

TRAVELING WAVE SOLUTIONS OF A REACTION-DIFFUSION EQUATION WITH STATE-DEPENDENT DELAY

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ABSTRACT. This paper is concerned with the traveling wave solutions of a reaction-diffusion equation with state-dependent delay. When the birth function is monotone, the existence and nonexistence of monotone traveling wave solutions are established. When the birth function is not monotone, the minimal wave speed of nontrivial traveling wave solutions is obtained. The results are proved by the construction of upper and lower solutions and application of the fixed point theorem.

1. Introduction. Differential equations with state-dependent delay are useful to describe population dynamics in which the amount of food available per biomass for a fixed food supply is a function of the total consumer biomass [1]. Intensive research efforts have been devoted to understanding the dynamics of differential equations with state-dependent delay. Andrewartha and Birch [2, pp. 370] studied a differential equation with state-dependent in which the duration of larval development of flies is a nonlinear increasing function of larval density. There is a wealth of literature on the research and we refer to Arino et al. [3], Cooke and Huang [4], Hartung et al. [6], Hu et al. [7], Magal and Arino [12], Mallet-Paret and Nussbaum [13], Walther [17] and the references cited therein. Despite that the dynamics of functional differential equations has been widely studied, the spatial-temporal patterns of differential equation with state-dependent delay are hardly understood.

In this paper, we consider the existence and nonexistence of traveling wave solutions of the following reaction-diffusion equation with state-dependent delay

$$u_t(x, t) = u_{xx}(x, t) - du(x, t) + b(u(x, t - \tau(u(x, t))))), \quad (1)$$

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where $x \in \mathbb{R}, t > 0$. In population dynamics, $d > 0$ is the death rate and $b : [0, \infty) \rightarrow [0, \infty)$ is the birth function satisfying

(B): $b(0) = 0, b(K) = K$ for some positive constant $K > 0$, and $b(u) > du, u \in (0, K)$ while $0 < b(u) < du, u > K$,

which will be imposed throughout this paper. In this model, time delay is not a constant and $\tau : [0, \infty) \rightarrow [0, \infty)$ satisfies the following assumptions:

(A1): $\tau(u)$ is C^1 for $u \in [0, \infty)$;

(A2): $0 \leq \tau'(u) < 1, u \in [0, \infty)$;

(A3): $0 \leq m = \tau(0) \leq \lim_{u \rightarrow \infty} \tau(u) = M < \infty$.

To continue our discussion, we first present the following definition.

Definition 1.1. A traveling wave solution of (1) is a special entire solution defined for all $x, t \in \mathbb{R}$ and taking the following form

$$u(x, t) = \phi(\xi), \xi = x + ct \in \mathbb{R},$$

where $c > 0$ is the wave speed and $\phi \in C^2(\mathbb{R}, \mathbb{R})$ is the wave profile. In particular, if $\phi(\xi)$ is monotone increasing, then it is a traveling wavefront.

By the above definition, ϕ and c must satisfy the following functional differential equation of second order

$$\phi''(\xi) - c\phi'(\xi) - d\phi(\xi) + b(\phi(\xi - c\tau(\phi(\xi)))) = 0, \xi \in \mathbb{R}. \quad (2)$$

In particular, to model precise transition processes in evolutionary systems by traveling wave solutions, we also consider the following asymptotic boundary conditions

$$\lim_{\xi \rightarrow -\infty} \phi(\xi) = 0, \lim_{\xi \rightarrow +\infty} \phi(\xi) = K \quad (3)$$

or a weaker version

$$\lim_{\xi \rightarrow -\infty} \phi(\xi) = 0, \liminf_{\xi \rightarrow +\infty} \phi(\xi) > 0. \quad (4)$$

When $\tau'(u) = 0$, the traveling wave solutions of (1) have been widely studied since Schaaf [14]. For monotone $b(u)$ with $\tau'(u) = 0$, comparison principle is admissible, the existence and nonexistence of monotone traveling wave solutions can be studied by monotone iteration, fixed point theorem and monotone semiflows, see Liang and Zhao [8], Ma [10], Wang et al. [19], Wu and Zou [20]. If $b(u)$ is locally monotone and $\tau'(u) = 0$ holds, then the traveling wave solutions of (1) can be studied by constructing auxiliary monotone equations, see Fang and Zhao [5], Ma [11], Wang [18].

However, when $\tau(u)$ is not a constant, because $\tau(u)$ is nondecreasing, the comparison principle in (1) does not hold even if $b(u)$ is monotone increasing. Therefore, the study of traveling wave solutions of (1) needs some new techniques. In this paper, similar to that in Wu and Zou [20], we first introduce an integral operator to study the existence of traveling wave solutions. Based on some estimations, we confirm the comparison principle on a proper subset of the space of continuous functions if $b(u)$ is monotone. Then the existence of (2)-(3) is proved by combining Schauder's fixed point theorem with upper and lower solutions if the wave speed is larger than a threshold c^* defined later. When the wave speed is c^* , we establish the existence of (2)-(3) by passing to a limit. If the wave speed is less than c^* , the nonexistence of (2)-(3) is confirmed. Therefore, c^* is the minimal wave speed of (2)-(3).

If $b(u)$ is not monotone, similar to Ma [11], we introduce two auxiliary monotone equations to confirm the existence of (2) with (4) for $c > c^*$. If $c = c^*$, the existence of nontrivial solutions of (2) is also proved by passing to a limit function. When $c < c^*$, we obtain an auxiliary equation with fixed time delay by (4). Applying the theory of asymptotic spreading, we establish the nonexistence of traveling wave solutions.

2. Monotone birth function. In this section, we investigate the existence of traveling wavefronts of (1), namely, existence of monotone solutions of (2)-(3), if $b : [0, \infty) \rightarrow [0, \infty)$ satisfies the following assumptions:

- (B1): $b(u)$ is C^1 for $u \in [0, K]$;
- (B2): $b'(0) > d$ and $b(u) < b'(0)u, u \in [0, K]$;
- (B3): $0 \leq b'(u) \leq b'(0), u \in [0, K]$;
- (B4): $0 \leq b'(0)u - b(u) < Lu^2$ for $u \in [0, K]$ and some $L > 0$.

It is easy to see that if $b(u) = pue^{-u}$, then (B1)-(B4) hold when $1 < p/d \leq e$. In particular, (B1)-(B4) will be imposed throughout this section without further illustration. Furthermore, we also denote

$$T = \sup_{u \in [0, K]} \tau'(u).$$

2.1. Some estimations. Let $\beta > d$ hold and define $H : C_{[0, K]} \rightarrow C$ by

$$H(\phi)(\xi) = \beta\phi(\xi) - d\phi(\xi) + b(\phi(\xi - c\tau(\phi(\xi)))). \tag{5}$$

Further define an operator $F : C_{[0, K]} \rightarrow C$ by

$$F(\phi)(\xi) = \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi} e^{\gamma_1(c)(\xi-s)} + \int_{\xi}^{\infty} e^{\gamma_2(c)(\xi-s)} \right] H(\phi)(s)ds, \tag{6}$$

where

$$\gamma_1(c) = \frac{c - \sqrt{c^2 + 4\beta}}{2}, \gamma_2(c) = \frac{c + \sqrt{c^2 + 4\beta}}{2}.$$

It is clear that a fixed point of F is a solution of (2), and a solution of (2) is a fixed point of F . Therefore, to study the existence of solutions of (2), it suffices to investigate the existence of fixed points of F .

Lemma 2.1. *If $\phi(\xi) \in C_{[0, K]}$, then $0 \leq H(\phi)(\xi) \leq \beta K$.*

Lemma 2.2. *Assume that $c > 0$ is fixed and*

$$4\beta \geq (1 + b'(0)KT)^2 c^2 + 4d.$$

If $\phi(\xi) \in C_{[0, K]}$, then $|c(F(\phi))'(\xi)| < \frac{\beta K}{1+b'(0)KT}$.

Proof. By the definition of F , we have

$$\begin{aligned} & c \left| \frac{d}{d\xi} F(\phi)(\xi) \right| \\ = & c \left| \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\gamma_1(c) \int_{-\infty}^{\xi} e^{\gamma_1(c)(\xi-s)} + \gamma_2(c) \int_{\xi}^{\infty} e^{\gamma_2(c)(\xi-s)} \right] H(\phi)(s)ds \right| \\ < & c \max \left\{ \left| \frac{\gamma_2(c)}{\gamma_2(c) - \gamma_1(c)} \int_{\xi}^{\infty} e^{\gamma_2(c)(\xi-s)} H(\phi)(s)ds \right|, \right. \end{aligned}$$

$$\left\{ \frac{\gamma_1(c)}{\gamma_2(c) - \gamma_1(c)} \int_{-\infty}^{\xi} e^{\gamma_1(c)(\xi-s)} H(\phi)(s) ds \right\} \leq \frac{c\beta K}{\gamma_2(c) - \gamma_1(c)} = \frac{\beta c K}{\sqrt{c^2 + 4\beta}}$$

$$< \frac{\beta K}{1 + b'(0)KT}.$$

The proof is complete. \square

Lemma 2.3. *Assume that $c > 0$ is fixed. If*

$$\beta \geq d(1 + b'(0)KT) + \frac{(1 + b'(0)KT)^2 c^2}{4},$$

then

$$\beta\phi(\xi) - d\phi(\xi) + b(\phi(\xi) - c\tau(\phi(\xi)))$$

is monotone nondecreasing in $\phi(\xi) \in C_{[0,K]}$ provided that

$$c|\phi(\xi_1) - \phi(\xi_2)| \leq \frac{\beta K}{1 + b'(0)KT} |\xi_1 - \xi_2|, \xi_1, \xi_2 \in \mathbb{R}.$$

Proof. Let $\phi_1(\xi), \phi_2(\xi)$ satisfy $\phi_1(\xi) \geq \phi_2(\xi)$ and

$$c|\phi_1(\xi_1) - \phi_1(\xi_2)| \leq \frac{\beta K}{1 + b'(0)KT} |\xi_1 - \xi_2|, \xi_1, \xi_2 \in \mathbb{R},$$

$$c|\phi_2(\xi_1) - \phi_2(\xi_2)| \leq \frac{\beta K}{1 + b'(0)KT} |\xi_1 - \xi_2|, \xi_1, \xi_2 \in \mathbb{R}.$$

By (B3), we have

$$b(\phi_1(\xi - c\tau(\phi_1(\xi)))) \geq b(\phi_2(\xi - c\tau(\phi_1(\xi))))$$

and

$$\begin{aligned} & b(\phi_1(\xi - c\tau(\phi_1(\xi)))) - b(\phi_2(\xi - c\tau(\phi_2(\xi)))) \\ &= b(\phi_1(\xi - c\tau(\phi_1(\xi)))) - b(\phi_2(\xi - c\tau(\phi_1(\xi)))) \\ & \quad + b(\phi_2(\xi - c\tau(\phi_1(\xi)))) - b(\phi_2(\xi - c\tau(\phi_2(\xi)))) \\ & \geq b(\phi_2(\xi - c\tau(\phi_1(\xi)))) - b(\phi_2(\xi - c\tau(\phi_2(\xi)))) \\ & \geq -b'(0) |\phi_2(\xi - c\tau(\phi_1(\xi))) - \phi_2(\xi - c\tau(\phi_2(\xi)))| \\ & \geq -\frac{b'(0)K\beta}{1 + b'(0)KT} |\tau(\phi_1(\xi)) - \tau(\phi_2(\xi))| \\ & \geq -\frac{b'(0)KT\beta}{1 + b'(0)KT} (\phi_1(\xi) - \phi_2(\xi)). \end{aligned}$$

Then the result follows from the definition of β . The proof is complete. \square

Lemma 2.4. *Assume that $c > 0$ is fixed. If*

$$\beta \geq d(1 + b'(0)KT) + \frac{(1 + b'(0)KT)^2 c^2}{4},$$

then

$$\beta\phi(\xi) - d\phi(\xi) + b(\phi(\xi) - c\tau(\phi(\xi)))$$

is monotone nondecreasing in ξ provided that $\phi(\xi) \in C_{[0,K]}$ is monotone nondecreasing in $\xi \in \mathbb{R}$ and

$$c|\phi(\xi_1) - \phi(\xi_2)| \leq \frac{\beta K}{1 + b'(0)KT} |\xi_1 - \xi_2|, \xi_1, \xi_2 \in \mathbb{R}.$$

Proof. If $\phi(\xi) \in C_{[0,K]}$ is monotone nondecreasing in $\xi \in \mathbb{R}$, then

$$\phi(\xi_1) - \phi(\xi_2) \geq 0$$

for any $\xi_1 \geq \xi_2$. From the monotonicity of b , we further have

$$b(\phi(\xi_1 - c\tau(\phi(\xi_1)))) \geq b(\phi(\xi_2 - c\tau(\phi(\xi_1))))$$

and

$$\begin{aligned} & b(\phi(\xi_1 - c\tau(\phi(\xi_1)))) - b(\phi(\xi_2 - c\tau(\phi(\xi_2)))) \\ &= b(\phi(\xi_1 - c\tau(\phi(\xi_1)))) - b(\phi(\xi_2 - c\tau(\phi(\xi_1)))) \\ & \quad + b(\phi(\xi_2 - c\tau(\phi(\xi_1)))) - b(\phi(\xi_2 - c\tau(\phi(\xi_2)))) \\ & \geq b(\phi(\xi_2 - c\tau(\phi(\xi_1)))) - b(\phi(\xi_2 - c\tau(\phi(\xi_2)))) \\ & \geq -b'(0) |\phi(\xi_2 - c\tau(\phi(\xi_1))) - \phi(\xi_2 - c\tau(\phi(\xi_2)))| \\ & \geq -\frac{b'(0)K\beta}{1+b'(0)KT} |\tau(\phi(\xi_1)) - \tau(\phi(\xi_2))| \\ & \geq -\frac{b'(0)KT\beta}{1+b'(0)KT} (\phi(\xi_1) - \phi(\xi_2)). \end{aligned}$$

Then the result is clear by the definition of β . The proof is complete. □

2.2. Upper and lower solutions. To investigate the existence of (2)-(3), we will use the upper and lower solutions defined as follows.

Definition 2.5. A continuous function $\bar{\phi}(\xi) \in C_{[0,K]}(\mathbb{R}, \mathbb{R})$ is an upper solution of (2) if there exist constants $T_i, i = 1, \dots, l$, such that $\bar{\phi}''(\xi), \bar{\phi}'(\xi)$ exist and are bounded for all $\xi \in \mathbb{R} \setminus \{T_i : i = 1, \dots, l\}$ and satisfy

$$\bar{\phi}''(\xi) - c\bar{\phi}'(\xi) - d\bar{\phi}(\xi) + b(\bar{\phi}(\xi - c\tau(\bar{\phi}(\xi)))) \leq 0, \quad \xi \in \mathbb{R} \setminus \{T_i : i = 1, \dots, l\}. \quad (7)$$

A lower solution can be similarly defined by inverting the inequality.

To construct upper and lower solutions, we define some constants. For $\lambda \geq 0, c \geq 0$, we first define

$$\Lambda(\lambda, c) = \lambda^2 - c\lambda - d + b'(0)e^{-\lambda m}.$$

Lemma 2.6. *There exists $c^* > 0$ such that $\Lambda(\lambda, c) > 0$ if $c \in [0, c^*), \lambda \geq 0$ while $\Lambda(\lambda, c) = 0$ has two positive distinct real roots $\lambda_1(c) < \lambda_2(c)$ if $c > c^*$. Moreover, $\Lambda(\lambda, c)$ also satisfies*

$$\Lambda(\lambda, c) \begin{cases} < 0, \lambda \in (\lambda_1(c), \lambda_2(c)), \\ > 0, \lambda \in [0, \lambda_1(c)) \cup (\lambda_2(c), \infty). \end{cases}$$

Lemma 2.7. *There exist $L_1, L_2 > 0$ such that*

$$c\lambda_1(c) < L_1, \lambda_1(c) < L_2, c > c^*.$$

Proof. Obviously, we have

$$\Lambda(2(b'(0) - d)/c, c) < 0$$

if $c > 0$ is large. Then we complete the proof. □

We now consider the existence of traveling wave solutions if $c > c^*$ is a constant and define continuous functions as follows

$$\bar{\phi}(\xi) = \min\{e^{\lambda_1(c)\xi}, K\}, \underline{\phi}(\xi) = \min\{e^{\lambda_1(c)\xi} - qe^{\eta\lambda_1(c)\xi}, 0\}, \quad (8)$$

where

$$1 < \eta < \min\left\{2, \frac{\lambda_2(c)}{\lambda_1(c)}\right\}, q = \frac{3L_2b'(0) + L}{-\Lambda(\eta\lambda_1(c), c)} + 1 + K > 1 + K.$$

Lemma 2.8. *For any $c > c^*$, there exists $\beta_1 > 0$ such that*

$$\begin{aligned} c|\bar{\phi}(\xi_1) - \bar{\phi}(\xi_2)| &\leq \frac{\beta K}{1 + b'(0)KT} |\xi_1 - \xi_2|, \\ c|\underline{\phi}(\xi_1) - \underline{\phi}(\xi_2)| &\leq \frac{\beta K}{1 + b'(0)KT} |\xi_1 - \xi_2| \end{aligned}$$

for any $\beta \geq \beta_1, c > c^*, \xi_1, \xi_2 \in \mathbb{R}$.

Proof. Let L_1 be defined by Lemma 2.7. If $\bar{\phi}(\xi) = e^{\lambda_1(c)\xi} < K$, then

$$c\bar{\phi}'(\xi) = c\lambda_1(c)e^{\lambda_1(c)\xi} < L_1K.$$

When $\bar{\phi}(\xi) = K < e^{\lambda_1(c)\xi}$, then $c\bar{\phi}'(\xi) = 0$ and

$$c|\bar{\phi}(\xi_1) - \bar{\phi}(\xi_2)| \leq 3L_1K|\xi_1 - \xi_2|, \xi_1, \xi_2 \in \mathbb{R}.$$

If $\underline{\phi}(\xi) = e^{\lambda_1(c)\xi} - qe^{\eta\lambda_1(c)\xi} > 0$, then

$$\begin{aligned} |c\underline{\phi}'(\xi)| &= c \left| \lambda_1(c)e^{\lambda_1(c)\xi} - q\eta\lambda_1(c)e^{\eta\lambda_1(c)\xi} \right| \leq c \left| \lambda_1(c)e^{\lambda_1(c)\xi} \right| + c \left| q\eta\lambda_1(c)e^{\eta\lambda_1(c)\xi} \right| \\ &< 3L_1K, \end{aligned}$$

which implies that

$$c|\underline{\phi}(\xi_1) - \underline{\phi}(\xi_2)| \leq 3L_1K|\xi_1 - \xi_2|, \xi_1, \xi_2 \in \mathbb{R}.$$

Let

$$3L_1K = \frac{\beta_1K}{1 + b'(0)KT}.$$

Then the proof is complete. \square

Remark 1. From the above proofs, we can fix $\beta > d$ such that Lemmas 2.1-2.8 hold for each fixed $c > c^*$.

Lemma 2.9. *If $\bar{\phi}(\xi)$ and $\underline{\phi}(\xi)$ are defined by (8), then $\bar{\phi}(\xi)$ is an upper solution of (2) while $\underline{\phi}(\xi)$ is a lower solution of (2).*

Proof. If $\bar{\phi}(\xi) = e^{\lambda_1(c)\xi} < K$, then

$$\begin{aligned} &\bar{\phi}''(\xi) - c\bar{\phi}'(\xi) - d\bar{\phi}(\xi) + b(\bar{\phi}(\xi - c\tau(\bar{\phi}(\xi)))) \\ &\leq \bar{\phi}''(\xi) - c\bar{\phi}'(\xi) - d\bar{\phi}(\xi) + b'(0)\bar{\phi}(\xi - c\tau(\bar{\phi}(\xi))) \\ &\leq \bar{\phi}''(\xi) - c\bar{\phi}'(\xi) - d\bar{\phi}(\xi) + b'(0)\bar{\phi}(\xi - cm) \\ &= e^{\lambda_1(c)\xi}\Delta(\lambda_1(c), c) \\ &= 0. \end{aligned}$$

When $\bar{\phi}(\xi) = K < e^{\lambda_1(c)\xi}$, then $\bar{\phi}(\xi - c\tau(\bar{\phi}(\xi))) \leq K$ and

$$\bar{\phi}''(\xi) - c\bar{\phi}'(\xi) - d\bar{\phi}(\xi) + b(\bar{\phi}(\xi - c\tau(\bar{\phi}(\xi)))) \leq -dK + b(K) = 0.$$

If $\underline{\phi}(\xi) = 0 > e^{\lambda_1(c)\xi} - qe^{\eta\lambda_1(c)\xi}$, then

$$\begin{aligned} &\underline{\phi}''(\xi) - c\underline{\phi}'(\xi) - d\underline{\phi}(\xi) + b(\underline{\phi}(\xi - c\tau(\underline{\phi}(\xi)))) \\ &= \underline{\phi}''(\xi) - c\underline{\phi}'(\xi) - d\underline{\phi}(\xi) + b(\underline{\phi}(\xi - cm)) \geq 0. \end{aligned}$$

If $\underline{\phi}(\xi) = e^{\lambda_1(c)\xi} - qe^{\eta\lambda_1(c)\xi} > 0$, then

$$|\underline{\phi}'(\xi)| = \left| \lambda_1(c)e^{\lambda_1(c)\xi} - q\eta\lambda_1(c)e^{\eta\lambda_1(c)\xi} \right| < \lambda_1(c)e^{\lambda_1(c)\xi} + \eta\lambda_1(c)qe^{\eta\lambda_1(c)\xi} < 3\lambda_1(c)e^{\lambda_1(c)\xi}$$

and

$$\begin{aligned} & \underline{\phi}''(\xi) - c\underline{\phi}'(\xi) - d\underline{\phi}(\xi) + b(\underline{\phi}(\xi - c\tau(\underline{\phi}(\xi)))) \\ & \geq \underline{\phi}''(\xi) - c\underline{\phi}'(\xi) - d\underline{\phi}(\xi) + b'(0)\underline{\phi}(\xi - c\tau(\underline{\phi}(\xi))) - L\underline{\phi}^2(\xi - c\tau(\underline{\phi}(\xi))) \\ & = \underline{\phi}''(\xi) - c\underline{\phi}'(\xi) - d\underline{\phi}(\xi) + b'(0)\underline{\phi}(\xi - cm) \\ & \quad + b'(0) [\underline{\phi}(\xi - c\tau(\underline{\phi}(\xi))) - \underline{\phi}(\xi - cm)] - L\underline{\phi}^2(\xi - c\tau(\underline{\phi}(\xi))) \\ & \geq -\Delta(\eta\lambda_1(c), c)qe^{\eta\lambda_1(c)\xi} - 3\lambda_1(c)b'(0)e^{2\lambda_1(c)\xi} - Le^{2\lambda_1(c)\xi} \\ & \geq -\Delta(\eta\lambda_1(c), c)qe^{\eta\lambda_1(c)\xi} - (3L_2b'(0) + L)e^{2\lambda_1(c)\xi} \\ & > 0 \end{aligned}$$

by the definition of q . The proof is complete. □

2.3. Existence of monotone traveling wave solutions: $c > c^*$. In this part, we assume that $c > c^*$ is fixed and prove the existence of fixed points of F by Schauder's fixed point theorem. Moreover, $\beta > 0$ is a fixed constant satisfying Remark 1. Let $\mu \in (0, -\gamma_1(c))$ be a constant and

$$B_\mu(\mathbb{R}, \mathbb{R}) = \left\{ \phi \in C(\mathbb{R}, \mathbb{R}) : \sup_{\xi \in \mathbb{R}} |\phi(t)| e^{-\mu|\xi|} < \infty \right\}$$

and

$$|\phi|_\mu = \sup_{\xi \in \mathbb{R}} |\phi(\xi)| e^{-\mu|\xi|} \text{ for } \phi \in B_\mu(\mathbb{R}, \mathbb{R}).$$

Then it is easy to show that $(B_\mu(\mathbb{R}, \mathbb{R}), |\cdot|_\mu)$ is a Banach space.

Define

$$\Gamma(\underline{\phi}, \bar{\phi}) = \left\{ \begin{array}{l} \phi(\xi) \in B_\mu(\mathbb{R}, \mathbb{R}) : \\ \begin{array}{l} (i) \ \underline{\phi}(\xi) \leq \phi(\xi) \leq \bar{\phi}(\xi); \\ (ii) \ \phi(\xi) \text{ is nondecreasing}; \\ (iii) \ c|\phi(\xi_1) - \phi(\xi_2)| \leq \frac{\beta K}{1+b'(0)KT} |\xi_1 - \xi_2|, \xi_1, \xi_2 \in \mathbb{R} \end{array} \end{array} \right\}.$$

It follows from the above results that Γ is nonempty and convex. Moreover, we can verify that Γ is closed and bounded with respect to the norm $|\cdot|_\mu$.

Lemma 2.10. *F admits the following properties.*

- (1): *If $\phi(\xi) \in \Gamma$, then $F(\phi)(\xi)$ is monotone increasing.*
- (2): *If $\phi_1(\xi), \phi_2(\xi) \in \Gamma$ with $\phi_1(\xi) \leq \phi_2(\xi)$, then $F(\phi_1)(\xi) \leq F(\phi_2)(\xi)$.*
- (3): *If $\phi(\xi) \in \Gamma$, then $c|(F(\phi)(\xi))'| < \frac{\beta K}{1+b'(0)KT}$.*

Lemma 2.10 is a consequence of Lemmas 2.1-2.3, and we omit the detailed proof here.

Lemma 2.11. $F : \Gamma \rightarrow \Gamma$.

Proof. From Lemma 2.10, it suffices to verify that

$$\underline{\phi}(\xi) \leq F(\underline{\phi})(\xi) \leq F(\bar{\phi})(\xi) \leq \bar{\phi}(\xi), \xi \in \mathbb{R}.$$

By the definition of upper solution, we see that

$$\begin{aligned} & F(\bar{\phi})(\xi) \\ &= \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi} e^{\gamma_1(c)(\xi-s)} + \int_{\xi}^{\infty} e^{\gamma_2(c)(\xi-s)} \right] H(\bar{\phi})(s) ds \\ &\leq \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi} e^{\gamma_1(c)(\xi-s)} + \int_{\xi}^{\infty} e^{\gamma_2(c)(\xi-s)} \right] \left(\beta \bar{\phi}(s) + c \bar{\phi}'(s) - \bar{\phi}''(s) \right) ds \\ &= \bar{\phi}(\xi) - \frac{\min \left\{ \lambda_1(c) K e^{\gamma_2(c) \left(\xi - \frac{\ln K}{\lambda_1(c)} \right)}, \lambda_1(c) K e^{\gamma_1(c) \left(\xi - \frac{\ln K}{\lambda_1(c)} \right)} \right\}}{\gamma_2(c) - \gamma_1(c)} \\ &\leq \bar{\phi}(\xi), \quad \xi \neq \frac{\ln K}{\lambda_1(c)}. \end{aligned}$$

Then the continuity implies that

$$\bar{\phi}(\xi) \geq F(\bar{\phi})(\xi), \xi \in \mathbb{R}.$$

In a similar way, we can prove that

$$F(\underline{\phi})(\xi) \geq \underline{\phi}(\xi), \xi \in \mathbb{R}.$$

The proof is complete. \square

Lemma 2.12. $F : \Gamma \rightarrow \Gamma$ is complete continuous in the sense of $|\cdot|_{\mu}$.

Proof. If $\phi_1, \phi_2 \in \Gamma$, then

$$\begin{aligned} & |b(\phi_1(\xi - c\tau(\phi_1(\xi)))) - b(\phi_2(\xi - c\tau(\phi_2(\xi))))| \\ &\leq |b(\phi_1(\xi - c\tau(\phi_1(\xi)))) - b(\phi_2(\xi - c\tau(\phi_1(\xi))))| \\ &\quad + |b(\phi_2(\xi - c\tau(\phi_1(\xi)))) - b(\phi_2(\xi - c\tau(\phi_2(\xi))))| \\ &\leq b'(0) |\phi_2(\xi - c\tau(\phi_1(\xi))) - \phi_1(\xi - c\tau(\phi_1(\xi)))| + b'(0) \beta K c T |\phi_2(\xi) - \phi_1(\xi)| \end{aligned}$$

and so

$$\begin{aligned} & |F(\phi_1)(\xi) - F(\phi_2)(\xi)| \\ &\leq \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi} e^{\gamma_1(c)(\xi-s)} ds + \int_{\xi}^{\infty} e^{\gamma_2(c)(\xi-s)} \right] |H(\phi_1)(s) - H(\phi_2)(s)| ds \\ &\leq \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi} e^{\gamma_1(c)(\xi-s)} ds + \int_{\xi}^{\infty} e^{\gamma_2(c)(\xi-s)} \right] \\ &\quad \left[(b'(0) \beta K c T + \beta - d) |\phi_2(s) - \phi_1(s)| + \right. \\ &\quad \left. b'(0) |\phi_2(s - c\tau(\phi_1(s))) - \phi_1(s - c\tau(\phi_1(s)))| \right] ds. \end{aligned}$$

Applying these estimations, we have

$$\begin{aligned} & |F(\phi_1)(\xi) - F(\phi_2)(\xi)| e^{-\mu|\xi|} \\ &\leq \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi} e^{(\gamma_1(c)+\mu)(\xi-s)} + \int_{\xi}^{\infty} e^{(\gamma_2(c)-\mu)(\xi-s)} \right] \\ &\quad (b'(0) \beta K c T + \beta - d) |\phi_2(s) - \phi_1(s)| e^{-\mu|s|} ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{e^{\mu c M} b'(0)}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi} e^{(\gamma_1(c) + \mu)(\xi - s)} ds + \int_{\xi}^{\infty} e^{(\gamma_2(c) - \mu)(\xi - s)} \right] \\
 & |\phi_2(s - c\tau(\phi_1(s))) - \phi_1(s - c\tau(\phi_1(s)))| e^{-\mu|s - c\tau(\phi_1(s))|} ds \\
 & \leq \frac{b'(0)\beta K c T + \beta - d + e^{\mu c M} b'(0)}{\gamma_2(c) - \gamma_1(c)} \left[\frac{1}{\gamma_2(c) - \mu} - \frac{1}{\gamma_1(c) + \mu} \right] |\phi_2 - \phi_1|_{\mu},
 \end{aligned}$$

which implies the continuity in the sense of $|\cdot|_{\mu}$.

Due to Lemma 2.1, it is easy to prove the compactness and we omit the details. The proof is complete. \square

We now state the main conclusion of this part.

Theorem 2.13. *For every $c > c^*$, (2) with (3) has a monotone solution.*

Proof. Applying Schauder’s fixed point theorem, it is not hard to show the existence of fixed point of F in Γ . Denote the fixed point by $\phi(\xi)$, then the monotonicity implies the existence of $\lim_{\xi \rightarrow -\infty} \phi(\xi)$, $\lim_{\xi \rightarrow +\infty} \phi(\xi)$. Moreover, $\lim_{\xi \rightarrow -\infty} \phi(\xi) = 0$ is obtained by the asymptotic behavior of upper and lower solutions. At the same time, the definition of $\underline{\phi}(\xi)$ leads to

$$0 < \lim_{\xi \rightarrow +\infty} \phi(\xi) \leq K.$$

Then it is easy to verify that $\lim_{\xi \rightarrow +\infty} \phi(\xi) = K$. The proof is complete. \square

2.4. Existence of monotone traveling wavefronts: $c = c^*$. In this part, we shall establish the existence of monotone solutions of (2)-(3) if $c = c^*$. We first present the main results as follows.

Theorem 2.14. *If $c = c^*$, then (2) with (3) has a monotone solution.*

Proof. Let $\{c_n\}_{n \in \mathbb{N}}$ be a decreasing sequence satisfying

$$c_n \rightarrow c^*, n \rightarrow \infty.$$

Note that $\{c_n\}_{n \in \mathbb{N}}$ is bounded, we can select a $\beta > 0$ such that Lemmas 2.2-2.4 hold for all $c_n, n \in \mathbb{N}$. In particular, we have

$$\gamma_1(c_n) \rightarrow \gamma_1(c^*), \gamma_2(c_n) \rightarrow \gamma_2(c^*), n \rightarrow \infty.$$

Then Theorem 2.13 implies that for each c_n , F with $c = c_n$ has a fixed point $\phi_n(\xi)$, where $\phi_n(\xi)$ is monotone increasing. Because a traveling wavefront is invariant in the sense of phase shift, we assume that

$$\phi_n(0) = K/2, n \in \mathbb{N}. \tag{9}$$

From Lemma 2.2, $\phi_n(\xi)$ is equicontinuous and uniformly bounded for all $n \in \mathbb{N}, \xi \in \mathbb{R}$. By Ascoli-Arzela Lemma and a standard nested subsequence argument, it follows that there exists a subsequence of $\{c_n\}_{n \in \mathbb{N}}$, still denoted by $\{c_n\}_{n \in \mathbb{N}}$, such that the corresponding $\{\phi_n(\xi)\}_{n \in \mathbb{N}}$ converges uniformly on every bounded interval of $\xi \in \mathbb{R}$, and hence pointwise for $\xi \in \mathbb{R}$ to a function $\phi^*(\xi)$. Moreover, it is evident that

$$\min\{e^{\gamma_1(c_n)(\xi - s)}, e^{\gamma_1(c_n)(\xi - s)}\} \rightarrow \min\{e^{\gamma_1(c^*)(\xi - s)}, e^{\gamma_1(c^*)(\xi - s)}\}, n \rightarrow \infty$$

and the convergence is uniform in $\xi, s \in \mathbb{R}$.

Let $n \rightarrow \infty$ in F , then the dominated convergence theorem implies that

$$F(\phi^*)(\xi) = \frac{1}{\gamma_2(c^*) - \gamma_1(c^*)} \left[\int_{-\infty}^{\xi} e^{\gamma_1(c^*)(\xi-s)} + \int_{\xi}^{\infty} e^{\gamma_2(c^*)(\xi-s)} \right] H(\phi^*)(s) ds$$

by the uniform continuity of $b(u), \tau(u)$. Therefore, $\phi^*(\xi)$ is a fixed point of F with $c = c^*$, and a bounded solution of (2) with $c = c^*$. In particular, Lemma 2.2 tells us that $\phi^*(\xi)$ is uniformly continuous.

By (9), we see that $\phi^*(0) = K/2$. Moreover, the monotonicity and boundedness of $\phi_n(\xi)$ indicate that $\phi^*(\xi)$ is monotone increasing and bounded for $\xi \in \mathbb{R}$, which further implies that $\lim_{\xi \rightarrow \pm\infty} \phi^*(\xi)$ exist. Denote

$$\lim_{\xi \rightarrow \pm\infty} \phi^*(\xi) = \phi_{\pm}.$$

Then the uniform continuity and the dominated convergence theorem in F lead to

$$\phi_{\pm} = \frac{\beta\phi_{\pm} - d\phi_{\pm} + b(\phi_{\pm})}{\beta},$$

and so $d\phi_{\pm} = b(\phi_{\pm})$. Again by (9), we have

$$0 \leq \phi_- \leq K/2 \leq \phi_+ \leq K,$$

which indicates that

$$\phi_- = 0, \phi_+ = K$$

by (B). The proof is complete. \square

2.5. Nonexistence of monotone traveling wavefronts: $c < c^*$. In this section, we investigate the nonexistence of traveling wavefronts for $c < c^*$. We first consider the following initial value problem

$$\begin{cases} u_t(x, t) = u_{xx}(x, t) - D_1 u(x, t) + D_2 u(x, t - m) - D_3 u^2(x, t), \\ u(x, s) = \psi(x, s), \end{cases} \quad (10)$$

in which $x \in \mathbb{R}, t > 0, s \in [-m, 0]$, D_1, D_2, D_3 are positive constants satisfying $D_2 > D_1$, $\psi(x, s)$ is uniformly continuous in $x \in \mathbb{R}, s \in [-m, 0]$. By Smith and Zhao [15], Thieme and Zhao [16], we have the following three lemmas.

Lemma 2.15. *If $\psi(x, s)$ satisfies*

$$0 \leq \psi(x, s) \leq \frac{D_2 - D_1}{D_3}, x \in \mathbb{R}, s \in [-m, 0],$$

then (10) has a unique mild solution $u(x, t)$ defined for all $x \in \mathbb{R}, t > 0$. Moreover, $u(x, t)$ is continuous in $x \in \mathbb{R}, t > 0$, and satisfies

$$0 \leq u(x, t) \leq \frac{D_2 - D_1}{D_3}, x \in \mathbb{R}, t > 0.$$

Lemma 2.16. *Assume that*

$$0 \leq w(x, t) \leq \frac{D_2 - D_1}{D_3}, x \in \mathbb{R}, t \geq -m,$$

and satisfies

$$\begin{cases} w_t(x, t) \geq (\leq) w_{xx}(x, t) - D_1 w(x, t) + D_2 w(x, t - m) - D_3 w^2(x, t), \\ w(x, s) \geq (\leq) \psi(x, s) \end{cases} \quad (11)$$

for $x \in \mathbb{R}, t > 0, s \in [-m, 0]$. Then $w(x, t) \geq (\leq) u(x, t), x \in \mathbb{R}, t > 0$.

Lemma 2.17. *Assume that $c > 0$ such that*

$$\lambda^2 - c\lambda - D_1 + D_2e^{-\lambda cm} > 0, \lambda \geq 0.$$

If $\psi(x, s)$ satisfies

$$0 \leq \psi(x, s) \leq \frac{D_2 - D_1}{D_3}, x \in \mathbb{R}, s \in [-m, 0], \psi(0, 0) > 0,$$

then

$$\liminf_{t \rightarrow \infty} \inf_{|x| < ct} u(x, t) = \limsup_{t \rightarrow \infty} \sup_{|x| < ct} u(x, t) = \frac{D_2 - D_1}{D_3}.$$

Theorem 2.18. *If $c < c^*$, then (2) with (3) does not have a monotone solution.*

Proof. Were the statement false, then (2) with (3) has a monotone solution $\phi(\xi)$ for some $c_1 < c^*$. Obviously, $\phi(\xi)$ satisfies

$$\lim_{\xi \rightarrow -\infty} \phi(\xi) = \lim_{\xi \rightarrow -\infty} \phi'(\xi) = 0. \tag{12}$$

In particular, we can select $\epsilon > 0$ such that

$$b'(0) > d + 2\epsilon$$

and

$$\lambda^2 - c\lambda - (d + \epsilon) + b'(0)e^{-\lambda cm} > 0$$

for all $\lambda > 0, 3c \leq 2c^* + c_1$. We first prove that there exists $N > 0$ such that

$$b(\phi(\xi - c_1\tau(\phi(\xi)))) \geq -\epsilon\phi(\xi) + b'(0)\phi(\xi - c_1m) - N\phi^2(\xi), \xi \in \mathbb{R}. \tag{13}$$

In fact, (B1)-(B4) and the monotonicity of $\phi(\xi)$ imply that

$$\begin{aligned} & b(\phi(\xi - c_1\tau(\phi(\xi)))) \\ & \geq b'(0)\phi(\xi - c_1\tau(\phi(\xi))) - L\phi^2(\xi - c_1\tau(\phi(\xi))) \\ & \geq b'(0)\phi(\xi - c_1\tau(\phi(\xi))) - L\phi^2(\xi) \\ & = b'(0)\phi(\xi - c_1m) - L\phi^2(\xi) + b'(0) [\phi(\xi - c_1\tau(\phi(\xi))) - \phi(\xi - c_1\tau(0))]. \end{aligned}$$

Due to (12), there exists $\xi_1 \in \mathbb{R}$ such that

$$b'(0) [\phi(\xi - c_1\tau(\phi(\xi))) - \phi(\xi - c_1\tau(0))] \geq -\epsilon\phi(\xi)$$

for all $\xi \leq \xi_1 + c_1M$. Therefore, we have

$$b(\phi(\xi - c_1\tau(\phi(\xi)))) \geq -\epsilon\phi(\xi) + b'(0)\phi(\xi - c_1m) - L\phi^2(\xi)$$

for all $\xi \leq \xi_1 + c_1M$. If $\xi \geq \xi_1 + c_1M$, then

$$\begin{aligned} & b'(0) [\phi(\xi - c_1\tau(\phi(\xi))) - \phi(\xi - c_1\tau(0))] \geq -b'(0)\phi(\xi - c_1\tau(0)) \\ & \geq -b'(0)\phi(\xi_1) = \frac{-b'(0)}{\phi(\xi_1)}\phi^2(\xi_1) \geq \frac{-b'(0)}{\phi(\xi_1)}\phi^2(\xi), \end{aligned}$$

which indicates (13).

From (13), $u(x, t) = \phi(x + c_1t)$ satisfies

$$\begin{cases} u_t(x, t) \geq u_{xx}(x, t) - (d + \epsilon)u(x, t) + b'(0)u(x, t - m) - Nu^2(x, t), \\ u(x, s) = \phi(x + c_1s), \end{cases} \tag{14}$$

for $x \in \mathbb{R}, t > 0, s \in [-m, 0]$. By Lemmas 2.15-2.17, we see that

$$\liminf_{t \rightarrow \infty} u(-(c_1 + c^*)t/2, t) \geq \frac{b'(0) - (d + \epsilon)}{N} > 0,$$

which implies a contradiction because of

$$\xi = -(c_1 + c^*)t/2 + c_1t \rightarrow -\infty, t \rightarrow \infty.$$

The proof is complete. \square

3. Nonmonotone birth function. In this section, we investigate the existence and nonexistence of positive traveling wave solutions of (1), namely, existence and nonexistence of positive solutions of (2) and (4), if $b : [0, \infty) \rightarrow [0, \infty)$ satisfies the following assumptions:

(C1): $b(u)$ is C^1 for $u \in [0, \infty)$;

(C2): $b'(0) > d$ and $b(u) < b'(0)u$, $u \in [0, \infty)$;

(C3): $0 \leq b'(0)u - b(u) < Lu^2$, $|b'(u)| \leq b'(0)$ for $u \in [0, K]$ and some $L > 0$.

Clearly, if $b(u) = pue^{-u}$, then (C1)-(C3) hold when $p > d$. In particular, (C1)-(C3) will be imposed throughout this section without further illustration. For convenience, we define

$$\mathcal{K} = \max_{u \in [0, K]} b(u), \mathcal{T} = \sup_{u \in [0, K]} \tau'(u).$$

We first state the nonexistence of traveling wave solutions.

Theorem 3.1. *If $c < c^*$, then (2) does not have a positive solution $\phi(\xi)$ satisfying (4) and $0 < \phi(\xi) \leq \mathcal{K}$.*

Proof. Were the statement false, then for some $c_1 < c^*$, (2) has a positive solution $\phi(\xi)$ satisfying (4). Then $\phi(\xi)$ is a fixed point of F and it is evident that

$$\lim_{\xi \rightarrow -\infty} \phi'(\xi) = 0.$$

Similar to the proof of Theorem 2.18, we can select $\epsilon > 0$ such that

$$b'(0) > d + 2\epsilon$$

and

$$\lambda^2 - c\lambda - (d + \epsilon) + b'(0)e^{-\lambda cm} > 0$$

for all $\lambda > 0$, $3c \leq 2c^* + c_1$.

Then there exists $N > 0$ such that

$$b(\phi(\xi - c_1\tau(\phi(\xi)))) \geq -\epsilon\phi(\xi) + b'(0)\phi(\xi - c_1m) - N\phi^2(\xi), \xi \in \mathbb{R},$$

although $b(u)$ is not monotone. In fact, because $\phi'(\xi) \rightarrow 0$, $\xi \rightarrow -\infty$, then there exists ξ_2 such that

$$b(\phi(\xi - c_1\tau(\phi(\xi)))) \geq b'(0)\phi(\xi - c_1m) - \frac{\epsilon}{2}\phi(\xi), \xi < \xi_2 + c_1M.$$

If $\xi \geq \xi_2 + c_1M$, then $\inf_{\xi \geq \xi_2 + c_1M} \phi(\xi) > 0$ such that there exists N satisfying

$$N\phi^2(\xi) \geq N \left(\inf_{t \geq \xi_0 + c_1M} \phi(t) \right)^2 \geq \mathcal{K} \geq b(\phi(\xi - c_1\tau(\phi(\xi))))), \xi \geq \xi_0 + c_1M.$$

Similar to the proof of Theorem 2.18, we can verify the result. \square

Remark 2. Theorem 3.1 still holds if $b(u)$, $u \in [0, K]$ is monotone.

For $u \in [0, \mathcal{K}]$, we define

$$\bar{b}(u) = \max_{v \in [0, u]} b(v), \underline{b}(u) = \min_{v \in [u, \mathcal{K}]} b(v).$$

It is easy to show both $\bar{b}(u)$ and $\underline{b}(u)$ are monotone and continuous and there exists $k > 0$ such that

$$0 < k \leq K \leq \mathcal{K}$$

and

$$\underline{b}(k) = dk, \bar{b}(\mathcal{K}) = d\mathcal{K}.$$

Consider the following two equations

$$V_t(x, t) = V_{xx}(x, t) - dV(x, t) + \bar{b}(V(x, t - \tau(V(x, t)))), \tag{15}$$

and

$$v_t(x, t) = v_{xx}(x, t) - dv(x, t) + \underline{b}(v(x, t - \tau(v(x, t)))), \tag{16}$$

in which all the parameters are the same as those in (1). Then the existence and nonexistence of monotone traveling wavefronts of (15) and (16) can be answered by the conclusions in Section 2.

Theorem 3.2. *If $c > c^*$, then (2) admits a positive solution satisfying (4).*

Proof. For any fixed $c > c^*$, we define

$$\bar{\phi}(\xi) = \min\{e^{\lambda_1(c)\xi}, \mathcal{K}\}, \underline{\phi}(\xi) = \varphi(\xi), \tag{17}$$

where $\lambda_1(c)$ is the same as that in Section 2, $\varphi(\xi)$ is a monotone traveling wavefront of (16) and satisfies

$$\lim_{\xi \rightarrow -\infty} \varphi(\xi)e^{-\lambda_1(c)\xi} = 1.$$

By the discussion in Section 2, we see that

$$\bar{\phi}(\xi) \geq \underline{\phi}(\xi), \xi \in \mathbb{R}.$$

Define

$$\Gamma^*(\underline{\phi}, \bar{\phi}) = \left\{ \phi(\xi) \in B_\mu(\mathbb{R}, \mathbb{R}) : \begin{array}{l} (i) \ \phi(\xi) \leq \phi(\xi) \leq \bar{\phi}(\xi); \\ (ii) \ c|\phi(\xi_1) - \phi(\xi_2)| \leq \frac{\beta\mathcal{K}}{1+b'(0)\mathcal{K}\tau}|\xi_1 - \xi_2|, \xi_1, \xi_2 \in \mathbb{R} \end{array} \right\}.$$

Similar to the discussion in Section 2, we can select β, μ such that

- (P1): Γ^* is nonempty, convex, bounded and closed;
- (P2): $F : \Gamma^* \rightarrow \Gamma^*$;
- (P3): $F : \Gamma^* \rightarrow \Gamma^*$ is complete continuous in the sense of decay norm $|\cdot|_\mu$,

in which the definition of F is the same as those in Section 2. Then there exists $\phi \in \Gamma^*$ such that ϕ is a fixed point of F and satisfies

$$k \leq \liminf_{\xi \rightarrow \infty} \phi(\xi) \leq \limsup_{\xi \rightarrow \infty} \phi(\xi) \leq \mathcal{K},$$

which implies (4). This completes the proof. □

Theorem 3.3. *If $c = c^*$, then (2) admits a nontrivial positive solution satisfying*

$$k \leq \liminf_{\xi \rightarrow +\infty} \phi(\xi) \leq \limsup_{\xi \rightarrow +\infty} \phi(\xi) \leq \mathcal{K}. \tag{18}$$

Proof. Let $\delta \in (0, k/2)$ be a positive constant and $\{c_n\}_{n \in \mathbb{N}}$ be a decreasing sequence satisfying

$$c_n \rightarrow c^*, n \rightarrow \infty.$$

Then Theorem 3.2 implies that for each c_n , F with $c = c_n$ has a fixed point $\phi_n(\xi)$ such that

$$\phi_n(0) = \delta, \phi_n(\xi) \geq \delta, \xi \geq 0, n \in \mathbb{N}, \quad (19)$$

then we can obtain the existence of $\phi^*(\xi)$ such that $\phi^*(\xi)$ is a positive solution of (2) with $c = c^*$ by a discussion similar to that in Section 2.

Denote

$$\liminf_{\xi \rightarrow +\infty} \phi^*(\xi) = \phi^- (\geq \delta > 0), \limsup_{\xi \rightarrow +\infty} \phi^*(\xi) = \phi^+ (\geq \delta > 0).$$

By the dominated convergence theorem in F , we see that

$$\phi^- \geq \frac{\beta\phi^- - d\phi^- + \underline{b}(\phi^-)}{\beta}$$

and

$$\phi^+ \leq \frac{\beta\phi^+ - d\phi^+ + \bar{b}(\phi^+)}{\beta}.$$

Then (18) is true since

$$d\phi^- \geq \underline{b}(\phi^-), d\phi^+ \leq \bar{b}(\phi^+).$$

Because of (19) and $\phi^*(0) < \frac{3\phi^-}{4}$, it is evident that $\phi^*(\xi)$ is not a constant. The proof is complete. \square

Remark 3. Similar to those in [5, 11, 18], we can obtain some sufficient conditions such that $\lim_{\xi \rightarrow \infty} \phi(\xi)$ exists even if b is not monotone.

Unfortunately, we cannot prove the existence of $\lim_{\xi \rightarrow -\infty} \phi(\xi)$ in Theorem 3.3. However, under some additional assumptions, we can formulate the limit behavior as follows.

Theorem 3.4. *Assume that $\mathcal{T} \geq 0$ is small enough. If $\phi(\xi)$ is defined by Theorem 3.3, then $\lim_{\xi \rightarrow -\infty} \phi(\xi) = 0$.*

Proof. Similar to Lemma 2.2, if we take

$$4\beta = (1 + b'(0)\mathcal{K}\mathcal{T})^2 c^{*2} + 4d,$$

then

$$|c^* \phi'(\xi)| < \frac{\beta K}{1 + b'(0)\mathcal{K}} =: L_3.$$

Therefore, we have

$$|b(\phi(\xi - c^* \tau(\phi(\xi)))) - b(\phi(\xi - cm))| \leq b'(0)L_3\mathcal{T}\phi(\xi)$$

and so

$$-d\phi(\xi) + b(\phi(\xi - c^* \tau(\phi(\xi)))) \geq -(d + b'(0)L_3\mathcal{T})\phi(\xi) + b(\phi(\xi - cm)).$$

If $\mathcal{T} > 0$ such that

$$d + b'(0)L_3\mathcal{T} < b'(0),$$

then there exists $\underline{k} \in (0, k]$ such that

$$(d + b'(0)L_3\mathcal{T})\underline{k} = \underline{b}(\underline{k}).$$

Take $4\delta = \underline{k}$ in the proof of Theorem 3.3. If $\limsup_{\xi \rightarrow -\infty} \phi(\xi) > 0$, then there exist a strictly decreasing sequence $\{\xi_j\}$ and a constant $\epsilon > 0$ such that

$$\xi_j < 0, \xi_j \rightarrow -\infty, j \rightarrow \infty$$

and

$$2\phi(\xi) > \limsup_{\xi \rightarrow -\infty} \phi(\xi), \xi \in [\xi_j - \epsilon, \xi_j + \epsilon], j \in \mathbb{N}.$$

Note that $u(x, t) = \phi(\xi)$ is an upper solution of

$$u_t(x, t) \geq u_{xx}(x, t) - (d + b'(0)L_3\mathcal{T})\phi(\xi) + b(\phi(\xi - cm)),$$

then the standard comparison principle and asymptotic spreading [16] imply that

$$4\phi(\xi_j) > 3\underline{k},$$

and a contradiction occurs. The proof is complete. \square

Remark 4. The proof of Theorem 3.4 is similar to Lin and Ruan [9, Section 5.5].

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