, satisfying*

$$a_i < b_i \le a_{i+1} < b_{i+1},$$

and such that $f(a_i) = 0$ and $f(b_i) = 0$, and $f(u) > 0$ on (a_i, b_i), for all $i = 1, ..., m$. Then there exists λ_0 such that for any $\lambda \ge \lambda_0$ the BVP (3.7) has m distinct convex solutions, $u_1, u_2, ..., u_m$, such that

$$a_i < \sup_{t\in[0,1]} u_i(t) \le b_i$$

for each $i = 1, ..., m$.

Furthermore, we may make $\|u_i\|$ as close to b_i as we wish for sufficiently large λ. Namely, for any $\eta > 0$ satisfying

$$\eta < \min_{1\le i\le m} \{b_i - a_i\},$$

there exits an λ_1 such that for any $\lambda \ge \lambda_1$, the m distinct solutions $\{u_i\}_{i=1}^m$ satisfy

$$(1 - \eta)b_i < \sup_{t\in[0,1]} u_i(t) \le b_i.$$

Remark. Note that f will satisfy the conditions in the theorem if there exist numbers $a_m > a_{m-1} > \cdots > a_1 > a_0 = 0$ such that $f(a_i) = 0$ for $i = 1, \dots, m$, and $f(u) > 0$ for $a_{i-1} < u < a_i$, $i = 1, \dots, m$.

To prove the theorem, for $i = 1, ..., m$ we define f_i by

$$f_i(u) = \begin{cases} f(u), & 0 \le u \le b_i, \\ 0, & b_i \le u, \end{cases}$$

and let the map $T_\lambda^i : K \to X$ be defined by

$$T_\lambda^i u(t) = \int_r^1 \big(\lambda \int_0^s n\tau^{n-1} f_i(v(\tau))d\tau\big)^{\frac{1}{n}} ds, \quad 0 \le r \le 1, \tag{4.11}$$

To get prepared for a proof of the theorem, we first prove two lemmas.

Lemma 4.2. *Let f be from Theorem (4.1). If $v \in K$ is a solution of 4.11, i.e., $T_\lambda^i v = v$, then v is a solution of (2.4) such that*

$$\sup_{t\in[0,1]} v(t) \le b_i$$

Proof. Notice that v satisfies

$$\begin{cases} \Big(\big(- v'(r)\big)^n\Big)' = \lambda n r^{n-1} f_i(v(r)) \text{ in } 0 < r < 1, \\ v'(0) = 0, \quad v(1) = 0. \end{cases} \tag{4.12}$$

If on the contrary that $\sup_{t\in[0,1]} v(t) = v(0) > b_i$, then there exists a $t_0 \in (0, 1)$ such that $v(t) > b_i$ for $t \in [0, t_0)$ and $v(t_0) = b_i$. It follows that

$$\Big(\big(- v'(r)\big)^n\Big)' = 0 \text{ for } r \in (0, t_0].$$

Thus, $\big(- v'(r)\big)^n$ is constant on $[0, t_0]$. Since $v'(0) = 0$, it follows that $v'(t) = 0$ for $t \in [0, t_0]$. Consequently, $v(t)$ is constant on $[0, t_0]$. This is a contradiction. Therefore $\sup_{t\in[0,1]} v(t) \le b_i$. On the other hand, since $f(u) \equiv f_i(u)$ for $0 \le u \le b_i$, v is a solution of (2.4). $\square$

Choose any ε such that

$$\max\left\{\frac{a_i}{b_i}\ :\ 1\le i\le m\right\}<\varepsilon<1.$$

Lemma 4.3. *Let the conditions of Theorem 4.1 be satisfied. For any $i\in\{1,...,m\}$, there exists r_i such that $[\varepsilon r_i, r_i]\subset(a_i,b_i)$. Furthermore, for any $v\in\partial\Omega_{r_i}$ we have*

$$\|T^i_\lambda v\|\ge\frac{1}{2}(1-\varepsilon)^2\lambda^{\frac{1}{n}}(\omega_{r_i})^{\frac{1}{n}},$$

where $\omega_{r_i}=\min_{\varepsilon r_i\le u\le r_i}\{f_i(u)\}>0.$

Proof. Based on the choice of ε, the existence of r_i with $[\varepsilon r_i, r_i]\subset(a_i,b_i)$ is obvious. Notice that for any $v\in K$ we have that $v(t)\ge v(0)(1-t)$ for all $t\in[0,1]$. In particular, we have $\varepsilon v(0)\le v(t)\le v(0)$ for all $t\in[0,1-\varepsilon]$. Let $v\in\partial\Omega_{r_i}$, then $f(v(t))\ge\omega_{r_i}$ for $t\in[0,1-\varepsilon]$. It follows that

$$\begin{aligned}\|T^i_\lambda v\|&\ge\int_0^{1-\varepsilon}\Big(n\lambda\int_0^s\tau^{n-1}f_i(v(\tau))d\tau\Big)^{\frac{1}{n}}ds\\&\ge\int_0^{1-\varepsilon}\Big(n\lambda\int_0^s\tau^{n-1}\omega_{r_i}d\tau\Big)^{\frac{1}{n}}ds\\&\ge\lambda^{\frac{1}{n}}(\omega_{r_i})^{\frac{1}{n}}\int_0^{1-\varepsilon}\Big(n\int_0^s\tau^{n-1}d\tau\Big)^{\frac{1}{n}}ds\\&>\frac{1}{2}(1-\varepsilon)^2\lambda^{\frac{1}{n}}(\omega_{r_i})^{\frac{1}{n}}.\end{aligned}$$

□

Now we are in position to prove Theorem 4.1

Proof. of Theorem 4.1. Define λ_0 by

$$\lambda_0=\frac{2^n}{(1-\varepsilon)^{2n}}\max\left\{\frac{r_i^n}{\omega_{r_i}}\ :\ i=1,2,\dots,m\right\}.$$

For each $i=1,...,m$ and $\lambda>\lambda_0$, by Lemma 4.3 we infer that

$$\|T^i_\lambda v\|>\|v\|\ \text{ for }\ \mathrm{v}\in\partial\Omega_{\mathrm{r_i}}.$$

On the other hand, for each fixed $\lambda>\lambda_0$ since $f_i(v)$ is bounded, there is an $R_i>r_i$ such that

$$\|T^i_\lambda v\|<\|v\|\ \text{ for }\ \mathrm{v}\in\partial\Omega_{\mathrm{R_i}}.$$

It follows from Lemma 2.1 that

$$i(T^i_\lambda,\Omega_{r_i},K)=0\ \text{ while }\ i(T^i_\lambda,\Omega_{R_i},K)=1,$$

and hence,

$$i(T^i_\lambda,\Omega_{R_i}\setminus\bar{\Omega}_{r_i},K)=1.$$

Thus, T^i_λ has a fixed point v_i in $\Omega_{R_i}\setminus\bar{\Omega}_{r_i}$. Lemma 4.2 implies that the fixed point v_i is a solution of (2.4) such that

$$a_i<r_i\le\|v_i\|\le b_i.$$

Consequently, (2.4) has m positive solutions, $v_1,v_2,...,v_m$ for each $\lambda>\lambda_0$, such that

$$a_i<\sup_{t\in[0,1]}u_i(t)\le b_i.$$

Since the choice of ε is arbitrary, the additional conditions can be easily satisfied by the m solutions if we choose $r_i < b_i$ sufficiently close to b_i. Therefore, the proof is complete. $\square$

5. **Eigenvalue problems.** In this section we will consider the eigenvalue problem 1.1 for which we introduce the following notations. For any function f, we define

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u^n}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u^n}. \tag{5.13}$$

Furthermore, for $r > 0$ and $0 < \alpha < 1$ we define

$$m_\alpha(r) = \min_{\alpha r \le t \le r} \{f(t)\}.$$

It is clear that $f_0 = 0$ and $f_\infty = \infty$ for $f = u^p$ with $p > n$, and $f_0 = \infty$ and $f_\infty = 0$ for $f = u^p$ with $0 < p < n$. In a recent paper [12], the second author obtained the existence of one nontrivial convex solution of (1.1) for the case $f_0 = 0$ and $f_\infty = \infty$, and the case $f_0 = \infty$ and $f_\infty = 0$.

In this section we shall give criteria of determining the number (none or one or two) of strictly convex solutions of the eigenvalue problem (1.1), based on appropriate combinations of superlinearity and sublinearity at zero and infinity. Our main results are the following.

Theorem 5.1. *We have the following conclusions.*

a. *If $f_0 = 0$ or $f_\infty = 0$, then (1.1) has a strictly convex solution for all $\lambda > \lambda_1$ for some $\lambda_1 > 0$.*
b. *If $f_0 = f_\infty = 0$, then (1.1) has two strictly convex solutions for all $\lambda > \lambda_2$ for some $\lambda_2 > 0$.*
c. *If $f_0 = \infty$ or $f_\infty = \infty$, then (1.1) has a strictly convex solution for all λ satisfying $0 < \lambda < \lambda_3$ for some $\lambda_3 > 0$.*
d. *If $f_0 = f_\infty = \infty$, then (1.1) has two strictly convex solutions for all λ satisfying $0 < \lambda < \lambda_4$ for some $\lambda_4 > 0$.*
e. *If $f_0 < \infty$ and $f_\infty < \infty$, then (1.1) has no nontrivial convex solution for all λ satisfying $0 < \lambda < \lambda_5$ for some $\lambda_5 > 0$.*
f. *If $f_0 > 0$ and $f_\infty > 0$, and $f(u) > 0$ for $u > 0$, then (1.1) has no nontrivial convex solution for all λ satisfying $\lambda > \lambda_6$ for some $\lambda_6 > 0$.*

Though we are able to provide explicit intervals of λ where (1.1) has one or two strictly convex solutions, the intervals are related to properties of f reflected by $m_\alpha(r)$. Finding the optimal intervals for the parameter λ so as to ensure existence of single or multiple solutions, and possible bifurcation points, may be addressed in future work.

Lemma 5.2. *For any $\eta > 0$ and $v \in K$, if $f(v(r)) \ge \big(v(r)\eta\big)^n$ for $r \in [0, 1-\alpha]$, then with T_λ from 2.5 we have*

$$\|T_\lambda v\| \ge \frac{1}{2}\eta\alpha(1-\alpha)^2\lambda^{\frac{1}{n}}\|v\|.$$

Proof. Note that $v(\tau) \geq \alpha\|v\|$ for $\tau \in [0, 1-\alpha]$. It follows that

$$\begin{aligned}
\|T_\lambda v\| &\geq \int_0^{1-\alpha} \Big(n\lambda \int_0^s \tau^{n-1} f(v(\tau)) d\tau\Big)^{\frac{1}{n}} ds \\
&\geq \int_0^{1-\alpha} \Big(n\lambda \int_0^s \tau^{n-1} \big(v(\tau)\eta\big)^n d\tau\Big)^{\frac{1}{n}} ds \\
&\geq \int_0^{1-\alpha} \Big(n\lambda \int_0^s \tau^{n-1} \big(\alpha\eta\|v\|\big)^n d\tau\Big)^{\frac{1}{n}} ds \\
&= \eta\|v\|\alpha\lambda^{\frac{1}{n}} \int_0^{1-\alpha} \Big(n \int_0^s \tau^{n-1} d\tau\Big)^{\frac{1}{n}} ds \\
&\geq \frac{1}{2}\eta\alpha(1-\alpha)^2\lambda^{\frac{1}{n}}\|v\|.
\end{aligned}$$

□

Define a new function

$$f^*(v) = \max_{0 \leq t \leq v} \{f(t)\}.$$

Note that $f_0^* = \lim_{v \to 0} \frac{f^*(v)}{v^n}$ and $f_\infty^* = \lim_{v \to \infty} \frac{f^*(v)}{v^n}$. We have

Lemma 5.3. *$f_0^* = f_0$ and $f_\infty^* = f_\infty$.*

Proof. Here we give a proof on the second equality since the first one can be similarly proved. First, it is obvious that

$$f_\infty^* \geq f_\infty.$$

So we need only to argue for the opposite direction for which we may assume that f is unbounded. There exists a sequence $v_k \to \infty$ such that

$$\lim_{k \to \infty} \frac{f^*(v_k)}{v_k^n} = f_\infty^*.$$

By the definition of f^*, there must exists a sequence q_k such that $q_k \leq v_k$ and

$$\lim_{k \to \infty} \frac{f(q_k)}{v_k^n} = f_\infty^*.$$

From the facts that $\frac{f(q_k)}{v_k^n} \leq \frac{f(q_k)}{q_k^n}$ and $q_k \to \infty$ (since f is assumed to be unbounded), we arrived at $f_\infty^* \leq f_\infty$. □

Lemma 5.4. *For $r > 0$, if there exists $\varepsilon > 0$ such that $f^*(r) \leq \varepsilon^n r^n$, then*

$$\|T_\lambda v\| \leq \frac{1}{2}\lambda^{\frac{1}{n}}\varepsilon\|v\| \quad \text{for} \quad v \in \partial\Omega_r.$$

Proof. From the definition of T_λ, we have that

$$\begin{aligned}
\|T_\lambda v\| \quad &\leq \quad \int_0^1 \Big(\lambda \int_0^s n\tau^{n-1} f(v(\tau)) d\tau\Big)^{\frac{1}{n}} ds \\
&\leq \quad \int_0^1 \lambda^{\frac{1}{n}} \Big(\int_0^s n\tau^{n-1} f^*(r) d\tau\Big)^{\frac{1}{n}} ds \\
&\leq \quad \int_0^1 \lambda^{\frac{1}{n}} \Big(\int_0^s n\tau^{n-1} \varepsilon^n r^n d\tau\Big)^{\frac{1}{n}} ds \\
&= \quad \frac{1}{2}\lambda^{\frac{1}{n}}\varepsilon\|v\|.
\end{aligned}$$

□

The following two lemmas allow us to estimate λ for which (2.4) has solutions.

Lemma 5.5. *For any $v \in \partial\Omega_r$, we have*

$$\|T_\lambda v\| \geq \frac{1}{2}(1-\alpha)^2 \lambda^{\frac{1}{n}} (m_\alpha(r))^{\frac{1}{n}}$$

Proof. Since $f(v(t)) \geq m_\alpha(r)$ for t $\in [0, 1-\alpha]$, it follows that

$$\begin{aligned}
\|T_\lambda v\| &\geq \int_0^{1-\alpha} \Big(n\lambda \int_0^s \tau^{n-1} f(v(\tau)) d\tau\Big)^{\frac{1}{n}} ds \\
&\geq \int_0^{1-\alpha} \Big(n\lambda \int_0^s \tau^{n-1} m(r) d\tau\Big)^{\frac{1}{n}} ds \\
&\geq \lambda^{\frac{1}{n}} (m_\alpha(r))^{\frac{1}{n}} \int_0^{1-\alpha} \Big(n \int_0^s \tau^{n-1} d\tau\Big)^{\frac{1}{n}} ds \\
&\geq \frac{1}{2}(1-\alpha)^2 \lambda^{\frac{1}{n}} (m_\alpha(r))^{\frac{1}{n}}.
\end{aligned}$$

□

Lemma 5.6. *For any $v \in \partial\Omega_r$, we have*

$$\|T_\lambda v\| \leq \frac{1}{2}\lambda^{\frac{1}{n}} f^*(r)^{\frac{1}{n}}.$$

Its verification is immediate and hence omitted.

Proof. of Theorem 5.1.

Part (a). Since f is not identically zero, we have $f(r_1) > 0$ for some $r_1 > 0$. Hence we can find some $\alpha \in (0, 1)$ such that $m_\alpha(r_1) > 0$. Choose

$$\lambda_1 = \frac{(2r_1)^n}{(1-\alpha)^{2n} m_\alpha(r_1)}.$$

By Lemma 5.5 we infer that

$$\|T_\lambda v\| > \|v\| \text{ for all } v \in \partial\Omega_{r_1} \text{ and } \lambda > \lambda_1.$$

Now we fix $\lambda > \lambda_1$. If $f_0 = 0$, then $f_0^* = 0$. And we can choose $0 < r_2 < r_1$ so that $f^*(r_2) \leq (\varepsilon r_2)^n$, where the constant $\varepsilon > 0$ satisfies

$$\frac{1}{2}\varepsilon\lambda^{\frac{1}{n}} < 1.$$

Thus, we have by Lemma 5.4 that

$$\|T_\lambda v\| \leq \frac{1}{2}\lambda^{\frac{1}{n}} \varepsilon\|v\| < \|v\| \quad for \quad v \in \partial\Omega_{r_2}.$$

If $f_\infty = 0$, then $f_\infty^* = 0$. And there is an $r_3 > r_1$ such that $f^*(r_3) \leq (\varepsilon r_3)^n$, where the constant $\varepsilon > 0$ satisfies

$$\frac{1}{2}\varepsilon\lambda^{\frac{1}{n}} < 1.$$

Thus, we have

$$\|T_\lambda v\| \leq \frac{1}{2}\lambda^{\frac{1}{n}} \varepsilon\|v\| < \|v\| \quad for \quad v \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0, \text{ and } i(T_\lambda, \Omega_{r_2}, K) = 1 \text{ or } i(T_\lambda, \Omega_{r_3}, K) = 1.$$

Thus, $i(T_\lambda, \Omega_{r_1} \setminus \bar{\Omega}_{r_2}, K) = -1$ or $i(T_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = 1$. Hence, T_λ has a fixed point in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ according to $f_0 = 0$ or $f_\infty = 0$, respectively. Consequently, (2.4) has a positive solution for $\lambda > \lambda_1$.

Part (b). With $\lambda_2 = \lambda_1$, the proof of (a) shows that in the current situation, 2.4 has a positive solution in either of

$$\Omega_{r_1} \setminus \bar{\Omega}_{r_2} \text{ and } \Omega_{r_3} \setminus \bar{\Omega}_{r_1}.$$

Part (c). Choose $\lambda_3 = \frac{(2r_1)^n}{f^*(r_1)}$. By Lemma 5.6 we infer that

$$\|T_\lambda v\| < \|v\| \text{ for all } v \in \partial\Omega_{r_1} \text{ and } 0 < \lambda < \lambda_3.$$

Now we fix λ. If $f_0 = \infty$, then there is $0 < r_2 < r_1$ such that $f(t) \geq (\eta t)^n$ for $0 \leq t \leq r_2$, where $\eta > 0$ is chosen so that

$$\frac{1}{2}\eta\alpha(1-\alpha)^2\lambda^{\frac{1}{n}} > 1.$$

Then $f(v(t)) \geq (\eta v(t))^n$ for all $v \in \partial\Omega_{r_2}$ and $t \in [0,1]$. Lemma 5.2 implies that

$$\|T_\lambda v\| > \|v\| \quad for \quad v \in \partial\Omega_{r_2}.$$

If $f_\infty = \infty$, there is an $M > 0$ such that $f(t) \geq (\eta t)^n$ for $t \geq M$, with the same chosen $\eta > 0$. Let $r_3 = \max\{2r_1, \frac{M}{\alpha}\}$. If $v \in \partial\Omega_{r_3}$, then

$$\min_{0\leq t\leq 1-\alpha} v(t) \geq \alpha\|v\| \geq M,$$

and hence,

$$f(v(t)) \geq (\eta v(t))^n \text{ for } t \in [0, 1-\alpha].$$

It follows from Lemma 5.2 that

$$\|T_\lambda v\| \geq \|v\| \quad for \quad v \in \partial\Omega_{r_3}.$$

Thus by Lemma 2.1 we have that

$$i(T_\lambda, \Omega_{r_1}, K) = 1, \text{ and } i(T_\lambda, \Omega_{r_2}, K) = 0 \text{ or } i(T_\lambda, \Omega_{r_3}, K) = 0,$$

and hence we obtained

$$i(T_\lambda, \Omega_{r_1} \setminus \overline{\Omega}_{r_2}, K) = 1 \text{ or } i(T_\lambda, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K) = -1.$$

Thus, T_λ has a fixed point in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ according to $f_0 = \infty$ or $f_\infty = \infty$, respectively. Consequently, (2.4) has a positive solution for all $0 < \lambda < \lambda_3$.

Part (d). Choose $\lambda_4 = \lambda_3$ and consider the same r_2 and r_3 as from (c). The proof of (c) shows that T_λ has a fixed point in either of $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ and v_2 in $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$. Consequently, (2.4) has two positive solutions.

Part (e) Since $f_0 < \infty$ and $f_\infty < \infty$, there exist positive numbers M and $r_1 < r_2$ such that

$$f(v) \leq Mv^n \text{ for } v \in [0, r_1] \cup [r_2, \infty).$$

Let

$$M_1 = \max\left\{M, \max_{r_1\leq v\leq r_2}\{\frac{f(v)}{v^n}\}\right\} > 0.$$

Thus, we have

$$f(v) \le M_1 v^n \text{ for all } v \in [0,\infty).$$

Define

$$\lambda_5 = \frac{2^n}{M_1}.$$

Assume that $v(t)$ is a positive solution of (2.4) for some $0 < \lambda < \lambda_5$. By Lemma 5.4 we have that

$$\|v\| = \|T_\lambda v\| \le \frac{1}{2}\lambda^{\frac{1}{n}} M^{\frac{1}{n}} \|v\| < \|v\|,$$

which is a contradiction.

Part (f). Since $f_0 > 0$ and $f_\infty > 0$, it follows that there exist positive numbers η, and $r_1 < r_2$ such that

$$f(v) \ge \eta v^n \text{ for all } v \in [0, r_1] \cup [r_2, \infty).$$

Let

$$\eta_1 = \min\left\{\eta, \min_{r_1 \le v \le r_2}\{\frac{f(v)}{v^n}\}\right\},$$

which is positive since we assumed, only for the consideration of (f) in the entire paper, that $f(u) > 0$ for $u > 0$. Thus, we have

$$f(v) \ge (\eta_1^{\frac{1}{n}} v)^n \text{ for all } v \in [0,\infty).$$

Let

$$\lambda_6 = \frac{2^n}{\eta_1 \alpha^n (1-\alpha)^{2n}}.$$

For some $\lambda > \lambda_6$ and assume $v(t)$ is a positive solution of (2.4). It follows from Lemma 5.2 that

$$\|v\| = \|T_\lambda v\| > \|v\|,$$

which is a contradiction. □

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