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Multiple Positive Solutions of Some Boundary Value Problems

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We study the existence of multiple positive solutions of the equations $-u''=f(t,u)$, subject to linear boundary conditions. We show that there are at least two positive solutions if $f(t,u)$ is superlinear at one end point (zero or infinity) and sublinear at the other. Applications of these results are provided to yield multiple positive solutions of some elliptic boundary value problems on an annulus.
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1. INTRODUCTION

In this paper we consider the following second order boundary value problem (BVP);

$$-u''=f(t,u), \qquad 0<t<1 \tag{1}$$

$$\begin{cases}\alpha u(0)-\beta u'(0)=0\\ \gamma u(1)+\delta u'(1)=0,\end{cases} \tag{2}$$

where f is continuous and $f(t,u)\geqslant 0$ for $t\in[0,1]$ and $u\geqslant 0$, α, β, γ, $\delta\geqslant 0$, and

$$\rho:=\gamma\beta+\alpha\gamma+\alpha\delta>0. \tag{3}$$

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The BVP (1) and (2) arises in a variety of different areas of applied mathematics and physics, see [1–3, 6, 12, 13] for some references along this line. Much attention has been paid to the existence of nontrivial solutions as can be seen in [4, 5, 7–11], for instance. There is also a vast literature on multiple solutions of this problem but, mainly under conditions which allow the application of variational methods combined with critical point theory. Our purpose here is to prove that the superlinearity of f at one end (zero or infinity) and sublinearity of f at the other end can sometimes imply the existence of at least two positive solutions of (1) and (2). To be precise, we introduce the following conditions on $f(t, u)$

(H1) $\lim_{u \to 0+} \min_{t \in [0,1]} (f(t, u)/u) = \infty$,
$\lim_{u \to +\infty} \min_{t \in [0,1]} (f(t, u)/u) = \infty$;

(H2) $\lim_{u \to 0+} \max_{t \in [0,1]} (f(t, u)/u) = 0$,
$\lim_{u \to +\infty} \max_{t \in [0,1]} (f(t, u)/u) = 0$;

(H3) There is a $p > 0$ such that $0 \leqslant u \leqslant p$ and $0 \leqslant t \leqslant 1$ implies

$$f(t, u) \leqslant \eta p,$$

where

$$\eta = \left(\int_0^1 G(s, s)\, ds \right)^{-1} = \frac{6\rho}{6\delta\beta + 3\gamma\beta + \alpha\gamma + 3\alpha\delta} \tag{4}$$

and $G(t, s)$ is the Green's function to $-u'' = 0$ subject to the boundary conditions (2).

(H4) There is a $p > 0$ such that $\sigma p \leqslant u \leqslant p$ implies

$$f(t, u) \geqslant \lambda p,$$

where $\lambda^{-1} = \int_{1/4}^{3/4} G(\frac{1}{2}, s)\, ds$, and

$$\sigma = \min \left\{ \frac{\gamma + 4\delta}{4(\gamma + \delta)}, \frac{\alpha + 4\beta}{4(\alpha + \beta)} \right\}.$$

The following well-known lemma is very crucial in our arguments, see [4, 7] for a proof and further discussion of the fixed point index.

LEMMA 1. *Let X be a Banach space, $K \subseteq X$ a cone in X. For $p > 0$, define $K_p = \{x \in K \mid |x| \leqslant p\}$. Assume that $F: K_p \to K$ is a compact map such that $Fx \neq x$ for $x \in \partial K_p = \{x \in K \mid |x| = p\}$.*

(i) *If $|x| \leqslant |Fx|$ for $x \in \partial K_p$, then*

$$i(F, K_p, K) = 0;$$

(ii) *If* $|x| \geqslant |Fx|$ *for* $x \in \partial K_p$, *then*

$$i(F, K_p, K) = 1.$$

The paper is organized as follows. In Section 2, we establish the existence of two positive solutions of (1) and (2) under the general conditions that $f(t, u)$ is superlinear at one end of a cone, and sublinear at the other. In Section 3, we apply these results to prove the existence of multiple positive radial solutions of some semilinear elliptic equations in an annulus subject to certain boundary conditions.

2. Multiple Positive Solutions

It is well known that the BVP (1) and (2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t, s) f(s, u(s))\, ds,$$

where

$$G(t, s) = \frac{1}{\rho} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leqslant s \leqslant t \leqslant 1 \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \leqslant t \leqslant s \leqslant 1 \end{cases}$$

and consequently, it is equivalent to the fixed point equation

$$u = Fu$$

in $X = C[0, 1]$, with $F: X \to X$ given by

$$Fu = \int_0^1 G(t, s) f(s, u(s))\, ds.$$

It is obvious that F is completely continuous. We define a cone in X by

$$K = \{u \in X \mid u(t) \geqslant 0 \text{ and } \min_{1/4 \leqslant t \leqslant 3/4} u(t) \geqslant \sigma\, |u|\}. \tag{5}$$

Lemma 2. $F(K) \subset K$.

Proof. A direct calculation shows that

$$\frac{G(t, s)}{G(s, s)} \geqslant \sigma \qquad \text{for all } \frac{1}{4} \leqslant t \leqslant \frac{3}{4} \quad \text{and} \quad s \in [0, 1],$$

where $\sigma > 0$ is from (H4). Hence, for $u \in K$ we have

$$\begin{aligned}\min_{1/4 \leqslant t \leqslant 3/4} (Fu)(t) &= \min_{1/4 t \leqslant 3/4} \int_0^1 G(t, s) f(s, u(s))\, ds \\ &\geqslant \sigma \int_0^1 G(s, s) f(s, u(s))\, ds \\ &\geqslant \sigma \max_{0 \leqslant t \leqslant 1} \int_0^1 G(t, s) f(s, u(s))\, ds \\ &= \sigma\, |Fu|,\end{aligned}$$

i.e., $Fu \in K$. This completes the proof.

Now, we can prove

THEOREM 3. *Assume that $f(t, u)$ satisfies* (H1) *and* (H3). *Then, BVP* (1) *and* (2) *has at least two positive solutions x_1 and x_2 such that*

$$0 < |x_1| < p < |x_2|.$$

Proof. Choose $M > 0$ such that

$$\sigma M \int_{1/4}^{3/4} G(\tfrac{1}{2}, s)\, ds > 1. \tag{6}$$

By (H1), there is an $r > 0$ such that $r < p$, and $0 \leqslant u \leqslant r$ implies $f(t, u) \geqslant Mu$. We claim that $|F(u)| > |u|$ for $u \in \partial K_r$. In fact, for $u \in \partial K_r$,

$$\begin{aligned}(F(u))(\tfrac{1}{2}) &= \int_0^1 G(\tfrac{1}{2}, s) f(s, u(s))\, ds \\ &\geqslant \sigma M \int_{1/4}^{3/4} G(\tfrac{1}{2}, s)\, |u|\, ds \\ &> |u|.\end{aligned}$$

Hence, Lemma 1 implies

$$i(F, K_r, K) = 0. \tag{7}$$

For the same $M > 0$ satisfying (6), (H1) implies that there is $R_1 > 0$ such that $f(t, u) \geqslant Mu$ for all $u \geqslant R_1$. Choose $R > \max\{p, R_1/\sigma\}$. Since, for $u \in \partial K_R$, $\min_{1/4 \leqslant t \leqslant 3/4} u(t) \geqslant \sigma\, |u| = R_1$, we have for such u

$$(F(u))(\tfrac{1}{2}) = \int_0^1 G(\tfrac{1}{2}, s) f(s, u(s))\, ds$$

$$\geqslant \sigma M \int_{1/4}^{3/4} G(\tfrac{1}{2}, s)\, |u|\, ds$$

$$> |u|,$$

i.e., $|u| < |Fu|$ for $u \in \partial K_R$. Thus, Lemma 1 implies

$$i(F, K_R, K) = 0. \tag{8}$$

On the other hand, by (H3) for $u \in \partial K_p$,

$$|F(u)| = \max_{0 \leqslant t \leqslant 1} \int_0^1 G(t, s) f(s, u(s))\, ds$$

$$\leqslant \int_0^1 G(s, s) f(s, u(s))\, ds$$

$$\leqslant \int_0^1 G(s, s) \eta\, |u|\, ds$$

$$= |u|,$$

where $\eta > 0$ is from (4). Hence, $|F(u)| \leqslant |u|$ for $u \in \partial K_p$. It is obvious that $Fu \neq u$ for $u \in \partial K_p$. An application of Lemma 1 again shows that

$$i(F, K_p, K) = 1. \tag{9}$$

Now, the additivity of the fixed point index and (7), (8), (9) together implies

$$i(F, K_R \backslash \mathring{K}_p, K) = -1$$

and

$$i(F, K_p \backslash \mathring{K}_r, K) = 1.$$

Consequently, F has a fixed point x_1 in $K_R \backslash \mathring{K}_p$, and a fixed point x_2 in $K_p \backslash \mathring{K}_r$. Both are solutions of BVP (1) and (2). It is clear that $x_1(t) > 0$ and $x_2(t) > 0$ for $t \in (0, 1)$. The proof is therefore complete.

Remarks. (1) This theorem includes as a particular case a result in [7] which proves that the following BVP has two positive solutions:

$$\begin{cases} -u'' = u^\alpha + u^\beta, & 0 < t < 1 \\ x(0) = x'(1) = 0, \end{cases}$$

where $0 < \alpha < 1 < \beta$.

(2) One can easily replace (H1) by the following weaker condition

(H1)* $\lim_{u \to 0^+} (f(t, u)/u) = \infty \ \forall t$, $\lim_{u \to +\infty} (f(t, u)/u) = \infty \ \forall t$;

since an application of Egorov's theorem implies in particular that for any $\varepsilon > 0$, there is a closed $A \subset [0, 1]$ with $\mu(A^c) < \varepsilon$ such that the above limits are uniform on $t \in A$. Here, $A^c = \mathbb{R} \backslash A$, μ denotes the Lebesgue on $\mathbb{R}$. Thus, some straightforward modification will carry through the same proof.

THEOREM 4. *Assume that* $f(t, u)$ *satisfies* (H2) *and* (H4). *Then the BVP* (1) *and* (2) *has at least two positive solutions* x_1 *and* x_2 *such that*

$$0 < |x_1| < p < |x_2|.$$

Proof. By (H2), for any $\varepsilon > 0$, there is an $M > 0$ such that

$$f(t, u) \leqslant M + \varepsilon u \qquad \text{for} \quad u \geqslant 0, t \in [0, 1]. \tag{10}$$

Let K denote the same cone as given by (5). By (10), for $u \in K$,

$$(F(u))(t) \leqslant \int_0^1 G(t, s)[M + \varepsilon u(s)] \, ds.$$

Consequently, by choosing $\varepsilon > 0$ sufficiently small and $R > p$ sufficiently large, we have

$$|F(u)| < |u| \qquad \text{for} \quad u \in \partial K_R.$$

Therefore, by Lemma 1,

$$i(F, K_R, K) = 1. \tag{11}$$

Similarly, for some small $r > 0$, $r < p$,

$$i(F, K_r, K) = 1. \tag{12}$$

On the other hand, for $u \in \partial K_p$ we have

$$\min_{1/4 \leqslant t \leqslant 3/4} u(t) \geqslant \sigma \, |u| = \sigma p.$$

Hence, by (H4), for $u \in \partial K_p$

$$\begin{aligned} (Fu)(\tfrac{1}{2}) &= \int_0^1 G(\tfrac{1}{2}, s) f(s, u(s)) \, ds \\ &\geqslant \lambda p \int_{1/4}^{3/4} G(\tfrac{1}{2}, s) \, ds \\ &= p = |u|. \end{aligned}$$

It is clear that $Fu \neq u$ for $u \in \partial K_p$. Lemma 1 implies

$$i(F, K_p, K) = 0. \tag{13}$$

As before, (11), (12), and (13) show that F has two positive fixed points, and consequently, the BVP (1) and (2) has two positive solutions. This completes the proof.

Remark 3. It is clear that in the case that $f(t, u) = a(t)\, g(u)$ with $a(t)$ not identically zero in any interval, (H2) can be weakened to

(H2)* $\lim_{u \to 0^+} (g(u)/u) = 0$, $\lim_{u \to +\infty} (g(u)/u) = 0$.

3. Multiple Positive Radial Solutions

Since the search for radially symmetric solutions of certain semilinear elliptic BVPs can be reduced to the one dimensional BVP (1) and (2), we will apply the results obtained in Section 2 to study the existence of multiple positive radial solutions of the following BVP in $\mathbb{R}^N$ for $N \geqslant 2$:

$$-\Delta u = g(|x|) f(u), \qquad R_1 < |x| < R_2 \tag{14}$$

$$\begin{cases} \alpha u(x) - \beta \dfrac{\partial u}{\partial n}(x) = 0, & |x| = R_1 \\[2mm] \gamma u(x) - \delta \dfrac{\partial u}{\partial n}(x) = 0, & |x| = R_2. \end{cases} \tag{15}$$

For radial solutions, (14)–(15) is equivalent to

$$-u''(r) = \frac{N-1}{r} u'(r) + g(r) f(u(r)), \qquad R_1 < r < R_2 \tag{16}$$

$$\begin{cases} \alpha u(R_1) - \beta \dfrac{\partial u}{\partial n}(R_1) = 0 \\[2mm] \gamma u(R_2) - \delta \dfrac{\partial u}{\partial n}(R_2) = 0. \end{cases} \tag{17}$$

Letting $s = -\int_r^{R_2} (1/t^{N-1})\, dt$, $v(s) = u(r(s))$, $m = -\int_{R_1}^{R_2} (1/t^{N-1})\, dt$, it is seen that (16)–(17) is equivalent to

$$v''(s) + r^{2(N-1)} g(r(s)) f(v(s)) = 0, \qquad m < s < 0 \tag{18}$$

$$\begin{cases} \alpha v(m) + \beta R_1^{1-N} v'(m) = 0 \\ \gamma v(0) - \delta R_2^{1-N} v'(0) = 0. \end{cases} \tag{19}$$

Now, let $t=(m-s)/m$, $z(t)=v(s)$, so that (18)–(19) is then equivalent to

$$-z''(t)=h(t)f(z(t)), \qquad 0<t<1 \tag{20}$$

$$\begin{cases} \alpha z(0)-\beta \dfrac{R_1^{1-N}}{m} z'(0)=0 \\ \gamma z(1)+\delta \dfrac{R_2^{1-N}}{m} z'(1)=0, \end{cases} \tag{21}$$

where $h(t)=m^2 r^{2(N-1)}[m(1-t)]\, g[r(m(1-t))]$.

Let G^* denote the Green's function of $-z''(t)=0$, $0\leqslant t\leqslant 1$, subject to boundary condition (21). Thus from (4), we may define

$$\eta^*=\left(\int_0^1 G^*(s,s)\,ds\right)^{-1}$$
$$=\frac{6(m\gamma\beta R_1^{1-N}+m^2\alpha\gamma+m\alpha\,\delta R_2^{1-N})}{6\delta\beta(R_1R_2)^{1-N}+3m\gamma\beta R_1^{1-N}+m^2\alpha\gamma+3m\alpha\,\delta R_2^{1-N}}.$$

Applying Theorem 3 to (20) and (21), we immediately have

THEOREM 5. *The BVP* (14) *and* (15) *has at least two positive radial solutions, provided that*

(1) $\lim_{u\to 0^+}(f(u)/u)=\infty$, $\lim_{u\to\infty}(f(u)/u)=\infty$, *and*

(2) *there is a* $p>0$ *such that for all* $r\in[R_1,R_2]$ *and* $0\leqslant u\leqslant p$,

$$m^2 r^{2(N-1)} g(r) f(u)\leqslant \eta^* p.$$

As a special case of (14) and (15), consider

$$\begin{cases} -\Delta u=g(|x|)f(u), & R_1<|x|<R_2 \\ u(x)=0 & \text{on } |x|=R_1 \text{ and } |x|=R_2. \end{cases} \tag{22}$$

Since in this special situation $\eta^*=6$, we have as a consequence of Theorem 5,

COROLLARY 6. *The BVP* (22) *has at least two positive radial solutions, provided that*

(1) $\lim_{u\to 0^+}(f(u)/u)=\infty$, $\lim_{u\to\infty} f(u)/u)=\infty$, *and*

(2) *there is a* $p>0$ *such that for all* $r\in[R_1,R_2]$ *and* $0\leqslant u\leqslant p$

$$g(r)f(u)\leqslant 6\left(\frac{R_2^N-R_1^N}{2-N}\right)^2 p.$$

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