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Positive periodic solutions of singular systems with a parameter

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ABSTRACT

The existence and multiplicity of positive periodic solutions for second-order non-autonomous singular dynamical systems are established with superlinearity or sublinearity assumptions at infinity for an appropriately chosen parameter. Our results provide a unified treatment for the problem and significantly improve several results in the literature. The proof of our results is based on the Krasnoselskii fixed point theorem in a cone.

1. Introduction

In a recent series of papers [3,4,11,12,15,25–27], the existence and multiplicity of positive periodic solutions for the singular systems

$$\ddot{x} + a(t)x = f(t, x) + e(t) \tag{1.1}$$

and

$$-\ddot{x} + a(t)x = f(t, x) + e(t) \tag{1.2}$$

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have been studied, where $a(t), e(t) \in C(\mathbb{R}, \mathbb{R}^n)$, $f(t, x) \in C(\mathbb{R} \times (\mathbb{R}^n \setminus \{0\}), \mathbb{R}^n)$ are T-periodic in t with a singularity at $x = 0$,

$$\lim_{x \to 0} f_i(t, x) = \infty, \quad i = 1, \ldots, n.$$

(1.1) and (1.2) represent singularities of repulsive type and attractive type respectively. One closely related example of the above systems is

$$\ddot{x} + ax + \nabla_x V(t, x) = e(t) \tag{1.3}$$

with $V(t, x) = (\frac{1}{\sqrt{\sum x_i^2}})^{\alpha+1}$, $\alpha > 0$, which was studied in [18]. A positive periodic solution of the above systems is of interest because it is a non-collision periodic orbit of the singular systems. Periodic solutions of singular systems have been studied over many years, see, for example, [1–5,10–15,17,18, 20–22,24–27,30]. One of the common assumptions to guarantee the existence of is a so-called strong force assumption (corresponds to the case $\alpha \geqslant 1$ in (1.3)), see, for example, [1,13] and references therein. However, more recently, the existence of positive periodic solutions of the singular systems has been established with a weak force condition [3,4,11,12,20,21,26,27].

The variational arguments have been the most used techniques to deal with the problem, see, for example, [1,18,22–24]. More recently, the method of lower and upper solutions, the Schauder's fixed point theorem and the Krasnoselskii fixed point theorem in a cone have been employed to investigate the existence of positive periodic solutions of the systems [2–4,11,12,14,15,19,25–27]. There is a rich literature on the use of the Krasnoselskii fixed point theorem for the existence of positive solutions of boundary value problems for general second-order differential equations (refer to [8,9,28] and many other papers).

Motivated by these recent developments, we investigate the existence and multiplicity of positive periodic solutions of the singular systems by the Krasnoselskii fixed point theorem. In this paper, we are able to obtain several existence results based on the Krasnoselskii fixed point theorem by constructing a cone defined on a product space. Similar cones have been proposed to study the existence of positive solutions of boundary value problems for systems of differential equations in several papers of the author and his co-authors [6,7,29]. We also note a related cone is used to study the existence of positive periodic solutions of singular periodic systems [11,26]. It seems that the Krasnoselskii fixed point theorem on compression and expansion of cones is quite effective in dealing with the problem. In fact, by choosing appropriate cones, the singularity of the systems is essentially removed and the associated operator becomes well defined for certain ranges of functions even when e_i is negative.

This paper is organized as follows. Main results are given in Section 2. In Section 3, we define a cone and discuss several properties of the equivalent operator on the cone. In order to simplify the proof in Section 3, we establish a series of lemmas and corollaries to estimate the operator. All the corollaries are the corresponding results for e_i taking negative values. The proof of the main results is presented in Sections 4 and 5.

2. Main results

In this section, we present our main results for the existence and multiplicity of positive periodic solutions of singular systems of repulsive type (1.1). For (1.2), all the results can be proved in the same way. First, we state a condition to guarantee the positiveness of the Green's function of the following scalar problems, $i = 1, 2, \ldots, n$,

$$x_i'' + a_i(t)x_i = e_i(t) \tag{2.4}$$

with periodic boundary conditions $x_i(0) = x_i(T)$, $x_i'(0) = x_i'(T)$, where $x = (x_1, x_2, \ldots, x_n)$, and $a_1, a_2, \ldots, a_n$ and $e_1, e_2, \ldots, e_n$ are T-periodic continuous functions. Let $G_i(t, s) \in C([0, T], \mathbb{R})$ be the

Green functions associated with (2.4). Now the periodic solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ of (2.4) is given by

$$x_i(t) = \int_0^T G_i(t,s)e_i(s)\,ds.$$

When $a_i(t) = k^2$, $0 < k < \frac{\pi}{T}$, the Green function G_i takes the following form,

$$G_i(t,s) = \begin{cases} \frac{\sin k(t-s)+\sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \leqslant s \leqslant t \leqslant T, \\ \frac{\sin k(s-t)+\sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \leqslant t \leqslant s \leqslant T. \end{cases}$$

We can verify that G_i is strictly positive. In fact, let $\hat{G}(x) = \frac{\sin(kx)+\sin k(T-x)}{2k(1-\cos kT)}$, $x \in [0,T]$. It is easy to check that $\hat{G}$ is increasing on $[0, \frac{T}{2}]$ and decreasing on $[\frac{T}{2}, T]$, and $G(t,s) = \hat{G}(|t-s|)$. Thus

$$0 < \frac{\sin kT}{2k(1-\cos kT)} = \hat{G}(0) \leqslant G(t,s) \leqslant \hat{G}\left(\frac{T}{2}\right) = \frac{\sin\frac{kT}{2}}{k(1-\cos kT)} = \frac{1}{2k\sin\frac{kT}{2}}$$

for $s,t \in [0,T]$. The same estimates can also be found in [11,19,25]. For a non-constant function $a_i(t)$, there is a criterion discussed in [25,31] to guarantee the positiveness of the Green's functions. Therefore, we always assume the following assumption (A) is true for systems of repulsive type (1.1) throughout the paper.

(A) The Green function $G_i(t,s)$, associated with (2.4), is positive for all $(t,s) \in [0,T] \times [0,T]$, $i = 1,2,\dots,n$.

Under hypothesis (A), we denote

$$0 < m_i = \min_{0 \leqslant s,t \leqslant T} G_i(t,s), \qquad M_i = \max_{0 \leqslant s,t \leqslant T} G_i(t,s),$$
$$0 < \sigma_i = \frac{m_i}{M_i}, \qquad \sigma = \min_{i=1,\dots,n}\{\sigma_i\} > 0. \tag{2.5}$$

We now examine the existence and multiplicity of positive periodic solutions of the following form, for $i = 1,\dots,n$

$$\ddot{x}_i + a_i(t)x_i = \lambda g_i(t) f_i(x) + \lambda e_i(t) \tag{2.6}$$

with $\lambda > 0$ is a positive parameter. By a positive T-periodic solution, we mean a positive T-periodic function in $C^2(\mathbb{R}, \mathbb{R}^n)$ solving corresponding systems and each component is positive for all t. Let $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}_+$, and denote by $|x| = \sum_{i=1}^n |x_i|$ the usual norm of $\mathbb{R}_+^n$ for $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. We will make the following assumptions:

(H1) $f_i(x)$ is a scalar continuous function defined for $|x| > 0$, and $f_i(x) > 0$ for $|x| > 0$, $i = 1,\dots,n$.
(H2) $a_i(t)$, $g_i(t)$, $e_i(t)$ are T-periodic continuous scalar functions in $t \in \mathbb{R}$, $a_i(t), g_i(t) \geqslant 0$, $t \in [0,T]$, $\int_0^T g_i(t)\,dt > 0$, $i = 1,\dots,n$.

We state our first theorem as follows.

Theorem 2.1. *Let* (A), (H1), (H2) *hold, and* $e_i(t) \geqslant 0$ *for* $t \in [0,T]$, $i = 1,\dots,n$. *Assume that* $\lim_{x\to 0} f_i(x) = \infty$, $i = 1,\dots,n$.

(a) *If* $\lim_{|x|\to\infty} \frac{f_i(x)}{|x|} = 0$, $i = 1, \ldots, n$, *then, for all* $\lambda > 0$, (2.6) *has a positive periodic solution.*
(b) *If* $\lim_{|x|\to\infty} \frac{f_i(x)}{|x|} = \infty$ *for* $i = 1, \ldots, n$, *then, for all sufficiently small* $\lambda > 0$, (2.6) *has two positive periodic solutions.*
(c) *There exists a* $\lambda_0 > 0$ *such that* (2.6) *has a positive periodic solution for* $0 < \lambda < \lambda_0$.

When $e_i(t)$ takes negative values, we give the following theorem. We need a stronger condition on g_i.

(H3) $g_i(t) > 0$ for $t \in [0, T]$, $i = 1, \ldots, n$.

Theorem 2.2. *Let* (A), (H1), (H2), (H3) *hold. Assume that* $\lim_{x\to 0} f_i(x) = \infty$, $i = 1, \ldots, n$.

(a) *If* $\lim_{|x|\to\infty} f_i(x) = \infty$ *and* $\lim_{|x|\to\infty} \frac{f_i(x)}{|x|} = 0$, $i = 1, \ldots, n$, *then there exists* $\lambda_0 > 0$ *such that* (2.6) *has a positive periodic solution for* $\lambda > \lambda_0$.
(b) *If* $\lim_{|x|\to\infty} \frac{f_i(x)}{|x|} = \infty$ *for* $i = 1, \ldots, n$, *then, for all sufficiently small* $\lambda > 0$, (2.6) *has two positive periodic solutions.*
(c) *There exists a* $\lambda_1 > 0$ *such that* (2.6) *has a positive periodic solution for* $0 < \lambda < \lambda_1$.

Now we apply Theorems 2.1, 2.2 to the following two-dimensional singular system, which has been examined in [4,12,14].

$$\begin{cases} \ddot{x} + a_1(t)x = \lambda\left(\sqrt{x^2 + y^2}\right)^{-\alpha} + \lambda\left(\sqrt{x^2 + y^2}\right)^{\beta} + \lambda e_1(t), \\ \ddot{y} + a_2(t)y = \lambda\left(\sqrt{x^2 + y^2}\right)^{-\alpha} + \lambda\left(\sqrt{x^2 + y^2}\right)^{\beta} + \lambda e_2(t), \end{cases} \tag{2.7}$$

with $\alpha, \beta > 0$, $a_1 \geqslant 0$, $a_2 \geqslant 0$, e_1, e_2 are T-periodic continuous in t. We only need to note the following inequality

$$\sqrt{x^2 + y^2} \leqslant |x| + |y| \leqslant \sqrt{2}\sqrt{x^2 + y^2}$$

since we use the summation norm in our theorems. For nonnegative e_1, e_2, Corollary 2.3 is an application of Theorem 2.1.

Corollary 2.3. *Assume that* a_1, a_2, e_1, e_2 *are* T*-periodic continuous in* t *and that* a_1, a_2 *satisfy the assumption* (A). *Also assume that* $e_1 \geqslant 0$ *and* $e_2 \geqslant 0$ *for* $t \in [0, T]$. *Let* $\alpha > 0$, $\beta > 0$, $\lambda > 0$.

(a) *If* $0 < \beta < 1$, *then, for all* $\lambda > 0$, (2.7) *has a positive periodic solution.*
(b) *If* $\beta > 1$, *then, for all sufficiently small* $\lambda > 0$, (2.7) *has two positive periodic solutions.*
(c) *There exists a* $\lambda_0 > 0$ *such that* (2.7) *has a positive periodic solution for* $0 < \lambda < \lambda_0$.

When e_1, e_2 take negative values, we have the following corollary from Theorem 2.2.

Corollary 2.4. *Assume that* a_1, a_2, e_1, e_2 *are* T*-periodic continuous in* t, *and that* a_1, a_2 *satisfy the assumption* (A). *Let* $\alpha > 0$, $\beta > 0$ *and* $\lambda > 0$.

(a) *If* $0 < \beta < 1$, *then there exists* $\lambda_0 > 0$ *such that* (2.7) *has a positive periodic solution for* $\lambda > \lambda_0$.
(b) *If* $\beta > 1$, *then, for all sufficiently small* $\lambda > 0$, (2.7) *has two positive periodic solutions.*
(c) *There exists a* $\lambda_1 > 0$ *such that* (2.7) *has a positive periodic solution for* $0 < \lambda < \lambda_1$.

We remark that the conclusions (b) of Theorems 2.1, 2.2 are still valid if at least one component of f satisfies $\lim_{|x|\to\infty} \frac{f_i(x)}{|x|} = \infty$. In addition, analogous results are true if one considers a system that not every component is singular at zero. For simplicity, every component of $f(t,x)$ is assumed to be singular at zero in this paper. Also we comment that Theorems 2.1 and 2.2 can be extended to the following more general system

$$\ddot{x}_i + a_i(t)x_i = \lambda f_i(t,x) + \lambda e_i(t) \tag{2.8}$$

if $f_i(t,x)$ satisfies $p_i(t)h_i(x) \leqslant f_i(t,x) \leqslant q_i(t)H_i(x)$, $i = 1,\dots,n$ with appropriate conditions on p_i, h_i, q_i, H_i.

In comparison with some related results in [3,4,11,12,15,25–27], the existence and multiplicity results in this paper can be applied to any periodic continuous function e_i. Of course, our results require the parameter λ sufficiently small or large. From Corollaries 2.3 and 2.4, for $\alpha > 0$, (2.7) always has a positive periodic solution(s) if the parameter λ is appropriately chosen according to $1 > \beta > 0$ or $\beta > 1$. These results further suggest that both a strong force assumption and weak singularity contribute to the existence of a positive solution(s) as long as certain conditions are met. Also it should be pointed out that, for the non-singular case ($\alpha \leqslant 0$), several possible combinations of superlinear and sublinear assumptions at zero and infinity were considered in [19] to obtain one or two positive periodic solutions of periodic boundary value problems. Finally, we provide a unified treatment of the problem for several important cases, and the conditions of our theorems are quite easy to verify.

We have formulated our arguments in a series of lemmas and corollaries to avoid repeated arguments in the proofs of the results. All the corollaries in Section 3 are the corresponding results for e_i which may take negative values. It seems, to some extend, that the lemmas and corollaries themselves are of importance, and reveal significant properties of the singular systems. We hope that they can be used in future research.

3. Preliminary results

We recall some concepts and conclusions of an operator in a cone. Let E be a Banach space and K be a closed, nonempty subset of E. K is said to be a cone if (i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geqslant 0$ and (ii) $u, -u \in K$ imply $u = 0$. The following well-known result of the fixed point theorem is crucial in our arguments.

Lemma 3.1. *(See [16].) Let X be a Banach space and K ($\subset X$) be a cone. Assume that Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$\mathcal{T} : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be completely continuous operator such that either

(i) $\|\mathcal{T}u\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|\mathcal{T}u\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|\mathcal{T}u\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|\mathcal{T}u\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$.

Then $\mathcal{T}$ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Consider the Banach space $X = \underbrace{C[0,T] \times \cdots \times C[0,T]}_{n}$, and for $x = (x_1,\dots,x_n) \in X$, let

$$\|x\| = \sum_{i=1}^{n} \sup_{t\in[0,T]} \left|x_i(t)\right|.$$

Denote by K the cone

$$K = \left\{ x = (x_1, \dots, x_n) \in X \colon x_i(t) \geqslant 0,\ t \in [0, T],\ i = 1, \dots, n, \text{ and } \min_{0 \leqslant t \leqslant T} \sum_{i=1}^{n} x_i(t) \geqslant \sigma \|x\| \right\}$$

where σ is defined in (2.5). Also, for $r > 0$, let

$$\Omega_r = \left\{ x \in K \colon \|x\| < r \right\}.$$

Note that $\partial \Omega_r = \{x \in K \colon \|x\| = r\}$.

Let us define $\mathcal{T}_\lambda = (\mathcal{T}_\lambda^1, \dots, \mathcal{T}_\lambda^n) : K \setminus \{0\} \to X$, where $\mathcal{T}_\lambda^i$, $i = 1, \dots, n$, are

$$\mathcal{T}_\lambda^i x(t) = \lambda \int_0^T G_i(t, s)\big(g_i(s) f_i\big(x(s)\big) + e_i(s)\big)\, ds, \quad 0 \leqslant t \leqslant T. \tag{3.9}$$

When e_i is nonnegative, $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative. If e_i takes negative values, we will choose $x(s)$ so that $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative. This is possible because $\lim_{x \to 0} f_i(x) = \infty$ or $\lim_{|x| \to \infty} f_i(x) = \infty$.

Now if x is a fixed point of $\mathcal{T}_\lambda$ in $K \setminus \{0\}$, then x is a positive solution of (2.6). Also note that each component $x_i(t)$ of any nonnegative periodic solution x is strictly positive for all t because of the positiveness of the Green functions and assumptions (H1) and (H2). We now look at several properties of the operator.

Lemma 3.2. *Assume* (A), (H1), (H2) *hold and* $e_i(t) \geqslant 0$, $t \in [0, T]$, $i = 1, \dots, n$. *Then* $\mathcal{T}_\lambda(K \setminus \{0\}) \subset K$ *and* $\mathcal{T}_\lambda : K \setminus \{0\} \to K$ *is completely continuous.*

Proof. If $x \in K \setminus \{0\}$, then $\min_{t \in [0,T]} \sum_{i=1}^{n} |x_i(t)| \geqslant \sigma \|x\| > 0$, and then $\mathcal{T}_\lambda$ is defined. Now we have that, for $i = 1, \dots, n$

$$\begin{aligned}
\min_{t \in [0,T]} \sum_{i=1}^{n} \mathcal{T}_\lambda^i x(t) &\geqslant \sum_{i=1}^{n} \min_{0 \leqslant t \leqslant T} \mathcal{T}_\lambda^i x(t) \\
&\geqslant \sum_{i=1}^{n} m_i \lambda \int_0^T \big(g_i(s) f_i\big(x(s)\big) + e_i(s)\big)\, ds \\
&= \sum_{i=1}^{n} \sigma_i \lambda M_i \int_0^T \big(g_i(s) f_i\big(u(s)\big) + e_i(s)\big)\, ds \\
&\geqslant \sum_{i=1}^{n} \sigma_i \sup_{0 \leqslant t \leqslant T} \mathcal{T}_\lambda^i x(t) \\
&\geqslant \sigma \sum_{i=1}^{n} \sup_{0 \leqslant t \leqslant T} \mathcal{T}_\lambda^i x(t) = \sigma \|\mathcal{T}_\lambda x\|.
\end{aligned}$$

Thus, $\mathcal{T}_\lambda(K \setminus \{0\}) \subset K$. It is easy to verify that $\mathcal{T}_\lambda$ is completely continuous. $\square$

If e_i takes negative values, we need to choose appropriate domains so that $g_i(s) f_i(x(s)) + e_i(s)$ becomes nonnegative. The proof of $\mathcal{T}_\lambda(K \setminus \{0\}) \subset K$ and $\mathcal{T}_\lambda(K \setminus \Omega_R) \subset K$ in Corollary 3.3 is the same as in Lemma 3.2.

Corollary 3.3. *Assume* (A), (H1), (H2), (H3) *hold.*

(a) *If* $\lim_{x\to 0} f_i(x) = \infty$, $i = 1,\dots,n$, *there is a* $\delta > 0$ *such that if* $0 < r < \delta$, *then* $\mathcal{T}_\lambda$ *is defined on* $\bar{\Omega}_r \setminus \{0\}$, $\mathcal{T}_\lambda(\bar{\Omega}_r \setminus \{0\}) \subset K$ *and* $\mathcal{T}_\lambda : \bar{\Omega}_r \setminus \{0\} \to K$ *is completely continuous.*

(b) *If* $\lim_{x\to\infty} f_i(x) = \infty$, $i = 1,\dots,n$, *there is a* $\Delta > 0$ *such that if* $R > \Delta$, *then* $\mathcal{T}_\lambda$ *is defined on* $K \setminus \Omega_R$, $\mathcal{T}_\lambda(K \setminus \Omega_R) \subset K$ *and* $\mathcal{T}_\lambda : K \setminus \Omega_R \to K$ *is completely continuous.*

Proof. We split $g_i(s) f_i(x(s)) + e_i(t)$ into the two terms $\frac{1}{2} g_i(s) f_i(x(s))$ and $\frac{1}{2} g_i(s) f_i(x(s)) + e_i(t)$. The first term is always nonnegative and used to carry out the estimates of the operator in the lemmas and corollaries in this section. We will make the second term $\frac{1}{2} g_i(s) f_i(x(s)) + e_i(t)$ nonnegative by choosing appropriate domains of f_i. The choice of the even split of $g_i(s) f_i(x(s))$ here is not necessarily optimal in terms of obtaining maximal λ-intervals for the existence of periodic solutions of the systems.

Noting that $g_i(t)$ is positive on $[0,T]$, $\lim_{x\to 0} f_i(x) = \infty$, $i = 1,\dots,n$, implies that there is a $\delta > 0$ such that

$$f_i(x) \geqslant 2\frac{\max_{t\in[0,T]}\{|e_i(t)|+1\}}{\min_{t\in[0,T]}\{g_i(t)\}}, \quad i = 1,\dots,n,$$

for $x \in \mathbb{R}^n_+$, $0 < |x| \leqslant \delta$. Now for $x \in \bar{\Omega}_r \setminus \{0\}$ and $0 < r < \delta$, noting that

$$\delta > r \geqslant \sum_{i=1}^{n} \bigl|x_i(t)\bigr| \geqslant \min_{t\in[0,T]} \sum_{i=1}^{n} \bigl|x_i(t)\bigr| \geqslant \sigma\|x\| > 0, \quad t \in [0,T],$$

and therefore, we have, for $t \in [0,T]$,

$$\begin{aligned} g_i(t) f_i\bigl(x(t)\bigr) + e_i(t) &\geqslant \frac{1}{2} g_i(t) f_i\bigl(x(t)\bigr) + e_i(t) \\ &\geqslant \frac{2}{2} g_i(t)\frac{\max_{t\in[0,T]}\{|e_i(t)|+1\}}{\min_{t\in[0,T]}\{g_i(t)\}} + e_i(t) \\ &> 0. \end{aligned}$$

Thus, it is clear that $\mathcal{T}^i_\lambda x(t)$ in (3.9) is well defined and positive, and now it is easy to see that $\mathcal{T}_\lambda(\bar{\Omega}_r \setminus \{0\}) \subset K$ and $\mathcal{T}_\lambda : \bar{\Omega}_r \setminus \{0\} \to K$ is completely continuous.

On the other hand, if $\lim_{x\to\infty} f_i(x) = \infty$, $i = 1,\dots,n$, there is an $R'' > 0$ such that

$$f_i(x) \geqslant 2\frac{\max_{t\in[0,T]}\{|e_i(t)|+1\}}{\min_{t\in[0,T]}\{g_i(t)\}}, \quad i = 1,\dots,n,$$

for $x \in \mathbb{R}^n_+$, $|x| \geqslant R''$. Now let $\Delta = \frac{R''}{\sigma}$. Then for $x \in K \setminus \Omega_R$, $R > \Delta$, we have that $\min_{0\leqslant t\leqslant T} \sum_{i=1}^n x_i(t) \geqslant \sigma\|x\| \geqslant R''$, and therefore,

$$g_i(t) f_i\bigl(x(t)\bigr) + e_i(t) \geqslant \frac{1}{2} g_i(t) f_i\bigl(x(t)\bigr) + e_i(t) > 0, \quad t \in [0,T].$$

Now $\mathcal{T}^i_\lambda x(t)$ in (3.9) is well defined and positive. It is clear that $\mathcal{T}_\lambda(K \setminus \Omega_R) \subset K$ and $\mathcal{T}_\lambda : K \setminus \Omega_R \to K$ is completely continuous. □

Now let

$$\Gamma = \min_{i=1,\dots,n}\left\{\frac{1}{2}m_i\sigma\int_0^T g_i(s)\,ds\right\} > 0.$$

Lemma 3.4. *Assume* (A), (H1), (H2) *hold and* $e_i(t) \geqslant 0$, $t \in [0,T]$, $i = 1,\dots,n$. *Let* $r > 0$ *and if there exist* $\eta > 0$ *and integer* j, $1 \leqslant j \leqslant n$, *such that*

$$f_j\big(x(t)\big) \geqslant \eta\sum_{i=1}^n x_i(t) \quad \text{for } t \in [0,T],$$

for $x(t) = (x_1(t),\dots,x_n(t)) \in \partial\Omega_r$, *then the following inequality holds,*

$$\|\mathcal{T}_\lambda x\| \geqslant \lambda\Gamma\eta\|x\|.$$

Proof. From the definition of $\mathcal{T}_\lambda x$ it follows that

$$\begin{aligned}
\|\mathcal{T}_\lambda x\| &\geqslant \max_{0\leqslant t\leqslant T}\mathcal{T}_\lambda^j x(t)\\
&\geqslant \frac{1}{2}\lambda m_j\int_0^T g_j(s)f_j\big(x(s)\big)\,ds\\
&\geqslant \frac{1}{2}\lambda m_j\int_0^T g_j(s)\eta\sum_{i=1}^n x_i(s)\,ds\\
&\geqslant \lambda m_j\frac{1}{2}\sigma\int_0^T g_j(s)\,ds\,\eta\|x\|\\
&= \lambda\Gamma\eta\|x\|. \qquad \square
\end{aligned}$$

If e_i takes negative values, we need to adjust δ and Δ in Corollary 3.3 to guarantee that $g_i(s)f_i(x(s)) + e_i(s)$ is nonnegative.

Corollary 3.5. *Assume* (A), (H1), (H2), (H3) *hold.*

(a) *If* $\lim_{x\to 0} f_i(x) = \infty$, $i = 1,\dots,n$, *then Lemma* 3.4 *is true if, in addition,* $0 < r < \delta$, *where* δ *is defined in Corollary* 3.3.
(b) *If* $\lim_{|x|\to\infty} f_i(x) = \infty$, $i = 1,\dots,n$, *then Lemma* 3.4 *is true if, in addition,* $r > \Delta$, *where* Δ *is defined in Corollary* 3.3.

Proof. We split $g_i(s)f_i(x(s)) + e_i(t)$ into the two terms $\frac{1}{2}g_i(s)f_i(x(s))$ and $\frac{1}{2}g_i(s)f_i(x(s)) + e_i(t)$. By choosing δ and Δ in Corollary 3.3, $g_i(s)f_i(x(s)) + e_i(t)$ becomes nonnegative. The estimate in Corollary 3.5 can be carried out by the first term as in Lemma 3.4. $\square$

Let $\hat{f}_i(\theta) : [1,\infty) \to \mathbb{R}_+$ be the function given by

$$\hat{f}_i(\theta) = \max\big\{f_i(u)\colon\ u \in \mathbb{R}_+^n \text{ and } 1 \leqslant |u| \leqslant \theta\big\}, \quad i = 1,\dots,n.$$

It is easy to see that $\hat{f}_i(\theta)$ is a nondecreasing function on $[1,\infty)$. The following lemma is essentially the same as Lemma 2.8 in [29]. The following proof is only for completeness.

Lemma 3.6. *(See [29].) Assume* (H1) *holds. If* $\lim_{|x|\to\infty}\frac{f_i(x)}{|x|}$ *exists* (*which can be infinity*), *then* $\lim_{\theta\to\infty}\frac{\hat{f}_i(\theta)}{\theta}$ *exists and* $\lim_{\theta\to\infty}\frac{\hat{f}_i(\theta)}{\theta}=\lim_{|x|\to\infty}\frac{f_i(x)}{|x|}$.

Proof. We consider the two cases, (a) $f_i(x)$ is bounded for $|x|\geqslant 1$ and (b) $f_i(x)$ is unbounded for $|x|\geqslant 1$. For case (a), it follows that $\lim_{\theta\to\infty}\frac{\hat{f}_i(\theta)}{\theta}=\lim_{|x|\to\infty}\frac{f_i(x)}{|x|}=0$. For case (b), for any $\delta>1$, let $M^i=\hat{f}_i(\delta)$ and

$$N_\delta^i=\inf\bigl\{|x|\colon\ x\in\mathbb{R}_+^n,\ |x|\geqslant\delta,\ f_i(x)\geqslant M^i\bigr\}\geqslant\delta>1,$$

then

$$\max\bigl\{f_i(x)\colon\ 1\leqslant|x|\leqslant N_\delta^i,\ x\in\mathbb{R}_+^n\bigr\}=M^i=\max\bigl\{f_i(x)\colon\ |x|=N_\delta^i,\ x\in\mathbb{R}_+^n\bigr\}.$$

Therefore, for any $\delta>1$, there exists an $N_\delta^i\geqslant\delta$ such that

$$\hat{f}_i(\theta)=\max\bigl\{f_i(x)\colon\ N_\delta^i\leqslant|x|\leqslant\theta,\ x\in\mathbb{R}_+^n\bigr\}\quad\text{for }\theta>N_\delta^i.$$

Now, suppose that $b_i=\lim_{|x|\to\infty}\frac{f_i(x)}{|x|}<\infty$. In other words, for any $\varepsilon>0$, there is a $\delta>1$ such that

$$b_i-\varepsilon<\frac{f_i(x)}{|x|}<b_i+\varepsilon\quad\text{for }x\in\mathbb{R}_+^n,\ |x|>\delta.\tag{3.10}$$

Thus, for $\theta>N_\delta^i$, there exist $x_1,x_2\in\mathbb{R}_+^n$ such that $|x_1|=\theta$, $\theta\geqslant|x_2|\geqslant N_\delta^i$ and $f_i(x_2)=\hat{f}_i(\theta)$. Therefore,

$$\frac{f_i(x_1)}{|x_1|}\leqslant\frac{\hat{f}_i(\theta)}{\theta}=\frac{f_i(x_2)}{\theta}\leqslant\frac{f_i(x_2)}{|x_2|}.\tag{3.11}$$

(3.10) and (3.11) yield that

$$b_i-\varepsilon<\frac{\hat{f}_i(\theta)}{\theta}<b_i+\varepsilon\quad\text{for }\theta>N_\delta^i.\tag{3.12}$$

Hence $\lim_{\theta\to\infty}\frac{\hat{f}_i(\theta)}{\theta}=\lim_{x\to\infty}\frac{f_i(x)}{|x|}$. Similarly, we can show $\lim_{\theta\to\infty}\frac{\hat{f}^i(\theta)}{\theta}=\lim_{x\to\infty}\frac{f_i(x)}{|x|}$ if $\lim_{|x|\to\infty}\frac{f_i(x)}{|x|}=\infty$. $\square$

Lemma 3.7. *Assume* (A), (H1), (H2) *hold and* $e_i(t)\geqslant 0$, $t\in[0,T]$, $i=1,\dots,n$.

Let $r>\max\{\frac{1}{\sigma},2\lambda\sum_{i=1}^n M_i\int_0^T|e_i(s)|\,ds\}$ *and if there exists an* $\varepsilon>0$ *such that*

$$\hat{f}_i(r)\leqslant\varepsilon r,\quad i=1,\dots,n,$$

then

$$\|\mathcal{T}_\lambda x\|\leqslant\lambda\hat{C}\varepsilon\|x\|+\frac{1}{2}\|x\|\quad\textit{for }x\in\partial\Omega_r,$$

where the constant $\hat{C}=\sum_{i=1}^n M_i\int_0^T g_i(s)\,ds$.

Proof. From the definition of $\mathcal{T}_\lambda$, we have for $x \in \partial\Omega_r$,

$$\begin{aligned}
\|\mathcal{T}_\lambda x\| &= \sum_{i=1}^{n} \max_{0\leqslant t\leqslant T} \mathcal{T}_\lambda^i x(t) \\
&\leqslant \sum_{i=1}^{n} \lambda M_i \int_0^T g_i(s) f_i\big(x(s)\big)\, ds + \lambda \sum_{i=1}^{n} M_i \int_0^T \big|e_i(s)\big|\, ds \\
&\leqslant \sum_{i=1}^{n} \lambda M_i \int_0^T g_i(s) \hat{f}_i(r)\, ds + \frac{r}{2} \\
&\leqslant \sum_{i=1}^{n} \lambda M_i \int_0^T g_i(s)\, ds r\varepsilon + \frac{r}{2} \\
&= \lambda \hat{C}\varepsilon\|x\| + \frac{1}{2}\|x\|. \qquad \square
\end{aligned}$$

If e_i takes negative values, we need to restrict the domain of $\mathcal{T}_\lambda$ to guarantee that $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative.

Corollary 3.8. *Assume* (A), (H1), (H2), (H3) *hold. If* $\lim_{x\to\infty} f_i(x) = \infty$, $i = 1, \ldots, n$, *Lemma* 3.7 *is true if, in addition,* $r > \Delta$, *where* Δ *is defined in Corollary* 3.3.

Proof. If we choose Δ defined in Corollary 3.3, then $\mathcal{T}_\lambda$ is well defined and $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative, and Corollary 3.8 can be shown in the same way as Lemma 3.7. $\square$

The conclusions of Lemmas 3.4 and 3.7 are based on the inequality assumptions between $f(x)$ and x. If these assumptions are not necessarily true, we will have the following results.

Lemma 3.9. *Assume* (A), (H1), (H2) *hold and* $e_i(t) \geqslant 0$, $t \in [0, T]$, $i = 1, \ldots, n$. *Let* $r > 0$. *Then*

$$\|\mathcal{T}_\lambda x\| \geqslant \lambda \sum_{i=1}^{n} \frac{m_i \hat{m}_r}{2} \int_0^T g_i(s)\, ds,$$

for all $x \in \partial\Omega_r$, *where* $\hat{m}_r = \min\{f_i(x)\colon\ x \in \mathbb{R}_+^n$ *and* $\sigma r \leqslant |x| \leqslant r,\ i = 1, \ldots, n\} > 0$.

Proof. If $x(t) \in \partial\Omega_r$, then $\sigma r \leqslant |x(t)| = \sum_{i=1}^n |x_i(t)| \leqslant r$, for $t \in [0, T]$. Therefore $f_i(x(t)) \geqslant \hat{m}_r$ for $t \in [0, T]$, $i = 1, \ldots, n$. By the definition of $\mathcal{T}_\lambda$, we have

$$\begin{aligned}
\|\mathcal{T}_\lambda x\| &= \sum_{i=1}^{n} \max_{0\leqslant t\leqslant T} \mathcal{T}_\lambda^i x(t) \\
&\geqslant \sum_{i=1}^{n} \frac{1}{2}\lambda m_i \int_0^T g_i(s) f_i\big(x(s)\big)\, ds \\
&\geqslant \lambda \sum_{i=1}^{n} \frac{m_i \hat{m}_r}{2} \int_0^T g_i(s)\, ds. \qquad \square
\end{aligned}$$

Now we consider the case that e_i may take negative values. We need to restrict the domain of $\mathcal{T}_\lambda$ to guarantee that $g_i(s)f_i(x(s))+e_i(s)$ is nonnegative. $\frac{1}{2}g_i(s)f_i(x(s))$ is used to carry out the estimates in Lemma 3.9.

Corollary 3.10. *Assume* (A), (H1), (H2), (H3) *hold.*

(a) *If* $\lim_{x\to 0} f_i(x)=\infty$, $i=1,\dots,n$, *Lemma* 3.9 *is true if, in addition,* $0<r<\delta$, *where* $\delta>0$ *is defined in Corollary* 3.3.
(b) *If* $\lim_{|x|\to\infty} f_i(x)=\infty$, $i=1,\dots,n$, *Lemma* 3.9 *is true if, in addition,* $r>\Delta$, *where* $\Delta>0$ *is defined in Corollary* 3.3.

Proof. By selecting δ and Δ defined in Corollary 3.3, $\mathcal{T}_\lambda$ is well defined and $g_i(s)f_i(x(s))+e_i(s)$ is nonnegative, and then Corollary 3.10 can be shown as Lemma 3.9. $\square$

Lemma 3.11. *Assume* (A), (H1), (H2) *hold and* $e_i(t)\geqslant 0$, $t\in[0,T]$, $i=1,\dots,n$. *Let* $r>0$. *Then*

$$\|\mathcal{T}_\lambda x\|\leqslant \lambda\left(\sum_{i=1}^n M_i\int_0^T g_i(s)\hat{M}_r\,ds+\sum_{i=1}^n M_i\int_0^T \left|e_i(s)\right|ds\right),$$

for all $x\in\partial\Omega_r$, *where* $\hat{M}_r=\max\{f_i(u)\colon\ u\in\mathbb{R}_+^n$ *and* $\sigma r\leqslant|u|\leqslant r,\ i=1,\dots,n\}>0$.

Proof. If $x\in\partial\Omega_r$, then $\sigma r\leqslant|x(t)|\leqslant r$, $t\in[0,T]$. Therefore $f_i(x(t))\leqslant\hat{M}_r$ for $t\in[0,T]$, $i=1,\dots,n$. Thus we have that

$$\begin{aligned}
\|\mathcal{T}_\lambda x\| &= \sum_{i=1}^n \max_{0\leqslant t\leqslant T}\mathcal{T}_\lambda^i x(t)\\
&\leqslant \sum_{i=1}^n \lambda M_i\int_0^T g_i(s)f_i\bigl(x(s)\bigr)\,ds+\lambda\sum_{i=1}^n M_i\int_0^T\left|e_i(s)\right|ds\\
&\leqslant \sum_{i=1}^n \lambda M_i\int_0^T g_i(s)f_i\bigl(x(s)\bigr)\,ds+\lambda\sum_{i=1}^n M_i\int_0^T\left|e_i(s)\right|ds\\
&\leqslant \sum_{i=1}^n \lambda M_i\int_0^T g_i(s)\hat{M}_r\,ds+\lambda\sum_{i=1}^n M_i\int_0^T\left|e_i(s)\right|ds\\
&\leqslant \lambda\left(\sum_{i=1}^n M_i\int_0^T g_i(s)\hat{M}_r\,ds+\sum_{i=1}^n M_i\int_0^T\left|e_i(s)\right|ds\right). \qquad\square
\end{aligned}$$

Again, if e_i takes negative values, we need to restrict r and R to guarantee $g_i(s)f_i(x(s))+e_i(s)$ is nonnegative.

Corollary 3.12. *Assume* (A), (H1), (H2), (H3) *hold.*

(a) *If* $\lim_{x\to 0} f_i(x)=\infty$, $i=1,\dots,n$, *Lemma* 3.11 *is true if, in addition,* $0<r<\delta$, *where* $\delta>0$ *is defined in Corollary* 3.3.

(b) *If* $\lim_{x\to\infty} f_i(x) = \infty$, $i = 1, \dots, n$, *Lemma* 3.11 *is true if, in addition,* $r > \Delta$, *where* $\Delta > 0$ *is defined in Corollary* 3.3.

Proof. By selecting δ and Δ defined in Corollary 3.3, $\mathcal{T}_\lambda$ is well defined and $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative, and then the corollary can be shown exactly as Lemma 3.11. □

4. Proof of Theorem 2.1

Proof of Theorem 2.1. Part (a). Since $e_i(t) \geqslant 0$, $\mathcal{T}_\lambda$ is defined on $K \setminus \{0\}$ and $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative. Noting $\lim_{|x|\to\infty} \frac{f_i(x)}{|x|} = 0$, $i = 1, \dots, n$, it follows from Lemma 3.6 that $\lim_{\theta\to\infty} \frac{\hat{f}_i(\theta)}{\theta} = 0$, $i = 1, \dots, n$. Therefore, we can choose $r_1 > \max\{\frac{1}{\sigma}, 2\lambda \sum_{i=1}^n M_i \int_0^T |e_i(s)|\, ds\}$ so that $\hat{f}_i(r_1) \leqslant \varepsilon r_1$, $i = 1, \dots, n$, where the constant $\varepsilon > 0$ satisfies

$$\lambda \hat{C} \varepsilon < \frac{1}{2},$$

and $\hat{C}$ is the positive constant defined in Lemma 3.7. We have by Lemma 3.7 that

$$\|\mathcal{T}_\lambda x\| \leqslant \left(\lambda \hat{C} \varepsilon + \frac{1}{2}\right) \|x\| < \|x\| \quad \text{for } x \in \partial \Omega_{r_1}.$$

On the other hand, by the condition $\lim_{x\to 0} f_i(x) = \infty$, there is a positive number $r_2 < r_1$ such that

$$f_i(x) \geqslant \eta |x|, \quad i = 1, \dots, n,$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n \setminus \{0\}$ and $|x| \leqslant r_2$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

It is easy to see that, for $x = (x_1, \dots, x_n) \in \partial \Omega_{r_2}$, $t \in [0, T]$,

$$f_i\big(x(t)\big) \geqslant \eta \sum_{i=1}^n x_i(t).$$

Lemma 3.4 implies that

$$\|\mathcal{T}_\lambda x\| \geqslant \lambda \Gamma \eta \|x\| > \|x\| \quad \text{for } x \in \partial \Omega_{r_2}.$$

By Lemma 3.1, $\mathcal{T}_\lambda$ has a fixed point $x \in \bar{\Omega}_{r_1} \setminus \Omega_{r_2}$. The fixed point $x \in \bar{\Omega}_{r_1} \setminus \Omega_{r_2}$ is the desired positive periodic solution of (2.6).

Part (b). Again since $e_i(t) \geqslant 0$, $\mathcal{T}_\lambda$ is defined on $K \setminus \{0\}$ and $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative. Fix two numbers $0 < r_3 < r_4$, there exists a $\lambda_0 > 0$ such that

$$\lambda_0 < \frac{r_3}{\sum_{i=1}^n M_i \int_0^T g_i(s) \hat{M}_{r_3}\, ds + \sum_{i=1}^n M_i \int_0^T |e_i(s)|\, ds},$$

and

$$\lambda_0 < \frac{r_4}{\sum_{i=1}^n M_i \int_0^T g_i(s) \hat{M}_{r_4}\, ds + \sum_{i=1}^n M_i \int_0^T |e_i(s)|\, ds},$$

where $\hat{M}_{r_3}$ and $\hat{M}_{r_4}$ are defined in Lemma 3.11. Thus, Lemma 3.11 implies that, for $0 < \lambda < \lambda_0$,

$$\|\mathcal{T}_\lambda x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_j}\ (j = 3, 4).$$

On the other hand, in view of the assumptions $\lim_{x\to\infty} \frac{f_i(x)}{|x|} = \infty$ and $\lim_{x\to 0} f_i(x) = \infty$, there are positive numbers $0 < r_2 < r_3 < r_4 < r_1'$ such that

$$f_i(x) \geqslant \eta|x|$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ and $0 < |x| \leqslant r_2$ or $|x| \geqslant r_1'$ where $\eta > 0$ is chosen so that

$$\lambda\Gamma\eta > 1.$$

Thus if $x = (x_1, \ldots, x_n) \in \partial\Omega_{r_2}$, then

$$f_i\big(x(t)\big) \geqslant \eta \sum_{i=1}^{n} x_i(t), \quad t \in [0, T].$$

Let $r_1 = \max\{2r_4, \frac{1}{\sigma} r_1'\}$. If $x = (x_1, \ldots, x_n) \in \partial\Omega_{r_1}$, then

$$\min_{0\leqslant t\leqslant T} \sum_{i=1}^{n} x_i(t) \geqslant \sigma\|x\| = \sigma r_1 \geqslant r_1',$$

which implies that

$$f_i\big(x(t)\big) \geqslant \eta \sum_{i=1}^{n} x_i(t) \quad \text{for } t \in [0, T].$$

Thus Lemma 3.4 implies that

$$\|\mathcal{T}_\lambda x\| \geqslant \lambda\Gamma\eta\|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_1},$$

and

$$\|\mathcal{T}_\lambda x\| \geqslant \lambda\Gamma\eta\|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_2}.$$

It follows from Lemma 3.1, that $\mathcal{T}_\lambda$ has two fixed points $x_1(t)$ and $x_2(t)$ such that $x_1(t) \in \bar{\Omega}_{r_3} \setminus \Omega_{r_2}$ and $x_2(t) \in \bar{\Omega}_{r_1} \setminus \Omega_{r_4}$, which are the desired distinct positive periodic solutions of (2.6) for $\lambda < \lambda_0$ satisfying

$$r_1 < \|x_1\| < r_3 < r_4 < \|x_2\| < r_2.$$

Part (c). First we note that $\mathcal{T}_\lambda$ is defined on $K \setminus \{0\}$ and $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative since $e_i(t) \geqslant 0$. Fix a number $r_3 > 0$. Lemma 3.11 implies that there exists a $\lambda_0 > 0$ such that we have, for $0 < \lambda < \lambda_0$,

$$\|\mathcal{T}_\lambda x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_3}.$$

On the other hand, in view of the assumption $\lim_{x\to 0} f_i(x) = \infty$, there is a positive number $0 < r_2 < r_3$ such that

$$f_i(x) \geqslant \eta|x|$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$ and $0 < |x| \leqslant r_2$ where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Thus if $x = (x_1, \dots, x_n) \in \partial \Omega_{r_2}$, then

$$f_i\big(x(t)\big) \geqslant \eta \sum_{i=1}^{n} x_i(t), \quad t \in [0, T].$$

Thus Lemma 3.4 implies that

$$\|\mathcal{T}_\lambda x\| \geqslant \lambda \Gamma \eta \|x\| > \|x\| \quad \text{for } x \in \partial \Omega_{r_2}.$$

Lemma 3.1 implies that $\mathcal{T}_\lambda$ has a fixed point $x \in \bar{\Omega}_{r_3} \setminus \Omega_{r_2}$. The fixed point $x \in \bar{\Omega}_{r_3} \setminus \Omega_{r_2}$ is the desired positive periodic solution of (2.6). $\square$

5. Proof of Theorem 2.2

Proof of Theorem 2.2. Part (a). Since $\lim_{|x| \to \infty} f_i(x) = \infty$, $i = 1, \dots, n$, by Corollary 3.3, there is a $\Delta > 0$ such that if $R > \Delta$, then $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative and $\mathcal{T}_\lambda : K \setminus \Omega_R \to K$ is defined. Now for a fixed number $r_1 > \Delta$, Corollary 3.10 implies that there exists a $\lambda_0 > 0$ such that, for $\lambda > \lambda_0$,

$$\|\mathcal{T}_\lambda x\| > \|x\| \quad \text{for } x \in \partial \Omega_{r_1}.$$

On the other hand, since $\lim_{|x| \to \infty} \frac{f_i(x)}{|x|} = 0$, $i = 1, \dots, n$, it follows from Lemma 3.6 that $\lim_{\theta \to \infty} \frac{\hat{f}_i(\theta)}{\theta} = 0$, $i = 1, \dots, n$. Therefore, we can choose

$$r_2 > \max\left\{2r_1, \frac{1}{\sigma}, 2\lambda \sum_{i=1}^{n} M_i \int_0^T \big|e_i(s)\big|\, ds\right\} > \Delta$$

so that $\hat{f}_i(r_2) \leqslant \varepsilon r_2$, $i = 1, \dots, n$, where the constant $\varepsilon > 0$ satisfies

$$\lambda \hat{C} \varepsilon < \frac{1}{2}.$$

We have, by Corollary 3.8, that

$$\|\mathcal{T}_\lambda x\| \leqslant \left(\lambda \hat{C} \varepsilon + \frac{1}{2}\right) \|x\| < \|x\| \quad \text{for } x \in \partial \Omega_{r_2}.$$

By Lemma 3.1, $\mathcal{T}_\lambda$ has a fixed point $x \in \bar{\Omega}_{r_2} \setminus \Omega_{r_1}$. The fixed point $x \in \bar{\Omega}_{r_2} \setminus \Omega_{r_1}$ is the desired positive periodic solution of (2.6).

Part (b). First, since $\lim_{x \to 0} f_i(x) = \infty$, $i = 1, \dots, n$, by Corollary 3.3, there is $\delta > 0$ such that if $0 < r < \delta$, $\mathcal{T}_\lambda$ is defined on $\bar{\Omega}_r \setminus \{0\}$ and $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative. Furthermore, $\mathcal{T}_\lambda(\bar{\Omega}_r \setminus \{0\}) \subset K$. Now for a fixed number $r_1 < \delta$, Corollary 3.12 implies that there exists a $\lambda_1 > 0$ such that we have, for $\lambda < \lambda_1$,

$$\|\mathcal{T}_\lambda x\| < \|x\| \quad \text{for } x \in \partial \Omega_{r_1}.$$

In view of the assumption $\lim_{x\to 0} f_i(x) = \infty$, there is a positive number $0 < r_3 < r_1$ such that

$$f_i(x) \geqslant \eta |x|$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$ and $0 < |x| \leqslant r_3$ where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Thus if $x = (x_1, \dots, x_n) \in \partial\Omega_{r_3}$, then

$$f_i\big(x(t)\big) \geqslant \eta \sum_{i=1}^{n} x_i(t), \quad t \in [0, T].$$

Thus Corollary 3.5 implies that

$$\|\mathcal{T}_\lambda x\| \geqslant \lambda \Gamma \eta \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_3}.$$

It follows from Lemma 3.1, $\mathcal{T}_\lambda$ has a fixed point $x_1(t)$ such that $x_1(t) \in \bar{\Omega}_{r_1} \setminus \Omega_{r_3}$ which is a positive periodic solutions of (2.6) for $\lambda < \lambda_1$ satisfying

$$r_3 < \|x_1\| < r_1.$$

On the other hand, since $\lim_{|x|\to\infty} \frac{f_i(x)}{|x|} = \infty$, $i = 1, \dots, n$, by Corollary 3.3, there is $\Delta > 0$ such that if $R > \Delta$, $\mathcal{T}_\lambda$ is defined on $K \setminus \Omega_R$ and $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative. Furthermore, $\mathcal{T}_\lambda(K \setminus \Omega_R) \subset K$. For a fixed number $r_2 > \max\{\Delta, r_1\}$, and Corollary 3.12 implies that there exists a $0 < \lambda_0 < \lambda_1$ such that we have, for $\lambda < \lambda_0$,

$$\|\mathcal{T}_\lambda x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_2}.$$

Since $\lim_{|x|\to\infty} \frac{f_i(x)}{|x|} = \infty$, $i = 1, \dots, n$, there is a positive number r' such that

$$f_i(x) \geqslant \eta |x|$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$ and $|x| \geqslant r'$ where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Let $r_4 = \max\{2r_2, \frac{1}{\sigma} r'\} > \Delta$. If $x = (x_1, \dots, x_n) \in \partial\Omega_{r_4}$, then

$$\min_{0\leqslant t\leqslant T} \sum_{i=1}^{n} x_i(t) \geqslant \sigma \|x\| = \sigma r_4 \geqslant r',$$

which implies that

$$f_i\big(x(t)\big) \geqslant \eta \sum_{i=1}^{n} x_i(t) \quad \text{for } t \in [0, T].$$

Again Corollary 3.5 implies that

$$\|\mathcal{T}_\lambda x\| \geqslant \lambda \Gamma \eta \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_4}.$$

It follows from Lemma 3.1, $\mathcal{T}_\lambda$ has a fixed point $x_2(t) \in \bar{\Omega}_{r_4} \setminus \Omega_{r_2}$, which is a positive periodic solutions of (2.6) for $\lambda < \lambda_0$ satisfying

$$r_2 < \|x_2\| < r_4.$$

Noting that

$$r_3 < \|x_1\| < r_1 < r_2 < \|x_2\| < r_4,$$

we can conclude that x_1 and x_2 are the desired distinct positive periodic solutions of (2.6) for $\lambda < \lambda_0$.

Part (c). Since $\lim_{x \to 0} f_i(x) = \infty$, $i = 1, \dots, n$, by Corollary 3.3, there is a $\delta > 0$ such that if $0 < r < \delta$, then $\mathcal{T}_\lambda$ is defined and $g_i(s) f_i(x(s)) + e_i(s)$ is nonnegative. Now for a fixed number $r_1 < \delta$, Corollary 3.12 implies that there exists a $\lambda_1 > 0$ such that we have, for $\lambda < \lambda_1$,

$$\|\mathcal{T}_\lambda x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_1}.$$

On the other hand, in view of the assumption $\lim_{x \to 0} f_i(x) = \infty$, there is a positive number $0 < r_2 < r_1 < \delta$ such that

$$f_i(x) \geqslant \eta |x|$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ and $0 < |x| \leqslant r_2$ where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Thus if $x = (x_1, \dots, x_n) \in \partial\Omega_{r_2}$, then

$$f_i\big(x(t)\big) \geqslant \eta \sum_{i=1}^{n} x_i(t), \quad t \in [0, T].$$

Thus Corollary 3.5 implies that

$$\|\mathcal{T}_\lambda x\| \geqslant \lambda \Gamma \eta \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_2}.$$

Lemma 3.1 implies that $\mathcal{T}_\lambda$ has a fixed point $x_1 \in \bar{\Omega}_{r_1} \setminus \Omega_{r_2}$. The fixed point $x_1 \in \bar{\Omega}_{r_1} \setminus \Omega_{r_2}$ is the desired positive periodic solution of (2.6). □

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