J Appl Math Comput (2012) 38:585–600
DOI 10.1007/s12190-011-0498-8

JAMC

An application of the Krasnoselskii theorem to systems of algebraic equations

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Received: 19 January 2011 / Published online: 30 June 2011

Abstract Based on the Krasnoselskii theorem, we study the existence, multiplicity and nonexistence of positive solutions of general systems of nonlinear algebraic equations under superlinearity and sublinearity conditions. Systems of nonlinear algebraic equations often arise from studies of differential and difference equations. Our results significantly extend and improve those in the literature. A number of examples and open questions are given to illustrate these results.

Keywords Krasnoselskii fixed point theorem · Nonlinear algebraic system · Positive solution · Cone

Mathematics Subject Classification (2000) 39A10 · 65H10

1 Introduction and main results

Solving algebraic equations is among the oldest problems in mathematics. Perhaps one of the most useful formulas is the classical quadratic formula

$$x_{12} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

which determines the number of solutions of the equation $Ax^2 + Bx + C = 0$. Numerous mathematical problems such as numerical solutions of differential equations, discrete boundary value problems and steady states of a complex dynamic system can

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be reduced to the study of the existence of positive solutions of systems of algebraic equations. There remains a great deal of algebraic problems, in particular, systems of algebraic equations, to be further investigated. A recent series of papers by Yang and Zhang [12], Zhang and Cheng [13], Zhang [14] and Zhang and Feng [15] studied the solutions of systems of nonlinear algebraic equations arising from systems of differential and difference equations and obtained a number of interesting results. In many nonlinear phenomena, only nonnegative solutions make sense and negative solutions may be translated into positive solutions. Such an algebraic problem can often be viewed as an eigenvalue problem. In this paper we shall study the positive solutions of the nonlinear algebraic system

$$\mathbf{x} = \lambda A\mathbf{F}(\mathbf{x}) \tag{1}$$

where $\lambda > 0$ is a parameter, $\mathbf{x} = \mathrm{col}(x_1, x_2, \ldots, x_n)$, $A = (a_{ij})$ is a $n \times n$ nonnegative matrix ($a_{ij} \geq 0$) and $\mathbf{F}(\mathbf{x}) = \mathrm{col}(f^{(1)}(\mathbf{x}), f^{(2)}(\mathbf{x}), \ldots, f^{(n)}(\mathbf{x}))$.

Zhang and Feng [15] studied (1) with a simpler form $f^{(i)}(\mathbf{x}) = f^{(i)}(x_i)$, $i = 1, \ldots, n$ and A is positive ($a_{ij} > 0$) based on the Krasnoselskii fixed point theorem. They proved the existence, multiplicity and nonexistence of positive solutions of the special form of (1) (also see [15] and references therein for a list of applications of (1)). It should be noted that two theorems in [15] need an additional condition (H2 below) to be valid (see Example 2).

In [12–14], the variational arguments are used to study the existence of solutions of the system of algebraic equations of the form

$$B\mathbf{x} = \lambda \mathbf{F}(\mathbf{x}) \tag{2}$$

where B is a $n \times n$ positive definite matrix and $f^{(i)}(\mathbf{x}) = f^{(i)}(x_i)$, $i = 1, \ldots, n$. Equation (2) can be converted into (1) by multiplying the inverse of B. Note that [14] corrects some errors in [13].

An analogous problem to the existence of positive solutions of (1) is the existence of positive solutions of systems of differential equations, which have been given considerable attention in the last few decades. This connection between algebraic equations and differential equations was observed in a frequently cited survey paper by Lions [6] where many types of bifurcation diagrams of positive solutions were discussed. One of the standard methods (e.g., [2, 8–11]) is to transform the differential equations into equivalent integral equations such as

$$x_i(t) = \lambda \int G_i(t,s) f^{(i)}(\mathbf{x}(\mathbf{s}))ds, \quad i = 1, \ldots, n. \tag{3}$$

If the integral kernels $G_i(t,s)$ take the Dirac delta function, then (1) can be heuristically viewed as special case of the general integral equations. Therefore it is not surprising to see the many methods in differential equations can be used to deal with (1). In particular, the Krasnoselskii's fixed point theorem on compression and expansion of a cone can be employed to prove the existence of positive solutions. In general, it is difficult to find exact intervals of λ for which differential equations have positive solutions. The first author posed a related question in [10]. However, for algebraic equations, it is possible to explicitly give optimal intervals of λ as shown in examples from Sect. 2. Often this phenomenon can be expressed as bifurcation diagrams such as Fig. 1. Perhaps, studying algebraic equations would help us further understand the number of positive solutions of differential equations.

We shall assume A is nonnegative ($a_{ij} \geq 0$) and every column of A has at least one positive element. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}^n_+ = \prod_{i=1}^n \mathbb{R}_+$,

$$m = \min_{i,j=1,\dots,n} a_{ij} \geq 0, \qquad M = \max_{i,j=1,\dots,n} a_{ij} > 0, \qquad \sigma = \frac{m}{nM} \geq 0.$$

For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we use the norm

$$\|\mathbf{x}\| = \sum_{i=1}^n |x_i|.$$

Denote K by the cone

$$K = \left\{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, x_i \geq \sigma \|\mathbf{x}\|, i = 1, \dots, n\right\}. \tag{4}$$

Note that $K = \mathbb{R}^n_+$ if $m = \sigma = 0$ or $n = 1$.

By *a positive solution* of the system of algebraic equations (1), we understand that a nontrivial vector $\mathbf{x} \in \mathbb{R}^n_+$ and satisfies (1). Note that some components of a positive solution of (1) may be zero if $K = \mathbb{R}^n_+$. In view of Lemma 2, all components of a positive solution of (1) are positive if $m > 0$ ($\sigma > 0$) or $n = 1$.

In order to state our results we use the notation as in [8] by the first author

$$\begin{aligned} f_0^{(i)} &= \lim_{\|\mathbf{u}\| \to 0} \frac{f^{(i)}(\mathbf{x})}{\|\mathbf{x}\|}, \qquad f_\infty^{(i)} = \lim_{\|\mathbf{x}\| \to \infty} \frac{f^{(i)}(\mathbf{x})}{\|\mathbf{x}\|}, \qquad \mathbf{x} \in K,\ i = 1, \dots, n, \\ \mathbf{F}_0 &= \sum_{i=1}^n f_0^{(i)}, \qquad \mathbf{F}_\infty = \sum_{i=1}^n f_\infty^{(i)}. \end{aligned} \tag{5}$$

It should be noted that there are differences between the cone K and that in [15]. We use the summation norm, which sometimes makes it easier to compute limits. We also use the whole first quadrant as a cone for $m = 0$. Similar cones have been proposed to study the existence of positive solutions of differential equations in several papers by the first author [2, 8–11] and other papers as well.

The assumptions for this paper are:

(H1) $A = (a_{ij})$ is an $n \times n$ nonnegative matrix ($a_{ij} \geq 0, i, j = 1, \dots, n$). Every column of A has at least one positive element. $f^{(i)} : \mathbb{R}^n_+ \to [0, \infty)$ is continuous, $i = 1, \dots, n$.

(H2) $f^{(i)}(\mathbf{x}) > 0$ for $\mathbf{x} \in K$ and $\|\mathbf{x}\| > 0$, $i = 1, \dots, n$.

Our main results for this paper are Theorems 1, 2, 3.

Theorem 1 *Assume* (H1) *holds.*

(a) *If* $\mathbf{F}_0 = 0$ *and* $\mathbf{F}_\infty = \infty$, *then for all* $\lambda > 0$ (1) *has at least a positive solution.*
(b) *If* $\mathbf{F}_0 = \infty$ *and* $\mathbf{F}_\infty = 0$, *then for all* $\lambda > 0$ (1) *has at least a positive solution.*

Theorem 2 *Assume* (H1)–(H2) *hold.*

(a) *If* $\mathbf{F}_0 = 0$ *or* $\mathbf{F}_\infty = 0$, *then there exists a* $\lambda_0 > 0$ *such that* (1) *has at least a positive solution for* $\lambda > \lambda_0$.
(b) *If* $\mathbf{F}_0 = \infty$ *or* $\mathbf{F}_\infty = \infty$, *then there exists a* $\lambda_0 > 0$ *such that* (1) *has at least a positive solution for* $0 < \lambda < \lambda_0$.
(c) *If* $\mathbf{F}_0 = \mathbf{F}_\infty = 0$, *then there exists a* $\lambda_0 > 0$ *such that* (1) *has at least two positive solutions for* $\lambda > \lambda_0$.
(d) *If* $\mathbf{F}_0 = \mathbf{F}_\infty = \infty$, *then there exists a* $\lambda_0 > 0$ *such that* (1) *has at least two positive solutions for* $0 < \lambda < \lambda_0$.
(e) *If* $\mathbf{F}_0 < \infty$ *and* $\mathbf{F}_\infty < \infty$, *then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (1) *has no positive solution.*
(f) *If* $\mathbf{F}_0 > 0$ *and* $\mathbf{F}_\infty > 0$, *then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (1) *has no positive solution.*

The results above significantly extend the corresponding ones in the literature. Not only do we extend the results in [15] in such a way that $f^{(i)}$ may depend on all x_i (Example 2), but also all components of the left side of (1) are completely independent as A is allowed to be a nonnegative matrix in this paper. While the solutions of (2) studied in [12–14] are not necessarily positive, some negative solutions may be converted into nonnegative solutions such as in the case that $f^{(i)}(\mathbf{x}) = f^{(i)}(x_i)$ is odd, which is one of the assumptions for the theorems in [13, 14]. In addition, (1) is more general than (2) in that $f^{(i)}$ in this paper may depend on all x_i and A is not necessarily invertible. Our results in this paper may improve/complement the corresponding results in [12–14]. The method used in [12–14] is based on the critical point theory and the Morse theory.

Further, by adapting the notion $\mathbf{F}_0$ and $\mathbf{F}_\infty$ from the first author [8], we provide a unified treatment for the existence of positive solutions of systems of algebraic equations and the results are better presented and easier to understand. Five examples and some related open questions are presented in Sect. 2. The proofs of Theorems 1, 2 shall be carried out in Sects. 3, 4, 5. While the general ideas for the proofs are similar to those from the first author [8] and Zhang and Feng [15], there are some substantial improvements and simplifications in this paper, in particular, in the proofs of Lemmas 2, 3, 4, 5 and 6, as the operator and cones are different.

Finally, we also give criteria for the existence of multiple positive solutions for general systems of algebraic equations (see Theorem 3). Related results on the number of positive solutions of systems of polynomials can be viewed as extensions of classic algebraic formulas to determine the positive solutions of algebraic equations as we demonstrate in examples.

2 Examples of algebraic equations and open questions

Example 1 We seek positive solutions of the polynomial equation

$$x = \lambda(a_N x^N + a_{N-1}x^{N-1} + \cdots + a_0), \quad \lambda > 0,\ a_i > 0,\ i = 1, \ldots, N. \tag{6}$$

This is a scalar equation and $f(x) = a_N x^N + a_{N-1}x^{N-1} + \cdots + a_0$. $K = [0, \infty)$ as $n = 1$. It is easy to see that

$$f_0 = \lim_{x\to 0} \frac{f(x)}{x} = \infty, \qquad f_\infty = \lim_{x\to\infty} \frac{f(x)}{x} = \infty.$$

Now according to Theorem 2, (6) has two positive solutions for sufficiently small $\lambda > 0$ and no solution for sufficiently large λ. Note that $f(x) > 0$ is a strictly convex function and $f(0) > 0$ and $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$. Interpreting the solution of $x = \lambda f(x)$ as the line $(\frac{1}{\lambda}x)$ crossing the convex curve $(f(x))$, we can see that there exists a $\lambda_0 > 0$ such that (6) has two positive solutions for $0 < \lambda < \lambda_0$; one solution for $\lambda = \lambda_0$ and no solution for $\lambda > \lambda_0$.

In fact, for $N = 2$, we are able to explicitly calculate λ_0. Indeed, let $A > 0$, $B > 0$, $C > 0$ and consider the following quadratic equations

$$x = \lambda(Ax^2 + Bx + C). \tag{7}$$

Equation (7) can be rewritten as $\lambda Ax^2 + (\lambda B - 1)x + \lambda C = 0$. From the quadratic formula we have

$$x_\lambda^\pm = \frac{-(\lambda B - 1) \pm \sqrt{(\lambda B - 1)^2 - 4\lambda^2 AC}}{2\lambda A}.$$

First, it follows that the necessary and sufficient conditions for (7) having positive solutions are

$$\lambda B - 1 < 0, \qquad (\lambda B - 1)^2 - 4\lambda^2 AC \geq 0.$$

On the other hand, $(\lambda B - 1)^2 - 4\lambda^2 AC = (\lambda B - 1 - 2\lambda\sqrt{AC})(\lambda B - 1 + 2\lambda\sqrt{AC})$. Because of $\lambda B - 1 < 0$, $(\lambda B - 1)^2 - 4\lambda^2 AC \geq 0$ if only if $\lambda B - 1 + 2\lambda\sqrt{AC} \leq 0$, which implies that

$$\lambda \leq \frac{1}{B + 2\sqrt{AC}}.$$

Let $\lambda_0 = \frac{1}{B+2\sqrt{AC}}$. For $\lambda > \lambda_0$ (7) has no positive solution; for $\lambda = \lambda_0$, (7) has one positive solution; for $\lambda < \lambda_0$, (7) has two positive solutions, $x_\lambda^- = \frac{-(\lambda B-1)-\sqrt{(\lambda B-1)^2-4\lambda^2 AC}}{2\lambda A}$ and $x_\lambda^+ = \frac{-(\lambda B-1)+\sqrt{(\lambda B-1)^2-4\lambda^2 AC}}{2\lambda A}$. Further calculation shows that

$$\lim_{\lambda\to 0} x_\lambda^+ = \infty.$$

Multiplying and dividing $-(\lambda B - 1) + \sqrt{(\lambda B - 1)^2 - 4\lambda^2 AC}$ to x_λ^-, we can obtain that

$$\lim_{\lambda\to 0} x_\lambda^- = 0.$$

This phenomenon can be summarized in Fig. 1.

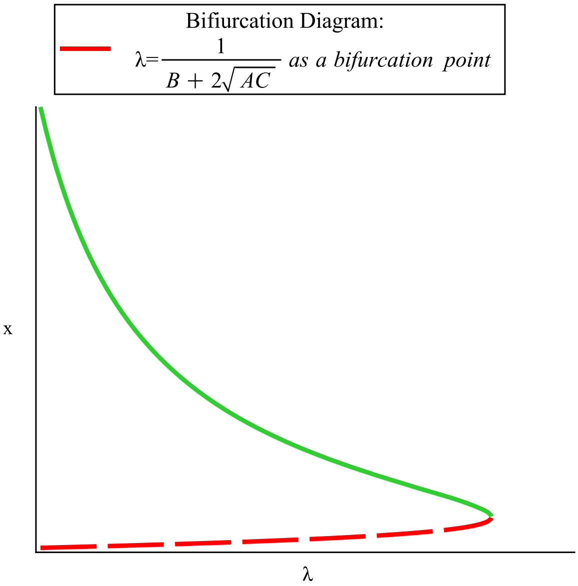

Fig. 1 For $\lambda > \frac{1}{B+2\sqrt{AC}}$, (7) has no positive solution. For $\lambda = \frac{1}{B+2\sqrt{AC}}$, (7) has one positive solution. For $\lambda < \frac{1}{B+2\sqrt{AC}}$, (7) has two positive solutions, $x_\lambda^\pm$. The *solid line* is the branch of x_λ^+ and the *dotted line* is the branch x_λ^-

An open question related to (6) is to determine the number of positive solutions for systems of higher degree polynomials in multiple variables, the maximum intervals of λ for which the systems have positive solutions and some similar bifurcation diagrams. Some coefficients of polynomials can be negative. Such a problem may be related to the finiteness of relative equilibria in the n-body problems of celestial mechanics, which can be reduced to a system of polynomial equations in multiple variables (Smale [7]).

Example 2 In [15], it is assumed that A is a positive matrix ($m > 0$) and F is of the special form $f^{(i)}(\mathbf{x}) = f^{(i)}(x_i)$. Then the existence, multiplicity and nonexistence of positive solutions of (1) were studied and its main results are analogous to Theorems 1, 2. As we indicate in Sect. 1, there are errors in Theorems 4.1 and 4.6. in [15], which can be corrected by adding a condition ($f^{(i)} > 0$ for $x_i > 0$). Theorem 4.1 in [15] states that (1) has one (two) positive solution(s) if $f_0^{(i)} = 0$ or $f_\infty^{(i)} = 0$ ($f_0^{(i)} = f_\infty^{(i)} = 0$), which are similar to Theorem 2(a, c). It is assumed in [15] that $f^{(i)}$ are only nonnegative. The result in [15] is not correct unless $f^{(i)} > 0$ for $x_i > 0$. A simple counterexample for the assertion is that $f^{(i)} \equiv 0$ for all i, then $f_0^{(i)} = f_\infty^{(i)} = 0$ for all i. But (1) has only a trivial solution. If $f^{(i)} > 0$ for $x_i > 0$, then $q(r) > 0$ in [15, p. 416, (iv)] and the proofs in [15] hold. The same issue arises in the proof of [15, Theorem 4.6] where c can be zero unless $f^{(i)} > 0$ for $x_i > 0$.

Once the condition ($f^{(i)} > 0$ for $x_i > 0$) is added to Theorems 4.1 and 4.6 in [15], its conclusions are all valid. Because of the special form of $f^{(i)}(\mathbf{x}) = f^{(i)}(x_i)$ in [15], $f_0^{(i)}$, $f_\infty^{(i)}$ in [15] are defined as $\lim_{x_i \to p} \frac{f^{(i)}(x_i)}{x_i}$, $p = 0, \infty$. Since the proofs are all carried out for $\mathbf{x} \in K$, we have

$$\|x\| \geq x_i \geq \sigma \|x\|, \quad i = 1, \ldots, n, \ \sigma > 0.$$

Thus $x_i \to 0$ or ∞ is equivalent to $\|\mathbf{x}\| \to 0$ or ∞ respectively. Therefore it is easily shown

$$\lim_{x_i \to p} \frac{f^{(i)}(x_i)}{x_i} = 0 \quad \text{implies} \quad \lim_{\|\mathbf{x}\| \to p} \frac{f^{(i)}(x_i)}{\|\mathbf{x}\|} = 0, \quad p = 0 \text{ or } \infty. \tag{8}$$

and

$$\lim_{x_i \to p} \frac{f^{(i)}(x_i)}{x_i} = \infty \quad \text{implies} \quad \lim_{\|\mathbf{x}\| \to p} \frac{f^{(i)}(x_i)}{\|\mathbf{x}\|} = \infty, \quad p = 0 \text{ or } \infty. \tag{9}$$

Therefore Theorems 1, 2 include the corresponding results in [15].

Example 3 Consider

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 x_2 \\ x_1 x_2^2 \end{pmatrix}, \quad \lambda > 0. \tag{10}$$

In this example, $f^{(1)} = x_1 x_2$, $f^{(2)} = x_1 x_2^2$, which is not covered in [15]. It is easy to see that $\sigma = \frac{1}{4}$ and

$$K = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_i \geq 0,\ x_i \geq \frac{1}{4}(x_1 + x_2), i = 1, 2 \right\}.$$

K is the region between the two lines $x_2 = 3x_1$ and $x_2 = \frac{1}{3}x_1$ in the first quadrant of the x_1, x_2 plane. For $(x_1, x_2) \in K$, we have

$$f_0^{(1)} = \lim_{x_1+x_2 \to 0} \frac{x_1 x_2}{x_1 + x_2} \leq \lim_{x_1+x_2 \to 0} \frac{x_1 x_2}{x_1} = \lim_{x_1+x_2 \to 0} x_2 \leq \lim_{x_1+x_2 \to 0} (x_1 + x_2) = 0$$

and since $x_2 > \frac{1}{4}(x_1 + x_2)$

$$f_\infty^{(1)} = \lim_{x_1+x_2 \to \infty} \frac{x_1 x_2}{x_1 + x_2} \geq \lim_{x_1+x_2 \to \infty} \frac{x_1 x_2}{4x_1} \geq \lim_{x_1+x_2 \to \infty} \frac{x_1 + x_2}{16} = \infty.$$

Thus $f_0^{(1)} = 0$ and $f_\infty^{(1)} = \infty$. In the same way, we have $f_0^{(2)} = 0$ and $f_\infty^{(2)} = \infty$. Now according to Theorem 1, (10) has positive solution for every $\lambda > 0$. In fact, by canceling x_1 from the first equation of (11), we have $x_2^2 + x_2 - \frac{1}{\lambda} = 0$ and therefore $x_2 = \frac{-1+\sqrt{1+\frac{4}{\lambda}}}{2} > 0$. Substituting this x_2 into the second equation, it follows that $x_1 = \frac{x_2}{\lambda(2x_2+x_2^2)} > 0$.

An open question is how the matrix A affects the maximum interval of λ for which (1) has positive solutions. A is not unique as we can rescale $\mathbf{F}$. Some estimates on the intervals may be given in terms of $\mathbf{F}_0$ and $\mathbf{F}_\infty$ as in the first author [10]. Note that, under the conditions of Theorem 1, (1) always has solutions.

Example 4 Consider the system of the two equations

$$\begin{cases} x_1 = \lambda(x_1 + x_2)^2 \\ x_2 = \lambda e^{x_1 + x_2} \end{cases} \quad \lambda > 0. \tag{11}$$

In this example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, K is $\mathbb{R}^2_+$, $f^{(1)} = (x_1 + x_2)^2$, $f^{(2)} = e^{x_1+x_2}$, which is not covered in [15]. For $(x_1, x_2) \in K$, we have

$$f_0^{(1)} = \lim_{x_1+x_2\to 0} \frac{(x_1 + x_2)^2}{x_1 + x_2} = 0, \qquad f_\infty^{(1)} = \lim_{x_1+x_2\to\infty} \frac{(x_1 + x_2)^2}{x_1 + x_2} = \infty$$

and

$$f_0^{(2)} = \lim_{x_1+x_2\to 0} \frac{e^{x_1+x_2}}{x_1 + x_2} = \infty, \qquad f_\infty^{(2)} = \lim_{x_1+x_2\to\infty} \frac{e^{x_1+x_2}}{x_1 + x_2} = \infty.$$

Thus $\mathbf{F}_0 = \infty$ and $\mathbf{F}_\infty = \infty$. Now according to Remark 2, (11) has two positive solutions for sufficiently small λ and no positive solution for sufficiently large λ. In fact, by adding the two equations together, we have $x_1 + x_2 = \lambda((x_1 + x_2)^2 + e^{x_1+x_2})$, which suggests that we consider this equation first,

$$x = \lambda(x^2 + e^x) \tag{12}$$

(12) is a scalar case. We can rewrite (12) as $\lambda = \frac{x}{x^2+e^x}$. Because the maximum value $\hat{M}$ of $\frac{x}{x^2+e^x}$ on $[0, \infty)$ is about 0.28. Then we see that for $\lambda > \hat{M}$, (12) has no positive solution; for $\lambda = \hat{M}$, (12) has one positive solution; for $\lambda < \hat{M}$, (12) has two positive solutions. See Fig. 2 for the graph of $\frac{x}{x^2+e^x}$. Now for $\lambda \le \hat{M}$, assume that x is the corresponding positive solutions of (12). Let $x_1 = \lambda x^2 > 0,\ x_2 = x - \lambda x^2 = \lambda e^x > 0$. It follows that $x_1 = \lambda(x_1 + x_2)^2$, $x_2 = \lambda e^{x_1+x_2}$ and therefore (x_1, x_2) is a solution of (11). In conclusion, for $\lambda > \hat{M}$, (11) has no positive solution; for $\lambda = \hat{M}$, (11) has one positive solution; for $\lambda < \hat{M}$, (11) has two positive solutions.

Example 5 Consider the system of equations

$$\begin{cases} x_1 = \lambda(x_1 + x_2)^2 \\ x_2 = \lambda(x_1 + x_2)^3 \end{cases} \quad \lambda > 0. \tag{13}$$

In this example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, K is $\mathbb{R}^2_+$ and $f^{(1)} = (x_1 + x_2)^2$, $f^2 = (x_1 + x_2)^3$, which is not covered in [15]. For $(x_1, x_2) \in K$, we have

$$f_0^{(1)} = \lim_{x_1+x_2\to 0} \frac{(x_1 + x_2)^2}{x_1 + x_2} = 0, \qquad f_\infty^{(1)} = \lim_{x_1+x_2\to\infty} \frac{(x_1 + x_2)^2}{x_1 + x_2} = \infty$$

and in the same way $f_0^{(2)} = 0$, $f_\infty^{(2)} = \infty$. Thus $\mathbf{F}_0 = 0$ and $\mathbf{F}_\infty = \infty$. Now according to Theorem 1, (13) has positive solutions for all $\lambda > 0$. In fact, by adding the two equations together, we have $x_1 + x_2 = \lambda((x_1 + x_2)^2 + (x_1 + x_2)^3)$, which suggests that we consider this equation first,

$$x = \lambda(x^2 + x^3) \tag{14}$$

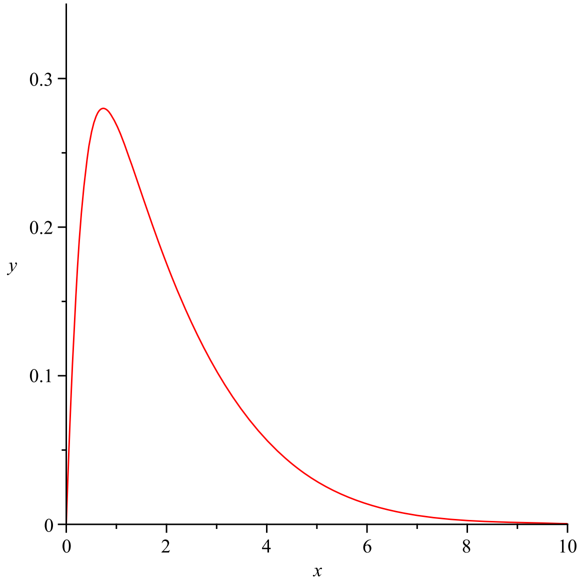

Fig. 2 The maximum value of $\frac{x}{x^2+e^x}$ is $\hat{M} \approx 0.28$. For $\lambda > \hat{M}$, (12) has no positive solution. For $\lambda = \hat{M}$, (12) has one positive solution. For $\lambda < \hat{M}$, (12) has two positive solutions

(14) is a scalar case. We can rewrite (14) as $\lambda = \frac{1}{x+x^2}$. For all $\lambda > 0$ (14) has one positive solution $x = \frac{-1+\sqrt{1+\frac{4}{\lambda}}}{2}$. Now for $\lambda > 0$, assume that $x = \frac{-1+\sqrt{1+\frac{4}{\lambda}}}{2}$ is the corresponding positive solutions of (14). Let $x_1 = \lambda x^2 > 0$, $x_2 = x - x_1 = \lambda x^3 > 0$. It follows that $x_1 = \lambda(x_1 + x_2)^2$, $x_2 = \lambda(x_1 + x_2)^3$ and therefore (x_1, x_2) is a solution of (13). In conclusion, for all $\lambda > 0$, (13) has one positive solution.

3 Preliminaries

We recall some concepts and conclusions of an operator in a cone. Let X be a Banach space and K be a closed, nonempty subset of X. K is said to be a cone if (i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geq 0$ and (ii) $u, -u \in K$ imply $u = 0$. We shall need the following well-known Krasnoselskii's fixed point theorem on cones to prove our theorems. We essentially use the version of the Krasnoselskii's theorem in conical shells, which can also be found in [1, 5]. In this paper, the Banach X is the finite dimensional space $\mathbb{R}^n$.

Lemma 1 (See [3–5]) *Let X be a Banach space and K $(\subset X)$ be a cone. Assume that Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$\mathcal{T} : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be completely continuous operator such that either

(i) $\|\mathcal{T}u\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|\mathcal{T}u\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|\mathcal{T}u\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|\mathcal{T}u\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then $\mathcal{T}$ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

In order to use the fixed point theorem, we shall let $X = \mathbb{R}^n$ and use the cone K defined in Sect. 1, and let

$$\Omega_r = \{\mathbf{x} \in K : \|\mathbf{x}\| < r\}.$$

Note that $\partial\Omega_r = \{\mathbf{x} \in K : \|\mathbf{x}\| = r\}$. Let $\mathbf{T}_\lambda : K \to X$ be a map with components $(T_\lambda^{(1)}, \ldots, T_\lambda^{(n)})$, where

$$T_\lambda^{(i)}\mathbf{x} = \lambda \sum_{j=1}^{n} a_{ij} f^{(j)}(\mathbf{x}). \tag{15}$$

Lemma 2 is necessary for applying the fixed point theorem to $\mathbf{T}_\lambda$. As a result of it, all positive solutions of (1) are in K. Thus all components of a positive solution of (1) are positive if $m > 0$.

Lemma 2 *Assume* (H1) *holds. Then* $\mathbf{T}_\lambda(\mathbb{R}^n_+) \subset K$ *and* $\mathbf{T}_\lambda : K \to K$ *is compact and continuous.*

Proof If $m = 0$, then $K = \mathbb{R}^n_+$, it is clear that $\mathbf{T}_\lambda(\mathbb{R}^n_+) \subset K$ as all components are nonnegative. For $m > 0$, let $\mathbf{x} \in \mathbb{R}^n_+$, then, for $i = 1, \ldots, n$

$$T_\lambda^{(i)}\mathbf{x} \le M\lambda \sum_{j=1,\ldots,n} f^{(j)}(\mathbf{x}); \qquad \sum_{i=1}^{n} T_\lambda^{(i)}\mathbf{x} \le M\lambda n \sum_{j=1,\ldots,n} f^{(j)}(\mathbf{x})$$

and therefore,

$$T_\lambda^{(i)}\mathbf{x} \ge m\lambda \sum_{j=1,\ldots,n} f^{(j)}(\mathbf{x}) = \sigma M\lambda n \sum_{j=1,\ldots,n} f^{(j)}(\mathbf{x}) \ge \sigma \sum_{i=1,\ldots,n} T_\lambda^{(i)}\mathbf{x} = \sigma\|T_\lambda\mathbf{x}\|.$$

Thus, $\mathbf{T}_\lambda(\mathbb{R}^n_+) \subset K$. Since K is a subset of $\mathbb{R}^n$, it is easy to verify that T_λ is compact and continuous. □

Theorem 3 gives conditions to have multiple positive solutions of (1) for a specific λ. These type of results are not included in [15]. Theorem 3 can be proved by applying the fixed point theorem on these cones repeatedly and its proof is omitted here. For the scalar case, it is just a consequence of the intermediate value theorem.

Theorem 3 *Assume* (H1) *holds and there exists a* $\lambda > 0$ *and a sequence of positive distinct numbers* r_i, $i = 1, \ldots, N+1$ *such that* $r_i < r_{i+1}$, $i = 1, \ldots, N$. *If either*

$$\|\mathbf{T}_\lambda\mathbf{x}\| > \|\mathbf{x}\| \quad \textit{for all } \|\mathbf{x}\| = r_i \textit{ and all odd } i \quad \textit{and}$$

$$\|\mathbf{T}_\lambda\mathbf{x}\| < \|\mathbf{x}\| \quad \textit{for all } \|\mathbf{x}\| = r_i \textit{ and all even } i$$

or

$$\|\mathbf{T}_\lambda\mathbf{x}\| < \|\mathbf{x}\| \quad \textit{for all } \|\mathbf{x}\| = r_i \textit{ and all odd } i \quad \textit{and}$$
$$\|\mathbf{T}_\lambda\mathbf{x}\| > \|\mathbf{x}\| \quad \textit{for all } \|\mathbf{x}\| = r_i \textit{ and all even } i$$

where $\mathbf{x} \in K$, *then* (1) *has at least* N *positive solutions for this* λ.

The following four lemmas will be repeatedly used in the proofs of the main theorems. There are some substantial improvements and simplifications in their proofs as the operator and cones are different from those in [8] and [15].

Lemma 3 *Assume* (H1) *holds. Let* $\mathbf{x} = (x_1, \ldots, x_n) \in K$ *and* $\eta > 0$. *If there exists a component* $f^{(j)}$ *of* $\mathbf{f}$ *such that*

$$f^{(j)}(\mathbf{x}) \geq \eta \|\mathbf{x}\|,$$

then

$$\|\mathbf{T}_\lambda \mathbf{x}\| \geq \lambda \Gamma \eta \|\mathbf{x}\|.$$

where $\Gamma > 0$ *is a constant.*

Proof From the fact that every column of A has a positive element, we can always find i such that $a_{ij} > 0$. From the definition of $\mathbf{T}_\lambda \mathbf{x}$ it follows that

$$\|\mathbf{T}_\lambda \mathbf{x}\| \geq T_\lambda^{(i)} \mathbf{x} \geq \lambda a_{ij} f^{(j)}(\mathbf{x}) \geq \lambda a_{ij} \eta \|\mathbf{x}\|.$$

Let $\Gamma = a_{ij} > 0$ and the lemma is proved. □

Lemma 4 *Assume* (H1) holds and let $\mathbf{x} = (x_1, \ldots, x_n) \in K$ and $\epsilon > 0$. If

$$f^{(i)}(\mathbf{x}) \leq \epsilon \|\mathbf{x}\|, \quad i = 1, \ldots, n,$$

then

$$\|\mathbf{T}_\lambda \mathbf{x}\| \leq \lambda \hat{C} \epsilon \|\mathbf{x}\|,$$

where the constant $\hat{C} = Mn^2$.

Proof From the definition of $\mathbf{T}_\lambda$, we have for $\mathbf{x} \in K$,

$$\|\mathbf{T}_\lambda \mathbf{x}\| = \sum_{i=1}^{n} T_\lambda^{(i)} \mathbf{x} \leq M\lambda \sum_{i=1}^{n} \sum_{j=1}^{n} f^{(j)}(\mathbf{x}) \leq Mn^2 \lambda \epsilon \|\mathbf{x}\| = \lambda \hat{C} \epsilon \|\mathbf{x}\|.$$

□

Lemma 5 *Assume* (H1)–(H2) *hold. If* $\mathbf{x} \in \partial\Omega_r$, $r > 0$, *then*

$$\|\mathbf{T}_\lambda \mathbf{x}\| \geq \lambda \hat{m}_r \Gamma'$$

where $\hat{m}_r = \min\{f^{(i)}(\mathbf{x}) : \mathbf{x} \in K$ *and* $\|\mathbf{x}\| = r,\ i = 1, \ldots, n\} > 0$ *and* $\Gamma' > 0$ *is a constant.*

Proof Since $f^{(i)}(\mathbf{x}) \geq \hat{m}_r$, $i = 1, \ldots, n$ and the fact that every column of A has a positive element, we can always find i such that $a_{i1} > 0$. It follows that $\|\mathbf{T}_\lambda \mathbf{x}\| \geq T_\lambda^{(i)} \mathbf{x} \geq \lambda a_{i1} f^{(1)}(\mathbf{x}) \geq \lambda a_{i1} \hat{m}_r$. Let $\Gamma' = a_{i1} > 0$ and the lemma is proved. □

Lemma 6 *Assume* (H1)–(H2) *hold. If* $\mathbf{x} \in \partial\Omega_r$, $r > 0$, *then*

$$\|\mathbf{T}_\lambda \mathbf{x}\| \le \lambda \hat{M}_r \hat{C},$$

where $\hat{M}_r = \max\{f^{(i)}(\mathbf{x}) : \mathbf{x} \in K \text{ and } \|\mathbf{x}\| \le r, i = 1, \dots, n\} > 0$ *and* $\hat{C}$ *is the positive constant defined in Lemma* 4.

Proof Since $f^{(i)}(\mathbf{x}) \le \hat{M}_r$, $i = 1, \dots, n$, a slight modification of the proof in Lemma 4 guarantees the result. □

4 Proof of Theorem 1

Now Lemmas 3, 4, 5 and 6 are used to prove Theorem 1 as in [8] (also see [15]).

Proof Part (a). $\mathbf{F}_0 = 0$ implies that $f_0^{(i)} = 0$, $i = 1, \dots, n$. Therefore, we can choose $r_1 > 0$ so that $f^{(i)}(\mathbf{x}) \le \epsilon\|\mathbf{x}\|$, $i = 1, \dots, n$ for $\mathbf{x} \in \partial\Omega_{r_1}$, where the constant $\epsilon > 0$ satisfies $\lambda\epsilon\hat{C} < 1$, and $\hat{C}$ is the positive constant defined in Lemma 4. We have by Lemma 4 that

$$\|\mathbf{T}_\lambda \mathbf{x}\| \le \lambda\epsilon\hat{C}\|\mathbf{x}\| < \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in \partial\Omega_{r_1}.$$

Now, since $\mathbf{F}_\infty = \infty$, there exists a component $f^{(i)}$ of $\mathbf{F}$ such that $f_\infty^{(i)} = \infty$. Therefore, there is an $\hat{H} > 0$ such that $f^{(i)}(\mathbf{x}) \ge \eta\|\mathbf{x}\|$ for $\mathbf{x} = (x_1, \dots, x_n) \in K$ and $\|\mathbf{x}\| \ge \hat{H}$, where $\eta > 0$ is chosen so that $\lambda\Gamma\eta > 1$. Let $r_2 = \max\{2r_1, \hat{H}\}$. If $\mathbf{x} = (x_1, \dots, x_n) \in \partial\Omega_{r_2}$, then $f^{(i)}(\mathbf{x}) \ge \eta\|\mathbf{x}\|$. It follows from Lemma 3 that

$$\|\mathbf{T}_\lambda \mathbf{x}\| \ge \lambda\Gamma\eta\|\mathbf{x}\| > \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in \partial\Omega_{r_2}.$$

Thus by Lemma 1 $\mathbf{T}_\lambda$ has a fixed point $\mathbf{x} \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$. The fixed point $\mathbf{x} \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ is the desired positive solution of (1).

Part (b). If $\mathbf{F}_0 = \infty$, there exists a component $f^{(i)}$ such that $f_0^{(i)} = \infty$. Therefore, there is an $r_1 > 0$ such that $f^{(i)}(\mathbf{x}) \ge \eta\|\mathbf{x}\|$ for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ and $\|\mathbf{x}\| \le r_1$, where $\eta > 0$ is chosen so that $\lambda\Gamma\eta > 1$. Lemma 3 implies that

$$\|\mathbf{T}_\lambda \mathbf{x}\| \ge \lambda\Gamma\eta\|\mathbf{x}\| > \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in \partial\Omega_{r_1}.$$

We now determine Ω_{r_2}. $\mathbf{F}_\infty = 0$ implies that $f_\infty^{(i)} = 0$, $i = 1, \dots, n$. Therefore there is an $r_2 > 2r_1$ such that $f^{(i)}(\mathbf{x}) \le \epsilon\|\mathbf{x}\|$, $i = 1, \dots, n$, $\mathbf{x} \in \partial\Omega_{r_2}$ where the constant $\epsilon > 0$ satisfies $\lambda\epsilon\hat{C} < 1$, and $\hat{C}$ is the positive constant defined in Lemma 4. Thus, we have by Lemma 4 that

$$\|\mathbf{T}_\lambda \mathbf{x}\| \le \lambda\epsilon\hat{C}\|\mathbf{x}\| < \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in \partial\Omega_{r_2}.$$

By Lemma 1, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (1). □

5 Proof of Theorem 2

Now Lemmas 3, 4, 5 and 6 are used to prove Theorem 1 as in [8] (also see [15]).

Proof Part (a). Fix a number $r_1 > 0$. Lemma 5 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{x}\| > \|\mathbf{x}\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{\mathrm{r}_1}, \lambda > \lambda_0.$$

If $\mathbf{F}_0 = 0$, then $f_0^{(i)} = 0$, $i = 1, \dots, n$. Therefore, we can choose $0 < r_2 < r_1$ so that $f^{(i)}(\mathbf{x}) \leq \epsilon\|\mathbf{x}\|$, $i = 1, \dots, n$, $\mathbf{x} \in \partial\Omega_{r_2}$ where the constant $\epsilon > 0$ satisfies $\lambda\epsilon\hat{C} < 1$, and $\hat{C}$ is the positive constant defined in Lemma 4. We have by Lemma 4 that

$$\|\mathbf{T}_\lambda \mathbf{x}\| \leq \lambda\epsilon\hat{C}\|\mathbf{x}\| < \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in \partial\Omega_{r_2}.$$

If $\mathbf{F}_\infty = 0$, then $f_\infty^{(i)} = 0$, $i = 1, \dots, n$. Therefore there is an $r_3 > 2r_1$ such that $f^{(i)}(\mathbf{x}) \leq \epsilon\|\mathbf{x}\|$, $i = 1, \dots, n$, $\mathbf{x} \in \partial\Omega_{r_3}$ where the constant $\epsilon > 0$ satisfies $\lambda\epsilon\hat{C} < 1$, and $\hat{C}$ is the positive constant defined in Lemma 4. Thus, we have by Lemma 4 that

$$\|\mathbf{T}_\lambda \mathbf{x}\| \leq \lambda\epsilon\hat{C}\|\mathbf{x}\| < \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in \partial\Omega_{r_3}.$$

It follows from Lemma 1 that $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ according to $\mathbf{F}_0 = 0$ or $\mathbf{F}_\infty = 0$, respectively. Consequently, (1) has a positive solution for $\lambda > \lambda_0$.

Part (b). Fix a number $r_1 > 0$. Lemma 6 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{x}\| < \|\mathbf{x}\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_1},\ 0 < \lambda < \lambda_0.$$

If $\mathbf{F}_0 = \infty$, there exists a component $f^{(i)}$ of $\mathbf{F}$ such that $f_0^{(i)} = \infty$. Therefore, there is a positive number $r_2 < r_1$ such that $f^{(i)}(\mathbf{x}) \geq \eta\|\mathbf{x}\|$ for $\mathbf{x} = (x_1, \dots, x_n) \in K$ and $\|\mathbf{x}\| \leq r_2$, where $\eta > 0$ is chosen so that $\lambda\Gamma\eta > 1$. Then $f^{(i)}(\mathbf{x}) \geq \eta\|\mathbf{x}\|$, for $\mathbf{x} = (x_1, \dots, x_n) \in \partial\Omega_{r_2}$. Lemma 3 implies that

$$\|\mathbf{T}_\lambda \mathbf{x}\| \geq \lambda\Gamma\eta\|\mathbf{x}\| > \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in \partial\Omega_{r_2}.$$

If $\mathbf{F}_\infty = \infty$, there exists a component $f^{(i)}$ of $\mathbf{F}$ such that $f_\infty^{(i)} = \infty$. Therefore, there is an $\hat{H} > 0$ such that $f^{(i)}(\mathbf{x}) \geq \eta\|\mathbf{x}\|$ for $\mathbf{x} = (x_1, \dots, x_n) \in K$ and $\|\mathbf{x}\| \geq \hat{H}$, where $\eta > 0$ is chosen so that $\lambda\Gamma\eta > 1$. Let $r_3 = \max\{2r_1, \hat{H}\}$. If $\mathbf{x} = (x_1, \dots, x_n) \in \partial\Omega_{r_3}$, then $f^{(i)}(\mathbf{x}) \geq \eta\|\mathbf{x}\|$. It follows from Lemma 3 that

$$\|\mathbf{T}_\lambda \mathbf{x}\| \geq \lambda\Gamma\eta\|\mathbf{x}\| > \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in \partial\Omega_{\mathrm{r}_3}.$$

It follows from Lemma 1 that $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ according to $\mathbf{F}_0 = \infty$ or $\mathbf{F}_\infty = \infty$, respectively. Consequently, (1) has a positive solution for $0 < \lambda < \lambda_0$.

Part (c). Fix two numbers $0 < r_3 < r_4$. Lemma 5 implies that there exists a $\lambda_0 > 0$ such that we have, for $\lambda > \lambda_0$,

$$\|\mathbf{T}_\lambda \mathbf{x}\| > \|\mathbf{x}\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_i} \ (i = 3, 4).$$

Since $\mathbf{F}_0 = 0$ and $\mathbf{F}_\infty = 0$, it follows from the proof of Theorem 2(a) that we can choose $0 < r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|\mathbf{T}_\lambda \mathbf{x}\| < \|\mathbf{x}\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_i} \ (i = 1, 2).$$

It follows from Lemma 1 that $\mathbf{T}_\lambda$ has two fixed points $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\mathbf{x}_1 \in \Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ and $\mathbf{x}_2 \in \Omega_{r_2} \setminus \bar{\Omega}_{r_4}$, which are the desired distinct positive solutions of (1) for $\lambda > \lambda_0$ satisfying

$$r_1 < \|\mathbf{x}_1\| < r_3 < r_4 < \|\mathbf{x}_2\| < r_2.$$

Part (d). Fix two numbers $0 < r_3 < r_4$. Lemma 6 implies that there exists a $\lambda_0 > 0$ such that we have, for $0 < \lambda < \lambda_0$,

$$\|\mathbf{T}_\lambda \mathbf{x}\| < \|\mathbf{x}\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_i} \ (i = 3, 4).$$

Since $\mathbf{F}_0 = \infty$ and $\mathbf{F}_\infty = \infty$, it follows from the proof of Theorem 2(b) that we can choose $0 < r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|\mathbf{T}_\lambda \mathbf{x}\| > \|\mathbf{x}\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_i} \ (i = 1, 2).$$

It follows from Lemma 1 that $\mathbf{T}_\lambda$ has two fixed points $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\mathbf{x}_1 \in \Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ and $\mathbf{x}_2 \in \Omega_{r_2} \setminus \bar{\Omega}_{r_4}$, which are the desired distinct positive solutions of (1) for $\lambda < \lambda_0$ satisfying

$$r_1 < \|\mathbf{x}_1\| < r_3 < r_4 < \|\mathbf{x}_2\| < r_2.$$

Part (e). Since $\mathbf{F}_0 < \infty$ and $\mathbf{F}_\infty < \infty$, then $f_0^{(i)} < \infty$ and $f_\infty^{(i)} < \infty$, $i = 1, \dots, n$. Therefore, for each $i = 1, \dots, n$, there exist positive numbers ϵ_1^i, ϵ_2^i, r_1^i and r_2^i such that $r_1^i < r_2^i$,

$$f^{(i)}(\mathbf{x}) \le \epsilon_1^i \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in K, \ \|\mathbf{x}\| \le r_1^i,$$

and

$$f^{(i)}(\mathbf{x}) \le \epsilon_2^i \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in K, \ \|\mathbf{x}\| \ge r_2^i.$$

Let

$$\epsilon^i = \max\left\{\epsilon_1^i, \epsilon_2^i, \max\left\{\frac{f^{(i)}(\mathbf{x})}{\|\mathbf{x}\|} : \mathbf{x} \in K, \ r_1^i \le \|\mathbf{x}\| \le r_2^i\right\}\right\} > 0$$

and $\epsilon = \max_{i=1,\dots,n}\{\epsilon^i\} > 0$. Thus, we have

$$f^{(i)}(\mathbf{x}) \le \epsilon \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in K, \ i = 1, \dots, n.$$

Assume $\mathbf{v}$ is a positive solution of (1). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0$, where $\lambda_0 = \frac{1}{\hat{C}\epsilon}$. In fact, for $0 < \lambda < \lambda_0$, since $\mathbf{T}_\lambda \mathbf{v} = \mathbf{v}$, we have

$$\|\mathbf{v}\| = \|\mathbf{T}_\lambda \mathbf{v}\| \le \lambda \hat{C}\epsilon \|\mathbf{v}\| < \|\mathbf{v}\|$$

which is a contradiction.

Part (f). Since $\mathbf{F}_0 > 0$ and $\mathbf{F}_\infty > 0$, there exist two components $f^{(i)}$ and $f^{(j)}$ of $\mathbf{F}$ such that $f_0^{(i)} > 0$ and $f_\infty^{(j)} > 0$. Therefore, there exist positive numbers η_1, η_2, r_1 and r_2 such that $r_1 < r_2$,

$$f^{(i)}(\mathbf{x}) \ge \eta_1 \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in K,\ \|\mathbf{x}\| \le r_1,$$

and

$$f^{(j)}(\mathbf{x}) \ge \eta_2 \|\mathbf{u}\| \quad \text{for } \mathbf{x} \in K,\ \|\mathbf{x}\| \ge r_2.$$

Let

$$\eta = \min\left\{\eta_1, \eta_2, \min\left\{\frac{f^{(j)}(\mathbf{x})}{\|\mathbf{x}\|} : \mathbf{x} \in K,\ r_1 \le \|\mathbf{x}\| \le r_2\right\}\right\} > 0.$$

Thus, we have

$$f^{(i)}(\mathbf{x}) \ge \eta \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in K,\ \|\mathbf{x}\| \le r_1 \tag{16}$$

and

$$f^{(j)}(\mathbf{x}) \ge \eta \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in K,\ \|\mathbf{x}\| \ge r_1. \tag{17}$$

Assume $\mathbf{v} = (v_1, \ldots, v_n)$ is a positive solution of (1). We will show that this leads to a contradiction for $\lambda > \lambda_0 = \frac{1}{\Gamma\eta}$. In fact, if $\|\mathbf{v}\| \le r_1$, (16) implies that $f^{(i)}(\mathbf{v}) \ge \eta\|\mathbf{v}\|$. On the other hand, if $\|\mathbf{v}\| > r_1$, then (17) implies that $f^{(j)}(\mathbf{v}) \ge \eta\|\mathbf{v}\|$. Since $\mathbf{T}_\lambda \mathbf{v} = \mathbf{v}$, it follows from Lemma 3 that, for $\lambda > \lambda_0$,

$$\|\mathbf{v}\| = \|\mathbf{T}_\lambda \mathbf{v}\| \ge \lambda \Gamma \eta \|\mathbf{v}\| > \|\mathbf{v}\|,$$

which is a contradiction. □

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