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On the existence of traveling waves for delayed reaction–diffusion equations ☆

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ABSTRACT

We study the existence of traveling wave solutions for reaction–diffusion equations with nonlocal delay, where reaction terms are not necessarily monotone. The existence of traveling wave solutions for reaction–diffusion equations with nonlocal delays is obtained by combining upper and lower solutions for associated integral equations and the Schauder fixed point theorem. The smoothness of upper and lower solutions is not required in this paper.

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1. Introduction

There have been extensive studies in traveling wave solutions for reaction–diffusion equations without delay in the literature, see, e.g., Murray [12]. Consider the following reaction–diffusion equation

$$w_t(t,x) = dw_{xx}(t,x) + h\big(w(t,x)\big), \quad x \in \mathbb{R},\ t \geqslant 0, \tag{1.1}$$

where h satisfies $h(0) = h(K) = 0$, $K > 0$, and $0 < h(w) \leqslant h'(0)w$, $w \in (0, K)$. A traveling wave solution of (1.1) is a special translation invariant solution of the form $w(t,x) = u(x + ct)$, where $u \in C^2(\mathbb{R}, \mathbb{R})$ is the profile of the wave that propagates through the one-dimensional spatial domain

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at a constant velocity $c > 0$. Substituting $w(t,x) = u(x+ct)$ into (1.1) and letting $\xi = x+ct$, we obtain the associated ordinary differential equation

$$du''(\xi) - cu'(\xi) + h\big(u(\xi)\big) = 0, \quad \xi \in \mathbb{R}. \tag{1.2}$$

It has been shown that there is a minimal wave speed $c_{min} = 2\sqrt{dh'(0)}$ such that for every $c > c_{min}$, there exists an increasing traveling wave solution of (1.1) with the form $w(t,x) = u(x+ct)$ and $u(-\infty) = 0$, $u(\infty) = K$.

Reaction–diffusion equations with delays often arise in biology and other disciplines. Schaaf [13] systematically studied scalar reaction–diffusion equations with a discrete delay. Wu and Zou [19], Ma [10,11], and Wang, Li and Ruan [17] and others obtained the existence of traveling wave solutions by constructing lower solutions and upper solutions of the associated ordinary differential equation and applying monotone iteration techniques or fixed point theorems. Al-Omari and Gourley [1], Gourley [6] studied the traveling wave solutions of an age-structured reaction–diffusion model with nonlocal delay and a nonlocal Fisher equation. Related results can also be found in [5,21].

In a recent paper [2], Boumenir and Nguyen revisited the existence of traveling wave solutions of reaction–diffusion equations with delays by the monotone iteration method. They pointed out upper and lower solutions of the associated ordinary differential equations are required to be smooth functions due to a failure of the Perron Theorem for weak solutions. A counterexample is given in [2] to explain the pitfalls of non-smooth upper solutions. However, as pointed in [2] it is often more difficult to construct smooth upper and lower solutions. More recently, Wu and Zou [20] addressed the problem by adding extra conditions on the upper and lower solutions at these points where smoothness is not satisfied. A related result can also be found in [22].

In this paper, we present a remedy to the problem due to the non-smooth upper and lower solutions. Instead of verifying upper and lower solutions through the associated ordinary differential equation, we carefully analyze and calculate the associated integrals and are able to verify upper and lower solutions through the associated integral equations. Smoothness is not required for the upper and lower solutions for the associated integral equations (see Definition 4.1). Identities between parameters are established to simplify the proof. The monotone iteration technique combined with upper and lower solutions has also been used to construct wave fronts for integral equations. See Diekmann [3], Weinberger [18], Thieme and Zhao [16], and more recently, Hsu and Zhao [8]. However, as remarked by Wu and Zou [19, Remark 5.2.9], the associated integral equations in this paper are derived from reaction–diffusion equations with delays and more complicated than those integral equations in [3,8,16,18].

One of the common assumptions to guarantee the existence of traveling wave solutions is quasi-monotonicity assumptions on reaction terms. In general, reaction–diffusion equations with delay are not necessarily monotone and even may not satisfy quasi-monotonicity assumptions. For equations without quasi-monotonicity assumptions, Ma [11] obtained the existence of traveling wave solutions of (1.7) by the Schauder's fixed point theorem. Hsu and Zhao [8] also established the existence of traveling waves for a class of nonmonotone discrete-time integrodifference equation models by the Schauder's fixed point theorem. Other related results for nonmonotone equations can also be found in [4,9,15].

In this paper, we consider a more general reaction–diffusion equation with nonlocal delays (1.3) where the term f may not be monotone or quasi-monotone:

$$w_t(t,x) = dw_{xx}(t,x) + g\left(w, \int_{\mathbb{R}} f\big(w(t-r,\tau)\big) J(x-\tau)\, d\tau\right), \tag{1.3}$$

where g, f, J satisfy (H1)–(H4). (1.3) includes several important reaction–diffusion systems from the literature. When $g(x,y) = -\alpha_1 x + \alpha_2 y$, $\alpha_1, \alpha_2 > 0$, and $J(x) = \delta(x)$, the Dirac delta function, (1.3) reduces the equation

$$w_t(t,x) = dw_{xx}(t,x) - \alpha_1 w(t,x) + \alpha_2 f\big(w(t-r,x)\big). \tag{1.4}$$

(1.3) also includes the nonlocal reaction–diffusion equation by So, Wu and Zou [14]

$$w_t(t,x) = dw_{xx}(t,x) - \alpha_1 w(t,x) + \alpha_2 \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\alpha_3}} e^{-\frac{(x-\tau)^2}{4\alpha_3}} f\big(w(t-r,\tau)\big)\,d\tau, \tag{1.5}$$

where $\alpha_3 > 0$. Clearly, (1.3) also includes the nonlocal reaction–diffusion equation by Gourley and Kuang [7]

$$w_t(t,x) = dw_{xx}(t,x) - \alpha_1 w^2(t,x) + \alpha_2 \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\alpha_3}} e^{-\frac{(x-\tau)^2}{4\alpha_3}} f\big(w(t-r,\tau)\big)\,d\tau. \tag{1.6}$$

Finally, with appropriate conditions on f_1, f_2 and $g(x,y) = -f_1(x) + f_2(x)y$, (1.3) also covers the reaction–diffusion equations (1.7) with nonlocal delays in Ma [11]

$$w_t(t,x) = dw_{xx}(t,x) - f_1(w) + f_2(w) \int_{\mathbb{R}} f\big(w(t-r,\tau)\big) J(x-\tau)\,d\tau. \tag{1.7}$$

2. Main results

We are interested in finding traveling waves $w(t,x) = u(x+ct)$ of (1.3), where $u \in C^2(\mathbb{R},\mathbb{R})$. To this end, we need to find a solution $u(\xi)$ where $\xi = x + ct$, for the following associated ordinary differential equation:

$$du''(\xi) - cu'(\xi) + g\left(u(\xi), \int_{\mathbb{R}} f\big(u(\xi-\tau-cr)\big) J(\tau)\,d\tau\right) = 0. \tag{2.8}$$

Based on the above examples, we make the following assumptions.

(H1) Let $r \geqslant 0$. $J(\tau) \geqslant 0$ is integrable on $\mathbb{R}$, and $J(\tau) = J(-\tau)$, $\tau \in (-\infty, +\infty)$, and

$$\int_{\mathbb{R}} J(\tau)\,d\tau = 1, \qquad \int_{\mathbb{R}} J(\tau)e^{\lambda\tau}\,d\tau < \infty,$$

for all $\lambda > 0$.

(H2) Let $K > 0$. f is Lipschitz continuous on $[0,K]$ and $f''(0)$ exists, $f(0) = 0$, $f'(0) > 0$, $f(x) > 0$ for $x \in (0,K]$, $f(x) \leqslant f(K)$ for $x \in [0,K]$, and there is a θ $(0 < \theta < K)$ such that f is increasing on $[0,\theta]$. Further assume

$$f(x) \leqslant f'(0)x, \quad x \in [0,K].$$

(H3) $g(x,y) \in C^2([0,K] \times [0, f(K)], \mathbb{R})$, $g_y(x,y) > 0$ for $(x,y) \in [0,K] \times [0,f(K)]$, $g(0,0) = 0$, $g(K, f(K)) = 0$ and $g(x, f(x)) > 0$ for $x \in (0,K)$; $g_x(0,0) + g_y(0,0)f'(0) > 0$. Further assume that

$$g(x,y) \leqslant g_x(0,0)x + g_y(0,0)y, \quad (x,y) \in [0,K] \times \big[0, f(K)\big].$$

(H4) There exists a positive $\theta_1 < K$ such that for each $y \in (0,\theta_1)$, $g(x,y) = 0$ has a solution $x \in (0,\theta)$ (θ is defined in (H2)).

We now can state our main results on the existence of a traveling wave to (1.3).

Theorem 2.1. *Assume* (H1)–(H4) *hold. Then there exists a* $c^* > 0$ *such that for* $c > c^*$, (1.3) *admits a traveling wave solution* $w(t,x) = u(x+ct)$ *such that* $0 < u(\xi) \leqslant K$, $\xi \in \mathbb{R}$, $\liminf_{\xi\to\infty} u(\xi) > 0$, $\lim_{\xi\to-\infty} u(\xi) = 0$. *If, in addition,* f *is nondecreasing on* $[0,K]$*, then* $u(\xi)$ *is nondecreasing on* $\mathbb{R}$ *and* $\lim_{\xi\to\infty} u(\xi) = K$.

Remark 2.2. The traveling wave solution $w(t,x) = u(x+ct)$ in Theorem 2.1 also satisfies $\lim_{\xi\to-\infty} u(\xi)e^{-\Lambda_1\xi} = 1$, where Λ_1 is defined in Lemma 3.1.

Remark 2.3. With appropriately choosing the parameters and f_1, f_2, it is easy to see that $g(x,y) = -\alpha_1 x + \alpha_2 y$, $g(x,y) = -\alpha_1 x^2 + \alpha_2 y$ and $g(x,y) = -f_1(x) + f_2(x)y$ satisfy (H3) and (H4). Thus, Theorem 2.1 covers corresponding results in the literature. It is worthwhile to note that the linearities $g(w(t,x), w(t-r,x))$ in Schaaf [13] do not cover some of the models above. For example, [13] requires $g(x,y) \geqslant 0$ for $(x,y) \in [0,K]^2$ $(f(x) = x)$. For the case that g admits an intermediate steady state, [13] requires $g_x(0,0) + g_y(0,0) < 0$ $(f(x) = x)$.

Remark 2.4. For the case that f is nondecreasing, Theorem 2.1 is valid even without assumption (H4). (H4) is only used in Section 6.

Remark 2.5. We believe it is equally important to present our theorems in a way that they can be easily verified. As such, some of our conditions can be stated in more general ways. For example, the assumption that $f''(0)$ exists in (H2) can be replaced by the following conditions: there exists some small number $\delta^1 > 0$, $\sigma^1 > 1$ and $a > 0$ such that $f(u) \geqslant f'(0)u - au^{\sigma_1}$, $u \in [0,\delta^1]$ [8,16,18]; or $\limsup_{u\to0^+} \frac{f'(0) - \frac{f(u)}{u}}{u^\nu} < \infty$, $\nu \in (0,1]$ [11].

Remark 2.6. With some additional assumptions (see [8,11]), the traveling wave solution $u(\xi)$ in Theorem 2.1 can satisfy $\lim_{\xi\to\infty} u(\xi) = K$. Also the assumption that $f(x) \leqslant f(K), x \in [0,K]$, in (H2) can be replaced with other conditions [8,11]. In this case, the function f^+ in Section 6 will be slightly different. Theorem 2.1 can be extended to an n-dimensional system of reaction–diffusion equations.

3. Preliminary results

Let

$$\Delta(c,\lambda) = d\lambda^2 - c\lambda + g_x(0,0) + g_y(0,0)f'(0)\int_{\mathbb{R}} e^{-\lambda(\tau+cr)} J(\tau)\,d\tau. \tag{3.9}$$

Then it is easy to verify the following properties:

$$\Delta(c,0) = g_x(0,0) + g_y(0,0)f'(0) > 0,$$

$\lim_{\lambda\to\infty} \Delta(c,\lambda) = \infty$ for all $c \geqslant 0$,

$$\frac{\partial^2\Delta(c,\lambda)}{\partial\lambda^2} = 2d + g_y(0,0)f'(0)\int_{\mathbb{R}} \tau^2 e^{-\lambda(\tau+cr)} J(\tau)\,d\tau > 0,$$

$$\frac{\partial\Delta(c,\lambda)}{\partial c} = -\lambda - \lambda r g_y(0,0)f'(0)\int_{\mathbb{R}} e^{-\lambda(\tau+cr)} J(\tau)\,d\tau < 0,$$

$\lim_{c\to\infty} \Delta(c,\lambda) = -\infty$ for $\lambda > 0$ and finally $\Delta(0,\lambda) > 0$. Based on these properties of $\Delta(c,\lambda)$, we state Lemma 3.1 which is similar to Lemma 2.1 [11] and Lemma 2.5 [13].

Lemma 3.1. *Assume that* (H1)–(H3) *hold. Then there exists a unique* $c^* > 0$ *such that*:

(1) *If* $c \geqslant c^*$, *then there exist two positive numbers* Λ_1, Λ_2 (*which are dependent on* c) *with* $\Lambda_1 \leqslant \Lambda_2$ *such that*

$$\Delta(c, \Lambda_1) = \Delta(c, \Lambda_2) = 0.$$

(2) *If* $c < c^*$, *then* $\Delta(c, \lambda) > 0$ *for all* $\lambda > 0$.
(3) *If* $c = c^*$, *then* $\Lambda_1 = \Lambda_2$; *and if* $c > c^*$, *then* $\Lambda_1 < \Lambda_2$,

$$\Delta(c, \lambda) < 0, \quad \text{for } \lambda \in (\Lambda_1, \Lambda_2), \qquad \Delta(c, \lambda) > 0, \quad \text{for } \lambda \in [0, \infty) \setminus [\Lambda_1, \Lambda_2].$$

Now let $\beta > \max_{(x,y)\in[0,K]\times[0,f(K)]} |g'_x(x, y)| > 0$. For $c > c^*$, the two solutions of the following equation

$$d\lambda^2 - c\lambda - \beta = 0 \tag{3.10}$$

are $-\lambda_1$ and λ_2 where

$$\lambda_1 = \frac{-c + \sqrt{c^2 + 4\beta d}}{2d} > 0, \qquad \lambda_2 = \frac{c + \sqrt{c^2 + 4\beta d}}{2d} > 0.$$

We choose β sufficiently large so that

$$\lambda_2 > \lambda_1 > \max\{2\Lambda_1, \Lambda_2\}. \tag{3.11}$$

Define an operator by

$$\mathcal{T}[u](\xi) = \frac{1}{d(\lambda_1 + \lambda_2)} \left(\int_{-\infty}^{\xi} e^{-\lambda_1(\xi - s)} H\big(u(s)\big)\, ds + \int_{\xi}^{\infty} e^{\lambda_2(\xi - s)} H\big(u(s)\big)\, ds \right), \tag{3.12}$$

where

$$H\big(u(s)\big) = \beta u(s) + g\left(u(s), \int_{\mathbb{R}} f\big(u(s - \tau - cr)\big) J(\tau)\, d\tau\right).$$

$\mathcal{T}[u]$ is defined on $\mathbb{R}$ if $H(u)$ is a bounded continuous function. In fact, the following identity holds:

$$\begin{aligned} &\frac{1}{d(\lambda_1 + \lambda_2)} \left(\int_{-\infty}^{\xi} e^{-\lambda_1(\xi - s)} \beta K\, ds + \int_{\xi}^{\infty} e^{\lambda_2(\xi - s)} \beta K\, ds \right) \\ &\quad = \frac{\beta K}{d(\lambda_1 + \lambda_2)} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) = \frac{\beta K}{d(\lambda_1 \lambda_2)} \\ &\quad = K. \end{aligned} \tag{3.13}$$

We shall show that a fixed point u of $\mathcal{T}$ or solution of the equation

$$u(\xi) = \mathcal{T}[u](\xi), \quad \xi \in \mathbb{R}, \tag{3.14}$$

is a traveling wave solution of (1.3). Similar results for different reaction terms can be found in [10,11,19] and others. More additional properties of $\mathcal{T}$ will be discussed in Section 5.

Lemma 3.2. *Assume* (H1)–(H3) *hold. If* $u \in C(\mathbb{R}, [0, K])$ *is a fixed point of* $\mathcal{T}[u]$,

$$u(\xi) = \mathcal{T}[u](\xi), \quad \xi \in \mathbb{R},$$

then $u \in C^2(\mathbb{R}, [0, K])$ *and is solution of* (2.8).

Proof. First if $u \in C(\mathbb{R}, [0, K])$, then $\int_{\mathbb{R}} f(u(t-\tau-cr))J(\tau)\,d\tau$ is a bounded continuous function. Indeed, for $s, t \in \mathbb{R}$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}} f\big(u(t-\tau-cr)\big)J(\tau)\,d\tau - \int_{\mathbb{R}} f\big(u(s-\tau-cr)\big)J(\tau)\,d\tau \right| \\
&\quad \leqslant \int_{\mathbb{R}} f\big(u(s-\tau-cr)\big)\big|J(t-s+\tau) - J(\tau)\big|\,d\tau \\
&\quad \leqslant f(K) \int_{\mathbb{R}} \big|J(t-s+\tau) - J(\tau)\big|\,d\tau.
\end{aligned}$$

The fact that $\int_{\mathbb{R}} |J(t-s+\tau) - J(\tau)|\,d\tau \to 0$ if $|t-s| \to 0$ implies that $\int_{\mathbb{R}} f(u(t-\tau-cr))J(\tau)\,d\tau$ and $H(u(s))$ are continuous functions on $\mathbb{R}$. Thus $\mathcal{T}[u](\xi)$ is defined and differentiable on $\mathbb{R}$. Direct calculations show

$$\big(\mathcal{T}[u](\xi)\big)' = \frac{1}{d(\lambda_1+\lambda_2)}\left(-\lambda_1 \int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)} H\big(u(s)\big)\,ds + \lambda_2 \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)} H\big(u(s)\big)\,ds\right)$$

and

$$\begin{aligned}
\big(\mathcal{T}[u](\xi)\big)'' = \frac{1}{d(\lambda_1+\lambda_2)}\Bigg(&\lambda_1^2 \int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)} H\big(u(s)\big)\,ds \\
&+ \lambda_2^2 \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)} H\big(u(s)\big)\,ds - \lambda_1 H\big(u(\xi)\big) - \lambda_2 H\big(u(\xi)\big)\Bigg).
\end{aligned}$$

Noting that $-\lambda_1, \lambda_2$ are solutions of (3.10), one can evaluate the following expression

$$\begin{aligned}
d\big(\mathcal{T}[u](\xi)\big)'' - c\big(\mathcal{T}[u](\xi)\big)' - \beta\mathcal{T}[u](\xi) &= \frac{d\lambda_1^2 + c\lambda_1}{d(\lambda_1+\lambda_2)} \int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)} H\big(u(s)\big)\,ds \\
&\quad + \frac{d\lambda_2^2 - c\lambda_2}{d(\lambda_1+\lambda_2)} \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)} H\big(u(s)\big)\,ds - H\big(u(\xi)\big) - \beta\mathcal{T}[u](\xi) \\
&= \beta\mathcal{T}[u](\xi) - H\big(u(\xi)\big) - \beta\mathcal{T}[u](\xi) \\
&= -H\big(u(\xi)\big).
\end{aligned}$$

Now if $u(\xi) = \mathcal{T}[u](\xi)$, $\xi \in \mathbb{R}$, then $u \in C^2(\mathbb{R}, [0, K])$ and is solution of (2.8). $\square$

We also need the following two lemmas to estimate $\mathcal{T}[u]$ in Section 4.

Lemma 3.3. *Assume* (H2)–(H3) *hold. There exist positive constants* D_1, D_2, D_3 *such that*

$$g(x,y) \geqslant g_x(0,0)x + g_y(0,0)y - D_1x^2 - D_2y^2, \quad \textit{for } x, y \in [0,K] \times \big[0, f(K)\big],$$

and

$$f(x) \geqslant f'(0)x - D_3x^2, \quad \textit{for } x \in [0,K].$$

Proof. According to Taylor's theorem in two variables, there exist positive constants D_1, D_2 such that

$$g(x,y) \geqslant g_x(0,0)x + g_y(0,0)y - D_1x^2 - D_2y^2, \quad \text{for } x, y \in [0,K] \times \big[0, f(K)\big].$$

Since $f''(0)$ exists, we can find an interval $[0,\eta]$, $0 < \eta < K$, and $D_4 > 0$ such that

$$f'(u) - f'(0) \geqslant -2D_4u, \quad u \in [0,\eta].$$

Integrating from 0 to u will produce, for $u \in [0,\eta]$,

$$f(u) \geqslant f'(0)u - D_4u^2.$$

For $u \in [\eta, K]$, we can always find positive constant D_5 such that

$$f(u) \geqslant f'(0)u - D_5u^2.$$

Now take $D_3 = \max\{D_4, D_5\}$ and we have, for $u \in [0,K]$,

$$f(u) \geqslant f'(0)u - D_3u^2. \qquad \square$$

4. Upper and lower solutions of integral equations

In this section, we give the definition of upper and lower solutions of (3.14) and show ϕ^+ and ϕ^- defined below are an upper and lower solution of (3.14).

Definition 4.1. A bounded continuous function $u(t) \in C(\mathbb{R}, [0,\infty))$ is an upper solution of (3.14) if

$$\mathcal{T}[u](\xi) \leqslant u(\xi), \quad \text{for all } \xi \in \mathbb{R};$$

a bounded continuous function $u(t) \in C(\mathbb{R}, [0,\infty))$ is a lower solution of (3.14) if

$$\mathcal{T}[u](\xi) \geqslant u(\xi), \quad \text{for all } \xi \in \mathbb{R}.$$

Let $c > c^*$, $\gamma > 1$, $q > 1$ and

$$\phi^+(\xi) = \min\big\{K, e^{\Lambda_1\xi}\big\}, \quad \xi \in \mathbb{R},$$

and

$$\phi^-(\xi) = \max\big\{0, e^{\Lambda_1\xi} - qe^{\gamma\Lambda_1\xi}\big\}, \quad \xi \in \mathbb{R}.$$

It is clear that if $\xi \geqslant \frac{\ln K}{\Lambda_1}$, $\phi^+(\xi) = K$, and $\xi < \frac{\ln K}{\Lambda_1}$, $\phi^+(\xi) = e^{\Lambda_1\xi}$. Similarly, if $\xi \geqslant \frac{\ln q}{(1-\gamma)\Lambda_1}$, $\phi^-(\xi) = 0$, and $\xi < \frac{\ln q}{(1-\gamma)\Lambda_1}$, $\phi^-(\xi) = e^{\Lambda_1\xi} - qe^{\gamma\Lambda_1\xi}$. By choosing $q > 1$ large so that $\frac{\ln K}{\Lambda_1} > \frac{\ln q}{(1-\gamma)\Lambda_1}$, we have $\phi^-(\xi) < \phi^+(\xi)$, $\xi \in \mathbb{R}$.

Before verifying the upper and lower solutions, we give an identity which will simply our proof in this section. First for $\Lambda > 0$ let

$$M(\Lambda) = \beta + g_x(0,0) + g_y(0,0) f'(0) \int_{\mathbb{R}} e^{-\Lambda(\tau + cr)} J(\tau)\, d\tau > 0. \tag{4.15}$$

Then we can show the following lemma.

Lemma 4.2. *Assume* (H1)–(H3) *hold. Then*

$$\frac{M(\Lambda_1)}{d(\lambda_1+\lambda_2)} \left(\frac{1}{\lambda_1 + \Lambda_1} + \frac{1}{\lambda_2 - \Lambda_1} \right) = 1. \tag{4.16}$$

Proof. Since Λ_1 is a zero of (3.9), it follows that

$$\begin{aligned} \frac{M(\Lambda_1)}{d(\lambda_1+\lambda_2)} \left(\frac{1}{\lambda_1 + \Lambda_1} + \frac{1}{\lambda_2 - \Lambda_1} \right) &= \frac{M(\Lambda_1)}{d(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)}{\lambda_1\lambda_2 + (\lambda_2 - \lambda_1)\Lambda_1 - \Lambda_1^2} \\ &= \frac{M(\Lambda_1)}{d} \frac{1}{\frac{\beta}{d} + \frac{c}{d}\Lambda_1 - \Lambda_1^2} \\ &= \frac{M(\Lambda_1)}{\beta + c\Lambda_1 - d\Lambda_1^2} \\ &= \frac{\beta + g_x(0,0) + g_y(0,0) f'(0) \int_{\mathbb{R}} e^{-\Lambda_1(\tau + cr)} J(\tau)\, d\tau}{\beta + c\Lambda_1 - d\Lambda_1^2} \\ &= 1. \qquad \square \end{aligned} \tag{4.17}$$

Lemma 4.3. *Assume* (H1)–(H3) *hold. For any $c > c^*$, ϕ^+ defined above is an upper solution of* (3.14).

Proof. Let $\xi^* = \frac{\ln K}{\Lambda_1}$. $\phi^+(\xi) = K$ if $\xi \geqslant \xi^*$, and $\phi^+(\xi) = e^{\Lambda_1 \xi}$ if $\xi < \xi^*$. Note that $\phi^+(\xi) \leqslant e^{\Lambda_1 \xi}$, $\xi \in \mathbb{R}$, and then

$$\int_{\mathbb{R}} \phi^+(\xi - \tau - cr) J(\tau)\, d\tau \leqslant e^{\Lambda_1 \xi} \int_{\mathbb{R}} e^{-\Lambda_1(\tau + cr)} J(\tau)\, d\tau, \quad \xi \in \mathbb{R}.$$

In view of (H2)–(H3) we have, for $s \in \mathbb{R}$,

$$\begin{aligned} H\big(\phi^+(s)\big) &= \beta\phi^+(s) + g\left(\phi^+(s), \int_{\mathbb{R}} f\big(\phi^+(s - \tau - cr)\big) J(\tau)\, d\tau\right) \\ &\leqslant \beta\phi^+(s) + g_x(0,0)\phi^+(s) + g_y(0,0) f'(0) \int_{\mathbb{R}} \phi^+(s - \tau - cr) J(\tau)\, d\tau \\ &\leqslant M(\Lambda_1) e^{\Lambda_1 s}. \end{aligned}$$

For $s \geqslant \xi^*$, because of (H2) and (H3), we have

$$H\big(\phi^+(s)\big) = \beta K + g\left(K, \int_{\mathbb{R}} f\big(\phi^+(s - \tau - cr)\big) J(\tau)\, d\tau\right) \leqslant \beta K + g\big(K, f(K)\big) = \beta K.$$

Thus, for $\xi \geqslant \xi^*$, we get

$$\mathcal{T}[\phi^+](\xi) \leqslant \frac{M(\Lambda_1)}{d(\lambda_1+\lambda_2)} \int_{-\infty}^{\xi^*} e^{-\lambda_1(\xi-s)} e^{\Lambda_1 s}\,ds + \frac{1}{d(\lambda_1+\lambda_2)}\left[\int_{\xi^*}^{\xi} e^{-\lambda_1(\xi-s)}\beta K\,ds + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}\beta K\,ds\right]. \tag{4.18}$$

Thus in view of (3.13), we add and subtract the term $\frac{\beta K}{d(\lambda_1+\lambda_2)}\int_{-\infty}^{\xi^*} e^{-\lambda_1(\xi-s)}\,ds$ at the left of (4.18). Now for $\xi \geqslant \xi^*$, noting that $e^{\Lambda_1\xi^*} = K$, (4.18) can be written as

$$\begin{aligned}
\mathcal{T}[\phi^+](\xi) &\leqslant K + \frac{1}{d(\lambda_1+\lambda_2)}\left(M(\Lambda_1)\int_{-\infty}^{\xi^*} e^{-\lambda_1(\xi-s)} e^{\Lambda_1 s}\,ds - \beta K \int_{-\infty}^{\xi^*} e^{-\lambda_1(\xi-s)}\,ds\right) \\
&= K + \frac{1}{d(\lambda_1+\lambda_2)}\left(M(\Lambda_1)\frac{e^{-\lambda_1\xi}e^{(\lambda_1+\Lambda_1)\xi^*}}{\lambda_1+\Lambda_1} - \beta K\frac{e^{-\lambda_1\xi}e^{\lambda_1\xi^*}}{\lambda_1}\right) \\
&= K + \frac{Ke^{-\lambda_1\xi}e^{\lambda_1\xi^*}}{d(\lambda_1+\lambda_2)}\left(\frac{M(\Lambda_1)}{\lambda_1+\Lambda_1} - \frac{\beta}{\lambda_1}\right) \\
&= K + \frac{Ke^{-\lambda_1\xi}e^{\lambda_1\xi^*}}{d(\lambda_1+\lambda_2)(\lambda_1+\Lambda_1)\lambda_1}\big(\lambda_1\big(M(\Lambda_1)-\beta\big) - \beta\Lambda_1\big).
\end{aligned} \tag{4.19}$$

And noting Λ_1 is a zero of (3.9), we have

$$\begin{aligned}
\lambda_1\big(M(\Lambda_1)-\beta\big) - \beta\Lambda_1 &= \lambda_1\big(c\Lambda_1 - d\Lambda_1^2\big) - \beta\Lambda_1 \\
&= \frac{4\beta d}{2d(\sqrt{c^2+4\beta d}+c)}\big(c\Lambda_1 - d\Lambda_1^2\big) - \beta\Lambda_1 \\
&= \frac{4\beta d(c\Lambda_1 - d\Lambda_1^2) - 2d(\sqrt{c^2+4\beta d}+c)\beta\Lambda_1}{2d(\sqrt{c^2+4\beta d}+c)} \\
&= \frac{2d\beta\Lambda_1(2c - 2d\Lambda_1 - \sqrt{c^2+4\beta d} - c)}{2d(\sqrt{c^2+4\beta d}+c)} \\
&= \frac{2d\beta\Lambda_1(c - 2d\Lambda_1 - \sqrt{c^2+4\beta d})}{2d(\sqrt{c^2+4\beta d}+c)} \\
&< 0.
\end{aligned} \tag{4.20}$$

Combining (4.19) and (4.20), we see that for $\xi \geqslant \xi^*$,

$$\mathcal{T}[\phi^+](\xi) \leqslant K. \tag{4.21}$$

Similarly, noting $e^{\Lambda_1\xi^*} = K$, one can see that, for $\xi \leqslant \xi^*$,

$$\begin{aligned}
\mathcal{T}[\phi^+](\xi) &\leqslant \frac{M(\Lambda_1)}{d(\lambda_1+\lambda_2)}\left(\int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)}e^{\Lambda_1 s}\,ds + \int_{\xi}^{\xi^*} e^{\lambda_2(\xi-s)}e^{\Lambda_1 s}\,ds\right) + \frac{1}{d(\lambda_1+\lambda_2)}\int_{\xi^*}^{\infty} e^{\lambda_2(\xi-s)}\beta K\,ds \\
&= \frac{M(\Lambda_1)}{d(\lambda_1+\lambda_2)}\left(\frac{e^{\Lambda_1\xi}}{\lambda_1+\Lambda_1} + \frac{e^{\Lambda_1\xi}}{\lambda_2-\Lambda_1} - \frac{e^{\lambda_2\xi}e^{-(\lambda_2-\Lambda_1)\xi^*}}{\lambda_2-\Lambda_1}\right) + \frac{\beta K}{d(\lambda_1+\lambda_2)}\frac{e^{\lambda_2\xi}e^{-\lambda_2\xi^*}}{\lambda_2} \\
&= \frac{e^{\Lambda_1\xi}M(\Lambda_1)}{d(\lambda_1+\lambda_2)}\left(\frac{1}{\lambda_1+\Lambda_1} + \frac{1}{\lambda_2-\Lambda_1}\right) \\
&\quad + \frac{M(\Lambda_1)e^{\lambda_2\xi-\lambda_2\xi^*}}{d(\lambda_1+\lambda_2)}\left(\frac{-K}{\lambda_2-\Lambda_1} + \frac{\beta K}{M(\Lambda_1)\lambda_2}\right).
\end{aligned} \tag{4.22}$$

Note that

$$\begin{aligned}
\frac{-K}{\lambda_2-\Lambda_1} + \frac{\beta K}{M(\Lambda_1)\lambda_2} &= K\frac{(-M(\Lambda_1)+\beta)\lambda_2 - \Lambda_1\beta}{(\lambda_2-\Lambda_1)\lambda_2 M(\Lambda_1)} \\
&= K\frac{-g_x(0,0)\lambda_2 - g_y(0,0)f'(0)\int_{\mathbb{R}} e^{-\Lambda_1(\tau+cr)}J(\tau)\,d\tau\,\lambda_2 - \Lambda_1\beta}{(\lambda_2-\Lambda_1)\lambda_2 M(\Lambda_1)} \\
&\leqslant 0.
\end{aligned} \tag{4.23}$$

Combining (4.16), (4.22) and (4.23) leads to for $\xi \leqslant \xi^*$,

$$\mathcal{T}[\phi^+](\xi) \leqslant e^{\Lambda_1\xi}.$$

And therefore, for $\xi \in \mathbb{R}$,

$$\mathcal{T}[\phi^+](\xi) \leqslant \phi^+(\xi). \tag{4.24}$$

This completes the proof of Lemma 4.3. □

Lemma 4.4. *Assume* (H1)–(H3) *hold. For any $c > c^*$, ϕ^- defined above is a lower solution of* (3.14) *if $1 < \gamma < \min\{2, \frac{\Lambda_2}{\Lambda_1}\}$ and q (which is independent of ξ) is sufficiently large.*

Proof. Again let $\xi^* = \frac{\ln q}{(1-\gamma)\Lambda_1}$. If $\xi \geqslant \xi^*$, $\phi^-(\xi) = 0$, and if $\xi < \xi^*$, $\phi^-(\xi) = e^{\Lambda_1\xi} - qe^{\gamma\Lambda_1\xi}$. For $\xi \in \mathbb{R}$, it follows that

$$\begin{aligned}
H\big(\phi^-(\xi)\big) &= \beta\phi^-(\xi) + g\left(\phi^-(\xi), \int_{\mathbb{R}} f\big(\phi^-(\xi-\tau-cr)\big)J(\tau)\,d\tau\right) \\
&\geqslant \beta\phi^-(\xi) + g\big(\phi^-(\xi), 0\big) \\
&= \beta\phi^-(\xi) + g\big(\phi^-(\xi), 0\big) - g(0,0) \\
&= \big(\beta + g_x(\zeta_1, 0)\big)\phi^-(\xi) \\
&\geqslant 0,
\end{aligned}$$

where $\zeta_1 \in [0, K]$.

Thus, for $\xi \geqslant \xi^*$,

$$\mathcal{T}[\phi^-](\xi) \geqslant \phi^-(\xi).$$

We now consider the case $\xi < \xi^*$. It is easy to see that

$$e^{\Lambda_1 \xi} \geqslant \phi^-(\xi) \geqslant e^{\Lambda_1 \xi} - q e^{\gamma \Lambda_1 \xi}, \quad \xi \in \mathbb{R}, \tag{4.25}$$

and, for $\xi \in \mathbb{R}$,

$$\begin{aligned} e^{\Lambda_1 \xi} \int_{\mathbb{R}} e^{-\Lambda_1(\tau + cr)} J(\tau)\, d\tau &\geqslant \int_{\mathbb{R}} \phi^-(\xi - \tau - cr) J(\tau)\, d\tau \\ &\geqslant e^{\Lambda_1 \xi} \int_{\mathbb{R}} e^{-\Lambda_1(\tau + cr)} J(\tau)\, d\tau - q e^{\gamma \Lambda_1 \xi} \int_{\mathbb{R}} e^{-\gamma \Lambda_1(\tau + cr)} J(\tau)\, d\tau. \end{aligned} \tag{4.26}$$

In view of Lemma 3.3, (4.25) and (4.26), we have, for $\xi \in \mathbb{R}$,

$$\begin{aligned} H\big(\phi^-(\xi)\big) &= \beta \phi^-(\xi) + g\bigg(\phi^-(\xi), \int_{\mathbb{R}} f\big(\phi^-(\xi - \tau - cr)\big) J(\tau)\, d\tau\bigg) \\ &\geqslant \beta \phi^-(\xi) + g_x(0,0) \phi^-(\xi) + g_y(0,0) \int_{\mathbb{R}} f\big(\phi^-(\xi - \tau - cr)\big) J(\tau)\, d\tau \\ &\quad - D_1\big(\phi^-(\xi)\big)^2 - D_2\bigg(\int_{\mathbb{R}} f\big(\phi^-(\xi - \tau - cr)\big) J(\tau)\, d\tau\bigg)^2 \\ &\geqslant \beta \phi^-(\xi) + g_x(0,0) \phi^-(\xi) + g_y(0,0) f'(0) \int_{\mathbb{R}} \phi^-(\xi - \tau - cr) J(\tau)\, d\tau \\ &\quad - D_1\big(\phi^-(\xi)\big)^2 - D_2\big(f'(0)\big)^2 \bigg(\int_{\mathbb{R}} \phi^-(\xi - \tau - cr) J(\tau)\, d\tau\bigg)^2 \\ &\quad - g_y(0,0) D_3 \int_{\mathbb{R}} \big(\phi^-(\xi - \tau - cr)\big)^2 J(\tau)\, d\tau \\ &\geqslant M(\Lambda_1) e^{\Lambda_1 \xi} - q M(\gamma \Lambda_1) e^{\gamma \Lambda_1 \xi} - \widehat{M} e^{2\Lambda_1 \xi}, \end{aligned} \tag{4.27}$$

where $M(\cdot)$ is defined in (4.15) and

$$\begin{aligned} \widehat{M} &= D_1 + D_2\big(f'(0)\big)^2 \bigg(\int_{\mathbb{R}} e^{-\Lambda_1(\tau + cr)} J(\tau)\, d\tau\bigg)^2 \\ &\quad + g_y(0,0) D_3 \int_{\mathbb{R}} e^{-2\Lambda_1(\tau + cr)} J(\tau)\, d\tau \\ &> 0. \end{aligned} \tag{4.28}$$

Observe that

$$g\bigg(0, \int_{\mathbb{R}} f\big(\phi^-(\xi - \tau - cr)\big) J(\tau)\, d\tau\bigg) \geqslant g(0,0) = 0$$

and we ignore the term $\int_{\xi^*}^{\infty} e^{\lambda_2(\xi-s)} H(\phi^-(s))\,ds$ in (4.29). Now we are able to estimate $\mathcal{T}[\phi^-]$ for $\xi \leqslant \xi^*$:

$$\begin{aligned}
\mathcal{T}[\phi^-](\xi) \geqslant{}& \frac{1}{d(\lambda_1+\lambda_2)}\Bigg(\int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)} M(\Lambda_1) e^{\Lambda_1 s}\,ds \\
&- q\int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)} M(\gamma\Lambda_1) e^{\gamma\Lambda_1 s}\,ds - \widehat{M}\int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)} e^{2\Lambda_1 s}\,ds \\
&+ \int_{\xi}^{\xi^*} e^{\lambda_2(\xi-s)} M(\Lambda_1) e^{\Lambda_1 s}\,ds - q\int_{\xi}^{\xi^*} e^{\lambda_2(\xi-s)} M(\gamma\Lambda_1) e^{\gamma\Lambda_1 s}\,ds - \widehat{M}\int_{\xi}^{\xi^*} e^{\lambda_2(\xi-s)} e^{2\Lambda_1 s}\,ds\Bigg) \\
={}& \frac{1}{d(\lambda_1+\lambda_2)}\Bigg(\frac{M(\Lambda_1)e^{\Lambda_1\xi}}{\lambda_1+\Lambda_1} - q\frac{M(\gamma\Lambda_1)e^{\gamma\Lambda_1\xi}}{\lambda_1+\gamma\Lambda_1} \\
&- \widehat{M}\frac{e^{2\Lambda_1\xi}}{\lambda_1+2\Lambda_1} + \frac{e^{\Lambda_1\xi^*-\lambda_2\xi^*+\lambda_2\xi} - e^{\Lambda_1\xi}}{\Lambda_1-\lambda_2} M(\Lambda_1) \\
&- q\frac{e^{\gamma\Lambda_1\xi^*-\lambda_2\xi^*+\lambda_2\xi} - e^{\gamma\Lambda_1\xi}}{\gamma\Lambda_1-\lambda_2} M(\gamma\Lambda_1) - \widehat{M}\frac{e^{2\Lambda_1\xi^*-\lambda_2\xi^*+\lambda_2\xi} - e^{2\Lambda_1\xi}}{2\Lambda_1-\lambda_2}\Bigg).
\end{aligned} \tag{4.29}$$

In view of the identity (4.16), we subtract two terms to make up a term $-qe^{\gamma\Lambda_1\xi}$ and thus we need to add the terms. Recall that $\gamma\Lambda_1 < 2\Lambda_1 < \lambda_2$. We ignore two positive terms

$$q\frac{M(\gamma\Lambda_1)}{(\lambda_2-\gamma\Lambda_1)d(\lambda_1+\lambda_2)} e^{(\gamma\Lambda_1-\lambda_2)\xi^*+\lambda_2\xi} \quad \text{and} \quad \frac{\widehat{M}}{(\lambda_2-2\Lambda_1)d(\lambda_1+\lambda_2)} e^{(2\Lambda_1-\lambda_2)\xi^*+\lambda_2\xi}.$$

Thus,

$$\begin{aligned}
\mathcal{T}[\phi^-](\xi) \geqslant{}& \frac{M(\Lambda_1)}{d(\lambda_1+\lambda_2)}\left(\frac{1}{\lambda_1+\Lambda_1} + \frac{1}{\lambda_2-\Lambda_1}\right)e^{\Lambda_1\xi} \\
&- \frac{M(\Lambda_1)}{d(\lambda_1+\lambda_2)}\left(\frac{1}{\lambda_1+\Lambda_1} + \frac{1}{\lambda_2-\Lambda_1}\right)qe^{\gamma\Lambda_1\xi} \\
&+ \frac{e^{\gamma\Lambda_1\xi}}{d(\lambda_1+\lambda_2)}\Bigg(q\left(\frac{M(\Lambda_1)}{\lambda_1+\Lambda_1} + \frac{M(\Lambda_1)}{\lambda_2-\Lambda_1} - \frac{M(\gamma\Lambda_1)}{\lambda_1+\gamma\Lambda_1} - \frac{M(\gamma\Lambda_1)}{\lambda_2-\gamma\Lambda_1}\right) \\
&- \frac{\widehat{M}}{(\lambda_1+2\Lambda_1)} e^{(2-\gamma)\Lambda_1\xi} - \frac{M(\Lambda_1)e^{(\Lambda_1-\lambda_2)\xi^*}}{(\lambda_2-\Lambda_1)} e^{(\lambda_2-\gamma\Lambda_1)\xi} \\
&- \frac{\widehat{M}}{(\lambda_2-2\Lambda_1)} e^{(2-\gamma)\Lambda_1\xi}\Bigg).
\end{aligned} \tag{4.30}$$

For $\xi \leqslant \xi^*$, $e^{(2-\gamma)\Lambda_1\xi}$, $e^{(\lambda_2-\gamma\Lambda_1)\xi}$ are bounded above. Because of the identity (4.16), (4.30) can be further simplified as

$$\begin{aligned}
\mathcal{T}[\phi^-](\xi) \geqslant{}& e^{\Lambda_1\xi} - qe^{\gamma\Lambda_1\xi} \\
&+ \frac{e^{\gamma\Lambda_1\xi}}{d(\lambda_1+\lambda_2)}\Bigg(q\left(\frac{M(\Lambda_1)}{\lambda_1+\Lambda_1} + \frac{M(\Lambda_1)}{\lambda_2-\Lambda_1} - \frac{M(\gamma\Lambda_1)}{\lambda_1+\gamma\Lambda_1} - \frac{M(\gamma\Lambda_1)}{\lambda_2-\gamma\Lambda_1}\right)
\end{aligned}$$

$$- \frac{\widehat{M}}{(\lambda_1 + 2\Lambda_1)} e^{(2-\gamma)\Lambda_1 \xi^*} - \frac{M(\Lambda_1) e^{(\Lambda_1 - \lambda_2)\xi^*}}{(\lambda_2 - \Lambda_1)} e^{(\lambda_2 - \gamma \Lambda_1)\xi^*}$$
$$- \frac{\widehat{M}}{(\lambda_2 - 2\Lambda_1)} e^{(2-\gamma)\Lambda_1 \xi^*} \bigg). \tag{4.31}$$

Now we only need to show that

$$\frac{M(\Lambda_1)}{\lambda_1 + \Lambda_1} + \frac{M(\Lambda_1)}{\lambda_2 - \Lambda_1} - \frac{M(\gamma \Lambda_1)}{\lambda_1 + \gamma \Lambda_1} - \frac{M(\gamma \Lambda_1)}{\lambda_2 - \gamma \Lambda_1} > 0.$$

Note $M(\Lambda_1) = \beta + c\Lambda_1 - d\Lambda_1^2$ because Λ_1 is a zero of (3.9). Lemma 3.1(3) implies that

$$\frac{M(\gamma \Lambda_1)}{\beta + c\gamma \Lambda_1 - d(\gamma \Lambda_1)^2} < 1.$$

Thus, we have

$$\begin{aligned}
&\frac{M(\Lambda_1)}{\lambda_1 + \Lambda_1} + \frac{M(\Lambda_1)}{\lambda_2 - \Lambda_1} - \frac{M(\gamma \Lambda_1)}{\lambda_1 + \gamma \Lambda_1} - \frac{M(\gamma \Lambda_1)}{\lambda_2 - \gamma \Lambda_1} \\
&\quad = \frac{(\lambda_1 + \lambda_2) M(\Lambda_1)}{\lambda_1 \lambda_2 + (\lambda_2 - \lambda_1)\Lambda_1 - \Lambda_1^2} - \frac{(\lambda_1 + \lambda_2) M(\gamma \Lambda_1)}{\lambda_1 \lambda_2 + (\lambda_2 - \lambda_1)\gamma \Lambda_1 - (\gamma \Lambda_1)^2} \\
&\quad = \frac{\frac{\sqrt{c^2 + 4\beta d}}{d} M(\Lambda_1)}{\frac{\beta}{d} + \frac{c}{d}\Lambda_1 - \Lambda_1^2} - \frac{\frac{\sqrt{c^2 + 4\beta d}}{d} M(\gamma \Lambda_1)}{\frac{\beta}{d} + \frac{c}{d}\gamma \Lambda_1 - (\gamma \Lambda_1)^2} \\
&\quad = \sqrt{c^2 + 4\beta d} \left(\frac{M(\Lambda_1)}{\beta + c\Lambda_1 - d\Lambda_1^2} - \frac{M(\gamma \Lambda_1)}{\beta + c\gamma \Lambda_1 - d(\gamma \Lambda_1)^2} \right) \\
&\quad = \sqrt{c^2 + 4\beta d} \left(1 - \frac{M(\gamma \Lambda_1)}{\beta + c\gamma \Lambda_1 - d(\gamma \Lambda_1)^2} \right) \\
&\quad > 0.
\end{aligned} \tag{4.32}$$

Finally, from (4.31) and (4.32), we conclude that there exists $q > 0$, which is independent of ξ, such that, for $\xi \leqslant \xi^*$,

$$\mathcal{T}[\phi^-](\xi) \geqslant e^{\Lambda_1 \xi} - q e^{\gamma \Lambda_1 \xi}. \tag{4.33}$$

And therefore,

$$\mathcal{T}[\phi^-](\xi) \geqslant \phi^-(\xi), \quad \xi \in \mathbb{R}.$$

This completes the proof. □

5. Proof of Theorem 2.1 with the monotonicity of *f*

In this section, we assume that f is nondecreasing on $[0, K]$ and prove Theorem 2.1. To this end, define the following Banach space

$$C_\rho = \Big\{ u\colon\ u \in C(\mathbb{R}),\ \sup_{\xi \in \mathbb{R}} \big| u(\xi) \big| e^{-\rho \xi} < \infty \Big\}$$

equipped with weighted norm

$$\|u\|_\rho = \sup_{\xi\in\mathbb{R}} \bigl|u(\xi)\bigr| e^{-\rho\xi},$$

where $C(\mathbb{R})$ is the set of all continuous functions on $\mathbb{R}$ and ρ is a positive constant such that $\rho < \Lambda_1$. It follows that $\phi^+ \in C_\rho$ and $\phi^- \in C_\rho$. Consider the following set

$$\mathcal{A} = \bigl\{u\colon\ u \in C_\rho,\ \phi^-(\xi) \leqslant u \leqslant \phi^+(\xi),\ \xi \in \mathbb{R}\bigr\}.$$

We shall show the following lemma.

Lemma 5.1. *Assume* (H1)–(H3) *hold and f is nondecreasing on* $[0, K]$. *Then $\mathcal{T}$ defined in* (3.12) *is monotone and therefore $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{A}$. Furthermore, $\mathcal{T}[u]$ is nondecreasing if $u \in \mathcal{A}$ and u is nondecreasing.*

Proof. In the same way as in Lemma 3.2, it can be verified that $H(u(\xi))$ and $\mathcal{T}[u](\xi)$ are bounded continuous functions on $\mathbb{R}$ if $u \in \mathcal{A}$. Note $\beta \geqslant \max_{(x,y)\in[0,K]\times[0,f(K)]} |g_x(x,y)|$, $g_y(x,y) \geqslant 0$, $(x,y) \in [0,K]\times[0,f(K)]$ and that f is nondecreasing. For any $u, v \in \mathcal{A}$ with $u(\xi) \geqslant v(\xi)$, $\xi \in \mathbb{R}$, we have, for $\xi \in \mathbb{R}$,

$$\begin{aligned} H\bigl(u(\xi)\bigr) - H\bigl(v(\xi)\bigr) &= \beta\bigl(u(\xi) - v(\xi)\bigr) + g_x(\zeta_1,\zeta_2)\bigl(u(\xi) - v(\xi)\bigr) \\ &\quad + g_y(\zeta_3,\zeta_4)\int_{\mathbb{R}} \bigl(f\bigl(u(\xi-\tau-cr)\bigr) - f\bigl(v(\xi-\tau-cr)\bigr)\bigr) J(\tau)\,d\tau \\ &\geqslant 0, \end{aligned} \tag{5.34}$$

where $\zeta_1, \zeta_3 \in [0,K]$, $\zeta_2, \zeta_4 \in [0, f(K)]$. Therefore $\mathcal{T}[u](\xi) \geqslant \mathcal{T}[v](\xi)$ for $\xi \in \mathbb{R}$.

If $u \in \mathcal{A}$ is nondecreasing, consider $\xi \in \mathbb{R}$ and $\xi_1 > 0$ and

$$\begin{aligned} \mathcal{T}[u](\xi+\xi_1) - \mathcal{T}[u](\xi) &= \frac{1}{d(\lambda_1+\lambda_2)}\left(\int_{-\infty}^{\xi+\xi_1} e^{-\lambda_1(\xi+\xi_1-s)} H\bigl(u(s)\bigr)\,ds + \int_{\xi+\xi_1}^{\infty} e^{\lambda_2(\xi+\xi_1-s)} H\bigl(u(s)\bigr)\,ds \right. \\ &\quad \left. - \int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)} H\bigl(u(s)\bigr)\,ds - \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)} H\bigl(u(s)\bigr)\,ds \right) \\ &= \frac{1}{d(\lambda_1+\lambda_2)}\left(\int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)} H\bigl(u(s+\xi_1)\bigr)\,ds - \int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)} H\bigl(u(s)\bigr)\,ds \right. \\ &\quad \left. + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)} H\bigl(u(s+\xi_1)\bigr)\,ds - \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)} H\bigl(u(s)\bigr)\,ds \right). \end{aligned} \tag{5.35}$$

It follows from (5.34) that $\mathcal{T}[u](\xi+\xi_1) - \mathcal{T}[u](\xi) \geqslant 0$ for $\xi \in \mathbb{R}$ and $\xi_1 > 0$. $\square$

Now we shall show that $\mathcal{T}[u]$ is continuous and maps a bounded set in $\mathcal{A}$ into a compact set.

Lemma 5.2. *Assume* (H1)–(H3) *hold. Then $\mathcal{T}: \mathcal{A} \to C_\rho$ is continuous with the weighted norm $\|.\|_\rho$.*

Proof. Let $L_f > 0$ be the Lipschitz constant of f on $[0, K]$ and

$$L_g = \max_{(x,y)\in[0,K]\times[0,f(K)]}\left\{\left|g_x(x,y)\right|, \left|g_y(x,y)\right|\right\}.$$

For any $u, v \in \mathcal{A}$, we have, for $\xi \in \mathbb{R}$,

$$\begin{aligned}
\left|H\big(u(\xi)\big) - H\big(v(\xi)\big)\right|e^{-\rho\xi} &\leqslant \beta\left|u(\xi) - v(\xi)\right|e^{-\rho\xi} + \left|g_x(\zeta_1,\zeta_2)\right|\left|u(\xi) - v(\xi)\right|e^{-\rho\xi} \\
&\quad + \left|g_y(\zeta_3,\zeta_4)\right| \int_{\mathbb{R}} \left|f\big(u(\xi-\tau-cr)\big) - f\big(v(\xi-\tau-cr)\big)\right|e^{-\rho\xi} J(\tau)\,d\tau \\
&\leqslant \beta\|u-v\|_\rho + L_g\|u-v\|_\rho + L_g L_f \int_{\mathbb{R}} e^{-\rho(\tau+cr)} J(\tau)\,d\tau\,\|u-v\|_\rho \\
&\leqslant L\|u-v\|_\rho,
\end{aligned} \tag{5.36}$$

where $\zeta_1, \zeta_3 \in [0, K]$, $\zeta_2, \zeta_4 \in [0, f(K)]$, $L = \beta + L_g + L_g L_f \int_{\mathbb{R}} e^{-\rho(\tau+cr)} J(\tau)\,d\tau$. Furthermore, we obtain

$$\begin{aligned}
\left|\mathcal{T}[u](\xi) - \mathcal{T}[v](\xi)\right|e^{-\rho\xi} &\leqslant \frac{1}{d(\lambda_1+\lambda_2)}\left(\int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)}\left|H\big(u(s)\big) - H\big(v(s)\big)\right|ds\right. \\
&\quad \left. + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}\left|H(u)(s) - H(v)(s)\right|ds\right)e^{-\rho\xi} \\
&\leqslant \frac{L\|u-v\|_\rho}{d(\lambda_1+\lambda_2)}\left(\int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)}e^{\rho s}\,ds + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}e^{\rho s}\,ds\right)e^{-\rho\xi} \\
&= \frac{\lambda_1+\lambda_2}{(\lambda_1+\rho)(\lambda_2-\rho)}\frac{L\|u-v\|_\rho}{d(\lambda_1+\lambda_2)},
\end{aligned} \tag{5.37}$$

and

$$\left\|\mathcal{T}[u] - \mathcal{T}[v]\right\|_\rho \leqslant \frac{L}{d(\lambda_1+\rho)(\lambda_2-\rho)}\|u-v\|_\rho.$$

Thus, $\mathcal{T}[u]$ is continuous. $\square$

Lemma 5.3. *Assume* (H1)–(H3) *hold. Then the set* $\mathcal{T}(\mathcal{A})$ *is relatively compact in* C_ρ.

Proof. Let $M_1 = \max_{u\in\mathcal{A},\,\xi\in\mathbb{R}} H(u(\xi)) > 0$. Recall that

$$\frac{1}{d(\lambda_1+\lambda_2)}\left[\int_{-\infty}^{t} e^{-\lambda_1(t-s)}ds + \int_{t}^{\infty} e^{\lambda_2(t-s)}\,ds\right] = \frac{1}{\beta}.$$

If $u \in \mathcal{A}$, $\xi \in \mathbb{R}$ and $\delta > 0$ (without loss of generality), we have

$$\begin{aligned}\mathcal{T}[u](\xi+\delta)-\mathcal{T}[u](\xi)&=\frac{1}{d(\lambda_1+\lambda_2)}\Bigg(\int_{-\infty}^{\xi+\delta} e^{-\lambda_1(\xi+\delta-s)}H\big(u(s)\big)\,ds+\int_{\xi+\delta}^{\infty} e^{\lambda_2(\xi+\delta-s)}H\big(u(s)\big)\,ds\\&\quad-\int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)}H\big(u(s)\big)\,ds-\int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}H\big(u(s)\big)\,ds\Bigg)\\&=\frac{1}{d(\lambda_1+\lambda_2)}\Bigg(\int_{-\infty}^{\xi} e^{-\lambda_1(\xi-s)}\big(e^{-\lambda_1\delta}H\big(u(s)\big)-H\big(u(s)\big)\big)\,ds\\&\quad+\int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}\big(e^{\lambda_2\delta}H\big(u(s)\big)-H\big(u(s)\big)\big)\,ds\\&\quad+\int_{\xi}^{\xi+\delta} e^{-\lambda_1(\xi+\delta-s)}H\big(u(s)\big)\,ds-\int_{\xi}^{\xi+\delta} e^{\lambda_2(\xi+\delta-s)}H\big(u(s)\big)\,ds\Bigg),\end{aligned} \tag{5.38}$$

and

$$\big|\mathcal{T}[u](\xi+\delta)-\mathcal{T}[u](\xi)\big|\leqslant \max\big\{\big|e^{-\lambda_1\delta}-1\big|,\big|e^{\lambda_2\delta}-1\big|\big\}\frac{M_1}{\beta}+\delta\frac{M_1}{d(\lambda_1+\lambda_2)}+\delta e^{\lambda_2\delta}\frac{M_1}{d(\lambda_1+\lambda_2)}.$$

Thus we establish that

$$\lim_{\delta\to 0}\big(\mathcal{T}[u](\xi+\delta)-\mathcal{T}[u](\xi)\big)=0,\quad \text{uniformly for all } u\in\mathcal{A},\ \xi\in\mathbb{R}. \tag{5.39}$$

Take any sequence $(u_n)\in\mathcal{A}$ and let $v_n=\mathcal{T}(u_n)$. From Lemma 5.1 and (5.39), (v_n) is uniformly bounded on $\mathbb{R}$ and uniformly equicontinuous. For $I_k=[-k,k]$, $k\in\mathbb{N}$, by Ascoli's theorem and the standard diagonal process, we can construct subsequences (u_{n_k}) of (u_n) such that there is a function $v\in C(-\infty,\infty)$ and $(v_{n_k}=\mathcal{T}[u_{n_k}])$ uniformly converges to v on each I_k for $k\in\mathbb{N}$. Now we need to show that $v\in\mathcal{A}$ and $\|v_{n_k}-v\|_\rho\to 0$ as $n_k\to\infty$. By Lemma 5.1, $\phi^-(\xi)\leqslant v(\xi)\leqslant\phi^+(\xi)$ for all $\xi\in\mathbb{R}$, and therefore $v\in\mathcal{A}$. Note that

$$\lim_{\xi\to\pm\infty}\big(\phi^+(\xi)-\phi^-(\xi)\big)e^{-\rho\xi}=0.$$

For any $\epsilon>0$, we can find M_0 such that if $|\xi|>M_0$, then, for all $k\in\mathbb{N}$,

$$\big|v_{n_k}(\xi)-v\big|e^{-\rho\xi}\leqslant\big(\phi^+(\xi)-\phi^-(\xi)\big)e^{-\rho\xi}<\epsilon.$$

On the other hand, on I_k, (v_{n_k}) uniformly converges to v. Thus there exists an $N>0$ such that, for $n_k>N$,

$$\big|v_{n_k}(\xi)-v\big|e^{-\rho\xi}<\epsilon,\quad \xi\in[-M_0,M_0].$$

Consequently, if $n_k>N$, the following inequality is true for all $\xi\in\mathbb{R}$

$$\big|v_{n_k}(\xi)-v\big|e^{-\rho\xi}<\epsilon.$$

Thus $\|v_{n_k}-v\|_\rho\to 0$ as $n_k\to\infty$. $\square$

Now we are in a position to prove Theorem 2.1 when f is monotone.
Define the following iteration

$$u_1 = \mathcal{T}[\phi^+], \qquad u_{n+1} = \mathcal{T}[u_n], \quad n > 1. \tag{5.40}$$

From Lemmas 4.3, 4.4, 5.1, u_n is nondecreasing on $\mathbb{R}$ and

$$\phi^-(\xi) \leqslant u_{n+1}(\xi) \leqslant u_n(\xi) \leqslant \phi^+(\xi), \quad \xi \in \mathbb{R},\ n \geqslant 1.$$

By Lemma 5.3 and monotonicity of (u_n), there is $u \in \mathcal{A}$ such that $\lim_{n\to\infty} \|u_n - u\|_\rho = 0$. Lemma 5.2 implies that $\mathcal{T}[u] = u$. Furthermore, u is nondecreasing. It is clear that $\lim_{\xi\to-\infty} u(\xi) = 0$. Assume that $\lim_{\xi\to\infty} u(\xi) = K'$. $K' > 0$ because of $u \in \mathcal{A}$. Applying l'Hospital's rule to (3.12), we get $K' = \frac{1}{\beta}(\beta K' + g(K', f(K')))$. By (H3), $K' = K$. Finally, note that

$$e^{\Lambda_1 \xi} - q e^{\gamma \Lambda_1 \xi} \leqslant u(\xi) \leqslant e^{\Lambda_1 \xi}, \quad \xi \in \mathbb{R}.$$

We immediately obtain

$$\lim_{\xi\to-\infty} u(\xi) e^{-\Lambda_1 \xi} = 1.$$

This completes the proof of Theorem 2.1 when f is monotone.

6. Proof of Theorem 2.1

Theorem 2.1 is proved when f is monotone in the last section. Now we need to prove it in the general case. In order to find traveling waves for (1.3), we will apply the Schauder's fixed point theorem. First there is an $x_0 > 0$ such that $f'(0)x_0 = f(K)$. Since $f(K) \leqslant f'(0)K$, we must have $x_0 \leqslant K$. Define the function

$$f^+(x) = \begin{cases} f'(0)x, & 0 \leqslant x \leqslant x_0, \\ f(K), & x_0 \leqslant x \leqslant K. \end{cases}$$

Then

$$f(x) \leqslant f^+(x) \leqslant f'(0)x, \quad x \in [0, K],$$

and

$$g\big(K, f^+(K)\big) = g\big(K, f(K)\big) = 0,$$

and $g(x, f^+(x)) \geqslant g(x, f(x)) > 0$ for $x \in (0, K)$.

According to (H2)–(H4), we can choose a positive $\sigma_0 < \theta$ such that $f(\sigma_0) < \min\{\theta_1, \min_{v\in[\theta,K]} f(v)\}$. Because of (H4) there exists $0 < \sigma < \theta$ such that $g(\sigma, f(\sigma_0)) = 0$. Note that $g(0, f(\sigma_0)) > g(0,0) = 0$ and we can assume that $\sigma > 0$ is the smallest number such that $g(\sigma, f(\sigma_0)) = 0$. Because g is nondecreasing with respect to the second variable, $g(\sigma, f(\sigma)) > g(\sigma, f(\sigma_0)) = 0$ implies that $f(\sigma) > f(\sigma_0)$, and furthermore $\sigma \geqslant \sigma_0$. Now define

$$f^-(x) = \begin{cases} f(x), & 0 \leqslant x \leqslant \sigma_0, \\ f(\sigma_0), & \sigma_0 \leqslant x \leqslant K. \end{cases}$$

It follows that $g(\sigma, f^-(\sigma)) = g(\sigma, f(\sigma_0)) = 0$. Now it is clear that both f^+ and f^- are nondecreasing on $[0, K]$ and further, for $0 \leqslant x \leqslant K$,

$$f^-(x) \leqslant f(x) \leqslant f^+(x) \leqslant f'(0)x.$$

In fact, if $x \in [0, \sigma_0]$, $f^-(x) = f(x)$. If $x \in [\sigma_0, \theta]$, $f^-(x) = f(\sigma_0) \leqslant f(x)$ because f is increasing on $[\sigma_0, \theta]$. If $x \in [\theta, K]$, $f^-(x) = f(\sigma_0) \leqslant \min_{v \in [\theta, K]} f(v) \leqslant f(x)$. In addition $g(x, f^-(x)) > 0$ for $x \in (0, \sigma)$ since σ is the smallest number such that $g(\sigma, f(\sigma_0)) = 0$, and for $x \in (0, \sigma_0)$, $g(x, f^-(x)) = g(x, f(x)) > 0$.

Now we examine two integral operators for f^- and f^+:

$$\begin{aligned} u(\xi) &= \mathcal{T}^+[u](\xi) \\ &= \frac{1}{d(\lambda_1 + \lambda_2)} \left[\int_{-\infty}^{\xi} e^{-\lambda_1(\xi - s)} H^+\big(u(s)\big)\, ds + \int_{\xi}^{\infty} e^{\lambda_2(\xi - s)} H^+\big(u(s)\big)\, ds \right] \end{aligned} \tag{6.41}$$

and

$$\begin{aligned} u(\xi) &= \mathcal{T}^-[u](\xi) \\ &= \frac{1}{d(\lambda_1 + \lambda_2)} \left[\int_{-\infty}^{\xi} e^{-\lambda_1(\xi - s)} H^-\big(u(s)\big)\, ds + \int_{\xi}^{\infty} e^{\lambda_2(\xi - s)} H^-\big(u(s)\big)\, ds \right], \end{aligned} \tag{6.42}$$

where

$$H^\pm\big(u(s)\big) = \beta u(s) + g\left(u(s), \int_{\mathbb{R}} f^\pm\big(u(s - \tau - cr)\big) J(\tau)\, d\tau\right).$$

As in Section 5, both $\mathcal{T}^+$ and $\mathcal{T}^-$ are monotone. In view of Section 5 and the fact that f^- is nondecreasing, there exists a nondecreasing fixed point u^- of (6.42) such that $\mathcal{T}^-[u^-] = u^-$, $\lim_{\xi \to \infty} u^-(\xi) = \sigma$, and $\lim_{\xi \to -\infty} u^-(\xi) = 0$. Furthermore, $\lim_{\xi \to -\infty} u^-(\xi) e^{-\Lambda_1 \xi} = 1$. According to Lemma 4.3, ϕ^+ is also an upper solution of $\mathcal{T}^+$ because the proof of Lemma 4.3 is still valid if f is replaced by f^+, and the corresponding upper solution of $\mathcal{T}^-$ is $\min\{\sigma, e^{\Lambda_1 \xi}\}$. It follows that $u^-(\xi) \leqslant \phi^+(\xi)$, $\xi \in \mathbb{R}$. Now let

$$\mathcal{B} = \big\{u\colon\ u \in C_\rho,\ u^-(\xi) \leqslant u(\xi) \leqslant \phi^+(\xi),\ \xi \in (-\infty, \infty)\big\},$$

where C_ρ is defined in Section 5. It is clear that $\mathcal{B}$ is a bounded nonempty closed convex subset in C_ρ. Furthermore, we have, for any $u \in \mathcal{B}$,

$$u^- = \mathcal{T}^-[u^-] \leqslant \mathcal{T}^-[u] \leqslant \mathcal{T}[u] \leqslant \mathcal{T}^+[u] \leqslant \mathcal{T}^+[\phi^+] \leqslant \phi^+.$$

Therefore, $\mathcal{T} : \mathcal{B} \to \mathcal{B}$. Note that the proof of Lemmas 5.2, 5.3 does not need the monotonicity of f. In the same way as in Lemmas 5.2, 5.3, we can show that $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is continuous and maps bounded sets into compact sets. Therefore, the Schauder fixed point theorem shows that the operator $\mathcal{T}$ has a fixed point u in $\mathcal{B}$, which is a traveling wave solution of (1.3) for $c > c^*$. Since $u^-(\xi) \leqslant u(\xi) \leqslant \phi^+(\xi)$, $\xi \in (-\infty, \infty)$, it is easy to see that $\lim_{\xi \to -\infty} u(\xi) = 0$, $\lim_{\xi \to -\infty} u(\xi) e^{-\Lambda_1 \xi} = 1$, $\liminf_{x \to \infty} u(x) \geqslant \sigma > 0$ and $0 < u^-(\xi) \leqslant u(\xi) \leqslant K$, $\xi \in (-\infty, \infty)$.

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