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Nonconstant radial positive solutions of elliptic systems with Neumann boundary conditions

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ABSTRACT

Let B_R be the ball of radius R in $\mathbb{R}^N$ with $N \geq 2$. We consider the nonconstant radial positive solutions of elliptic systems of the form

$$\begin{aligned} &-\Delta u + u = f(u,v) && \text{in } B_R, \\ &-\Delta v + v = g(u,v) && \text{in } B_R, \\ &\partial_\nu u = \partial_\nu v = 0 && \text{on } \partial B_R, \end{aligned}$$

where f and g are nondecreasing in each component. With few assumptions on the nonlinearities, we apply bifurcation theory to show the existence of at least one nonnegative, nonconstant and nondecreasing solution.

1. Introduction

Very recently, Bonheure, Serra and Tilli [3] considered the Neumann problem

$$\begin{aligned} -\Delta u + u &= a(|x|)f(u,v) && \text{in } B_R, \\ -\Delta v + v &= b(|x|)g(u,v) && \text{in } B_R, \qquad (1.1) \\ \partial_\nu u = \partial_\nu v &= 0 && \text{on } \partial B_R, \end{aligned}$$

where B_R is the ball of radius R in $\mathbb{R}^N$ with $N \geq 2$, a, b, f and g satisfy the assumptions:

(A) $a, b \in L^1(0, R)$ are nonnegative, nondecreasing and not identically zero;

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(H1) $f, g \in C(\mathbb{R}^+ \times \mathbb{R}^+)$ are nonnegative and nondecreasing in each variable;
(H2)

$$\lim_{s+t\to 0^+} \frac{f(s,t)+g(s,t)}{s+t} = 0, \qquad \lim_{s+t\to\infty} \frac{f(s,t)+g(s,t)}{s+t} = \infty.$$

Applying the cone of nonnegative and nondecreasing functions and the fixed point index theory [1,12], they proved the following

Theorem A *([3, Theorem 1.1]). Under assumptions (A), (H1) and (H2), problem (1.1) admits at least one solution (u, v) with u and v both nonnegative and nondecreasing.*

If both a and b are constant, one cannot expect that the solution is nonconstant without further assumptions. Indeed, arguing as in [2, Proposition 4.1], one can provide examples of systems of the form (1.1) with $a = b = 1$ whose unique positive solutions are constants. In [3], Bonheure, Serra and Tilli also considered the existence of nonconstant solutions of the system

$$\begin{aligned} -\Delta u + u &= f(u,v) && \text{in } B_R, \\ -\Delta v + v &= g(u,v) && \text{in } B_R, \\ \partial_\nu u = \partial_\nu v &= 0 && \text{on } \partial B_R. \end{aligned} \tag{1.2}$$

They proved the following

Theorem B *([3, Theorem 1.2]). Assume that f and g satisfy (H1) and (H2) and are differentiable. Assume the only constant nontrivial solution of (1.2) is (α, β). Let $P := (\alpha, \beta)$. Let λ_j^r be the j-th radial eigenvalues of $-\Delta + I$ with Neumann boundary condition $\partial_\nu u = 0$ on ∂B_R. Let the matrix*

$$M_P := \begin{pmatrix} f_u(\alpha,\beta) & f_v(\alpha,\beta) \\ g_u(\alpha,\beta) & g_v(\alpha,\beta) \end{pmatrix} \tag{1.3}$$

(where $f_u = \frac{\partial f}{\partial u}$, $f_v = \frac{\partial f}{\partial v}$, $g_u = \frac{\partial g}{\partial u}$, $g_v = \frac{\partial g}{\partial v}$) have two real eigenvalues $\underline{\lambda}_P$, $\overline{\lambda}_P$ with $\underline{\lambda}_P \le \overline{\lambda}_P$. If

$$\underline{\lambda}_P \notin \{\lambda_1^r, \lambda_2^r\} \quad \text{and} \quad \overline{\lambda}_P > \lambda_2^r, \tag{1.4}$$

then problem (1.2) admits at least one nonnegative, nonconstant and nondecreasing solution.

Notice that they overcome the lack of compactness by considering the cone of nonnegative, nondecreasing radial functions of $H^1(B_R)$, which was firstly introduced in [20]. Their approach is based on Dancer's abstract results on the local fixed point index for a map defined between wedges (see Dancer [7,8]). However, the condition (H2) seems unduly restrictive.

The purpose of the present paper is to show the existence of nonnegative, nonconstant and nondecreasing solutions of (1.2) when the nonlinearity is asymptotically linear growth at infinity and no growth restriction at the origin. Our approach is based upon a global results for the solution set of

$$x = A(\mu, x),$$

where $A : [0, \infty) \times W \to W$ is completely continuous, and W is a wedge in a real Banach space $\mathbb{E} := C^1[0, R] \times C^1[0, R]$ such that $W - W$ is dense in $\mathbb{E}$, see Dancer [7].

We shall make the following assumptions:

(A0) f and g are differentiable, and the only constant nontrivial solution of (1.2) is (α, β), and

$$f_u(\alpha,\beta) \geq 0, \quad f_v(\alpha,\beta) \geq 0, \quad g_u(\alpha,\beta) \geq 0, \quad g_v(\alpha,\beta) \geq 0.$$

Moreover, there exist $\xi, \zeta : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(u,v) - \alpha = f_u(\alpha,\beta)(u-\alpha) + f_v(\alpha,\beta)(v-\beta) + \xi(u-\alpha, v-\beta),$$
$$g(u,v) - \beta = g_u(\alpha,\beta)(u-\alpha) + g_v(\alpha,\beta)(v-\beta) + \zeta(u-\alpha, v-\beta),$$

where

$$\xi(t,s) = o(\sqrt{t^2+s^2}), \quad \zeta(t,s) = o(\sqrt{t^2+s^2}), \quad \text{as } (t,s) \to (0,0);$$

(A1) $f, g \in C(\mathbb{R}^+ \times \mathbb{R}^+)$ are nonnegative and strictly increasing in each variable, where $\mathbb{R}^+ = [0, \infty)$;
(A2) $f, g \in C(\mathbb{R}^+ \times \mathbb{R}^+)$ are locally Lipschitz in $\mathbb{R}^+ \times \mathbb{R}^+$;
(A3) $\underline{\lambda}_P \cdot \lambda_2^r \neq \lambda_1^r \cdot \overline{\lambda}_P$, $\overline{\lambda}_P > \lambda_2^r$;
(A4) there exists a matrix

$$M_\infty := \begin{pmatrix} h_1 & h_2 \\ k_1 & k_2 \end{pmatrix},$$

such that

$$f(t,s) = h_1 t + h_2 s + \hat{\xi}(t,s),$$
$$g(t,s) = k_1 t + k_2 s + \hat{\zeta}(t,s),$$
$$\hat{\xi}(t,s) = o(\sqrt{s^2+t^2}), \quad \hat{\zeta}(t,s) = o(\sqrt{s^2+t^2}), \quad \text{as } \sqrt{t^2+s^2} \to \infty \text{ in } \mathbb{R}^+ \times \mathbb{R}^+,$$

and

$$h_1, \ k_2 \in [0, \infty), \qquad h_2, \ k_1 \in (0, \infty);$$

(A5) there exists $\delta_* > 0$, such that one of the following conditions hold:
  (i) $y\xi(y,z) < 0$ and $z\zeta(y,z) < 0$ for $0 < |y| + |z| < \delta_*$;
  (ii) $y\xi(y,z) > 0$ and $z\zeta(y,z) > 0$ for $0 < |y| + |z| < \delta_*$.

Theorem 1.1. *Assume that (A0)–(A5) hold. Then problem (1.2) admits at least one nonnegative, nonconstant and nondecreasing solution* (u, v).

Remark 1.1. It can also be of interest to compare our results to those concerning a single equation. Miciano and Shivaji [17] showed the existence and multiplicity of positive solutions for a class of semipositone Neumann problems via quadrature method. In [2], Bonheure, Noris and Weth used a variational approach to obtain the first existence result for nonconstant solutions of

$$\begin{aligned} -\Delta u + u &= h(u) \quad && \text{in } B_R, \\ \partial_\nu u &= 0 && \text{on } \partial B_R, \end{aligned} \tag{1.5}$$

where $u_0 = h(u_0)$ is the unique positive fixed point, $h'(u_0) > \lambda_2^r$, $h'(0) = 0$, $\liminf_{s\to\infty} \frac{h(s)}{s} > 1$. We may use the global bifurcation theory due to Dancer [7] to establish the existence of at least one nonnegative, nonconstant and nondecreasing solution of (1.5) when the nonlinearity h is asymptotically linear growth at infinity and no growth restriction at the origin.

For other results on the global bifurcation structure of positive solutions of nonlinear elliptic systems, see Cheng and Zhang [4], Tian and Zhang [21], Zou [22], Ma, Gao and Lu [16] and the references therein.

The rest of the paper is organized as follows. In Section 2, we state some results on the spectrum structure of the linear Neumann problem and give some preliminary results. In Section 3, we introduce some functional setting and state some global results on the solution set of abstract operator equations. Finally in Section 4 we prove our main results on the existence of nonconstant radial positive solutions by applying the abstract global result due to Dancer [7].

2. Some preliminary results

Let us consider the linear eigenvalue problem

$$\begin{aligned} -\Delta u(x) &= \mu a(|x|)u(x), \quad &&\text{in } B_R, \\ \partial_\nu u &= 0, &&\text{on } \partial B_R, \end{aligned} \tag{2.1}$$

where $a \in C[0,R]$ satisfies

$$a(r) > 0, \qquad r \in [0,R]. \tag{2.2}$$

Lemma 2.1. *Assume that (2.2) is fulfilled. Then the radial eigenvalues of (2.1) are as follows:*

$$0 = \mu_0^r < \mu_1^r < \mu_2^r < \cdots \to \infty. \tag{2.3}$$

Moreover, for each $k \in \mathbb{N}^* := \{0,1,\cdots\}$, the radial eigenvalue μ_k^r is simple, and the radial eigenfunction ψ_k^r, being regarded as a function of r, possesses exactly k simple zeros in $[0,R]$, and ψ_k^r is radially monotone if and only if $k \in \{0,1\}$.

It is easy to see that Lemma 2.1 is an immediate consequence of the following results on singular linear eigenvalue problems, see [15, Theorem 2.2] for detail.

Lemma 2.2 *([15, Theorem 2.2]). Assume that (2.2) is fulfilled. Then the eigenvalues of the problem*

$$\begin{aligned} &-u''(r) - \frac{N-1}{r}u'(r) = \mu a(r)u(r), \qquad r \in (0,R), \\ &u'(0) = 0 = u'(R) \end{aligned} \tag{2.4}$$

are as follows:

$$0 = \mu_0^r < \mu_1^r < \mu_2^r < \cdots \to \infty.$$

Moreover, for each $k \in \mathbb{N}^*$, μ_k^r is simple, and the eigenfunction ψ_k^r possesses exactly k simple zeros in $[0,R]$, and ψ_k^r is monotone if and only if $k \in \{0,1\}$.

Lemma 2.3. *Let $(\eta_n, (y_n, z_n))$ be a sequence of solutions of the problem*

$$\begin{aligned} &-(r^{N-1}y_n')' + r^{N-1}y_n = \eta_n r^{N-1}P(y_n, z_n), \quad r \in (0, R), \\ &-(r^{N-1}z_n')' + r^{N-1}z_n = \eta_n r^{N-1}Q(y_n, z_n), \quad r \in (0, R), \\ &y_n'(0) = 0 = y_n'(R), \quad z_n'(0) = 0 = z_n'(R), \end{aligned} \tag{2.5}$$

where $|\eta_n| \le \hat{\eta}$ ($\hat{\eta}$ is a positive constant), there exist nonnegative constants p_i and q_i for $i = 1, 2$ such that $P : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy

$$P(s,t) \le p_1|s| + p_2|t|, \qquad Q(s,t) \le q_1|s| + q_2|t|.$$

Then $\|(y_n', z_n')\|_\infty \to \infty$ as $n \to \infty$ implies $\|(y_n, z_n)\|_\infty \to \infty$ as $n \to \infty$.

Proof. Assume on the contrary that $\|(y_n, z_n)\|_\infty \not\to \infty$ as $n \to \infty$. Then, after taking a subsequence and relabeling, if necessary, it follows that

$$\|(y_n, z_n)\|_\infty \le M_0$$

for some $M_0 > 0$. From (2.5), we get

$$y_n' = -\eta_n \int_0^r (\frac{s}{r})^{N-1} P(y_n, z_n) ds + \int_0^r (\frac{s}{r})^{N-1} y_n ds$$

and

$$z_n' = -\eta_n \int_0^r (\frac{s}{r})^{N-1} Q(y_n, z_n) ds + \int_0^r (\frac{s}{r})^{N-1} z_n ds,$$

which imply that

$$\begin{aligned} \|y_n'\|_\infty &\le (\hat{\eta}(p_1 + p_2)M_0 + M_0)R, \\ \|z_n'\|_\infty &\le (\hat{\eta}(q_1 + q_2)M_0 + M_0)R. \end{aligned}$$

However, this is a contradiction. □

Lemma 2.4. *Let (A0) hold. Then for every $\lambda \in (0, \infty)$, the matrix λM_P has two real eigenvalues $\gamma_1^P(\lambda)$ and $\gamma_2^P(\lambda)$ with $\gamma_1^P(\lambda) \le \gamma_2^P(\lambda)$.*

Proof. Denote

$$a := f_u(\alpha, \beta), \quad b := f_v(\alpha, \beta), \quad c := g_u(\alpha, \beta), \quad d := g_v(\alpha, \beta).$$

Since the characteristic equation

$$(\lambda a - \gamma)(\lambda d - \gamma) - \lambda^2 cb = 0,$$

i.e.

$$\gamma^2 - \lambda(a + d)\gamma + \lambda^2(ad - cb) = 0$$

has roots

$$\gamma_1^P(\lambda) = \frac{\lambda(a+d) - \lambda\sqrt{(a-d)^2 + 4cb}}{2}, \ \gamma_2^P(\lambda) = \frac{\lambda(a+d) + \lambda\sqrt{(a-d)^2 + 4cb}}{2}.$$

Then $\gamma_1^P(\lambda) \leq \gamma_2^P(\lambda)$. □

Remark 2.1. Obviously,

$$\gamma_1^P(1) = \underline{\lambda}_P, \qquad \gamma_2^P(1) = \overline{\lambda}_P. \qquad \square$$

Lemma 2.5. *Let (A0) hold. Assume that*

$$\gamma_2^P(1) > \lambda_2^r.$$

Then there exists a unique $\lambda_{*,2} \in (0,1)$*, such that the larger eigenvalue of* $\lambda_{*,2}M_P$ *is* λ_2^r*, i.e.*

$$\gamma_2^P(\lambda_{*,2}) = \lambda_2^r.$$

Moreover,

$$\lambda_{*,2} = \frac{\lambda_2^r}{\gamma_2^P(1)}.$$

Proof. It is easy to see that $\gamma_2^P(\lambda)$ is strictly increasing continuous function on $\lambda \in (0,\infty)$. Combining this with the fact $\gamma_2^P(0) = 0$ and using the assumption $\gamma_2^P(1) > \lambda_2^r$, it deduces that there exists a unique $\lambda_{*,2} \in (0,1)$, such that $\gamma_2^P(\lambda_{*,2}) = \lambda_2^r$. It is easy to check that the larger eigenvalue of the matrix $\frac{\lambda_2^r}{\gamma_2^P(1)}M_P$ is λ_2^r, and $\lambda_{*,2} = \frac{\lambda_2^r}{\gamma_2^P(1)}$. □

3. Functional setting and preliminary results

3.1. Crandall–Rabinowitz theorem

Let X and Y be two real Banach space, V be an open neighborhood of 0 in X. Let $J = (a,b) \subset \mathbb{R}$ be an open interval, and let $F : J \times V \to Y$ be a twice continuously Fréchet differentiable mapping. Let $F_x, F_\lambda, F_{\lambda x}$, etc., denote the various derivative of F with respect to $\lambda \in J$ and $x \in V$. The null space and range of a linear operator A are denoted by $N(A)$ and $R(A)$. Let dim and codim denote, respectively, dimension and codimension.

Lemma 3.1 *([5, Crandall–Rabinowitz theorem]). Suppose that* $\lambda_0 \in J$ *and also that*

(B1) $F(\lambda, 0) = 0$ *for all* $\lambda \in J$*;*
(B2) $\dim N(F_x(\lambda_0, 0)) = \operatorname{codim} R(F_x(\lambda_0, 0)) = 1$*;*
(B3) $F_{x\lambda}(\lambda_0, 0)x_0 \notin R(F_x(\lambda_0, 0))$*, where* $x_0 \in X$ *spans* $N(F_x(\lambda_0, 0))$*.*

Let Z *be any complement of* $\operatorname{span}\{x_0\}$ *in* X*. Then there exist a neighborhood* U *of* $(\lambda_0, 0)$ *in* $\mathbb{R} \times X$*, an open interval* $\hat{J}$ *containing* 0 *and continuously differentiable functions* $\lambda : \hat{J} \to \mathbb{R}$ *and* $\psi : \hat{J} \to Z$ *such that*

$$\lambda(0) = \lambda_0, \quad \psi(0) = 0,$$

and, if $x(s) = sx_0 + s\psi(s)$*, then*

$$F(\lambda(s), x(s)) = 0.$$

Moreover, $F^{-1}(0)$ *near* $(\lambda_0, 0)$ *consists precisely of the curves* $x = 0$ *and* $(\lambda(s), x(s)), s \in \hat{J}$*, i.e.,*

$$F^{-1}(0) \cap U = \big\{\big(\lambda(s), y(s)\big) : s \in \hat{J}\big\} \cup \big\{(\lambda, 0) : (\lambda, 0) \in U\big\}.$$

3.2. Preliminaries in wedge

Let W be a *wedge* in X, namely a closed convex subset of X such that $\alpha W \subset W$ for every $\alpha \geq 0$. Recall that a wedge is said to be a *cone* if $W \cap -W = \{0\}$.

To apply the abstract results it is necessary to assume that

$$W - W \text{ is dense in } X. \tag{3.1}$$

Definition 3.1. Let W be a wedge satisfying (3.1), and let $y \in W$. We define

$$W_y := \{x \in X \mid \exists \gamma > 0 \text{ such that } y + \gamma x \in W\}$$

and

$$S_y := \{x \in \overline{W}_y \mid -x \in \overline{W}_y\}.$$

Note (for all details we refer to [7,8]) that the set W_y is convex, contains W and $\pm y$, and $\alpha W_y \subset W_y$ for every $\alpha \geq 0$. Thus $\overline{W}_y$ is a wedge containing W and $\pm y$.

Concerning S_y, it can be easily proved that it is a closed subspace of X containing y.

Still following [7,8], we introduce the following notion.

Definition 3.2. We say a compact operator $L : X \to X$ mapping $\overline{W}_y$ into itself has property α if

$$\text{there exist } t \in (0, 1) \text{ and } \omega \in \overline{W}_y \setminus S_y \text{ such that } \omega - tL\omega \in S_y.$$

We can now turn to the statement of the main result.

Let $\Phi : W \to X$ be a (nonlinear) map satisfying

(C1) Φ is completely continuous,
(C2) $\Phi(W) \subset W$,
(C3) $\Phi(y) = y$,
(C4) Φ is differentiable at y in "W" (see [7]),
(C5) $\Phi'(y) =: L$ is compact from X to X.

Under these assumptions, it can be proved that L maps $\overline{W}_y$ into $\overline{W}_y$.

Denoting by

$$i_W(\Phi, y)$$

the local fixed point index of y in W, see for example [11], the results by Dancer that we need, precisely Theorem 1 in [7] and Proposition 1 in [8] can be collected in a single statement as follows.

Lemma 3.2 *([7,8]). Let X be a Banach space and let $W \subset X$ be a wedge satisfying (3.1). Let $\Phi : W \to X$ satisfy (C1)–(C5). Then the following statements hold:*

(i) *If $I - L$ is invertible and L has property α, then $i_W(\Phi, y) = 0$;*
(ii) *If $I - L$ is not invertible but $\mathrm{Ker}(I - L) \cap \overline{W}_y = \{0\}$, then $i_W(\Phi, y) = 0$.*

Lemma 3.3 *([7, Theorem 1]). Let X be a Banach space and let $W \subset X$ be a wedge satisfying (3.1). Let $\Phi : W \to X$ satisfy (C1)–(C5). Let $I - L$ be invertible. Then*

$$i_W(\Phi, y) = i_{S_y}(L, 0) = i_X(L, 0)$$

if L does not have property α on $\overline{W}_y$.

Lemma 3.4 *([7, Proposition 1]). Assume that $A : [0, \infty) \times W \to W$ is completely continuous, that $\mu > 0$, and that y is an isolated solution in W of $x = A(\mu, x)$ with $i_W(A(\mu, \cdot), y) \neq 0$. Suppose $\epsilon, \delta > 0$ are such that $x \neq A(\lambda, x)$ if either*

(1) *$x \in W$, $0 < \|x - y\| < \epsilon$ and $\lambda = \mu$, or*
(2) *$x \in W$, $\|x - y\| = \epsilon$ and $|\lambda - \mu| \leq \delta$.*

Let T denote the component of

$$\{(\lambda, x) \in [0, \infty) \times W : x = A(\lambda, x)\} \backslash \{(\lambda, x) : \|x - y\| \leq \epsilon,\ \mu - \delta < \lambda < \mu\}$$

containing (μ, y). Then

(i) *T is unbounded, or*
(ii) *$\inf\{\lambda : (\lambda, x) \in T\} = 0$, or*
(iii) *$T \cap \{(\lambda, z) : \lambda = \mu - \delta,\ \|z - y\| < \epsilon\} \neq \emptyset$.*

3.3. Index Jump Principle in wedge

Let $W \subset X$ be a Wedge. We consider the parameter-dependent equation

$$x - H(\mu, x) = 0, \quad \mu \in \mathbb{R},\ x \in W \tag{3.2}$$

and require the following

(D1) The operator $H : U(\mu_0, 0) \to W$ is compact with $H(\mu, 0) \equiv 0$;
(D2) For $\mu_1 < \mu_2$ we have $i_W(H(\mu_1, \cdot), 0) \neq i_W(H(\mu_2, \cdot), 0)$.

Here we naturally require the indices to be defined and in (D1) we let

$$U(\mu_0, 0) := [\mu_1, \mu_2] \times U(0),$$

where $\mu_1 < \mu_0 < \mu_2$, $U(0)$ is a neighborhood of the origin in W.

The index jump condition (D2) can also be expressed in the form

$$\deg_W(I - H(\mu_1, \cdot), U(0), 0) \neq \deg_W(I - H(\mu_2, \cdot), U(0), 0).$$

Lemma 3.5 *(Index Jump Principle).*

(a) *If (D1) is satisfied and if* $(\mu_0, 0)$ *is not a bifurcation point of the equation (3.2), then* $i_W(H(\mu,\cdot),0)$ *is defined and constant on a neighborhood of* $\mu = \mu_0$.

(b) *If (D1) and (D2) are satisfied, then the equation (3.2) has a bifurcation point* $(\mu, 0)$ *with* $\mu_1 < \mu < \mu_2$.

Proof. (a) If $(\mu_0, 0)$ is not a bifurcation point, then there exists a neighborhood V of $(\mu_0, 0)$ in which $(\mu_0, 0)$ is the only solution of (3.2) in $W \cap V$. Homotopy invariance yields the constancy of $i_W(H(\mu,\cdot),0)$.

(b) Suppose that such a bifurcation point does not exist. Then $i_W(H(\mu,\cdot),0)$ as a function of μ is locally constant by (a) and integer valued, and thus constant on $[\mu_1, \mu_2]$. But it contradicts (D2). □

3.4. Leray–Schauder degree of the linear Neumann system

In this subsection, we attempt to compute the Leray–Schauder degree of the linear system

$$(-\Delta + I)\begin{pmatrix} y \\ z \end{pmatrix} = \lambda M_P \begin{pmatrix} y \\ z \end{pmatrix} \quad \text{in } B_R, \quad \partial_\nu y = \partial_\nu z = 0 \text{ on } \partial B_R, \tag{3.3}$$

where $\lambda > 0$.

Let $X := C^1[0, R]$. Then it is a Banach space under the norm

$$\|u\|_1 = \max\{\|u\|_\infty,\ \|u'\|_\infty\}.$$

Let $\mathbb{E} := C^1[0, R] \times C^1[0, R]$. Then it is a Banach space under the norm

$$\|(u,\ v)\|^2 = \|u\|_1^2 + \|v\|_1^2.$$

Let

$$\lambda_j^r := 1 + \mu_{j-1}^r, \qquad j \in \mathbb{N},$$

where μ_{j-1}^r is given in Lemma 2.1. By the similar argument to establish [9, Proposition 1.1] (in which the Leray–Schauder degree of the corresponding Dirichlet problem was considered), with obvious changes, we may get the following

Lemma 3.6. *Let* V *be a bounded open subset in* $\mathbb{E}$ *with* $0 \in V$. *Assume that* $\sigma(\lambda M_P) \cap \sigma(-\Delta + I) = \emptyset$. *Here and after,* $\sigma(-\Delta + I)$ *denotes the spectrum of the operator* $-\Delta + I$ *subject to the Neumann condition. Denote by* B_λ *the Jordan canonical form of* λM_P. *Then*

$$\deg_{LS}(I - (-\Delta + I)^{-1}\lambda M_P, V, 0) = \deg_{LS}(I - (-\Delta + I)^{-1}B_\lambda, V, 0).$$

Moreover,

(i) *If*

$$B_\lambda = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}.$$

Then

$$\deg_{LS}(I-(-\Delta+I)^{-1}B_\lambda, V, 0) = (-1)^{m(\gamma_1)+m(\gamma_2)},$$

where $m(\gamma_i)$ is the sum of the algebraic multiplicity of the eigenvalues $\lambda_j^r \in \sigma(-\Delta+I)$ such that $\lambda_j^r < \gamma_i$, and $m(\gamma_i) = 0$ if all eigenvalues λ_j^r of $-\Delta+I$ are such that $\lambda_j^r > \gamma_i$.

(ii) *If*

$$B_\lambda = \begin{pmatrix} \gamma & 0 \\ 1 & \gamma \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \gamma & -\eta \\ \eta & \gamma \end{pmatrix} \text{ for } \eta \neq 0.$$

Then

$$\deg_{LS}(I-(-\Delta+I)^{-1}B_\lambda, V, 0) = 1. \qquad \square$$

4. Proof of the main results

4.1. Equivalent operator equation in cones

Since (A4) implies that f and g are asymptotically linear growth at infinity, we only need to work in the Banach space $\mathbb{E}$ rather than $H_r^1(B_R) \times H_r^1(B_R)$.

Let us define the wedge $W = K \times K$ where K is the cone of nonnegative radially nondecreasing functions defined in

$$K = \{u \in X \mid 0 \le u(r) \le u(s), \ \forall\, 0 \le r < s \le R\}$$

with $X = C^1[0,R]$. Of course, in this case W is not just a wedge, but a true cone.

First of all, it is easy to see that W satisfies $\mathbb{E} = \overline{W-W}$; this is a consequence of the fact that $K-K$ is dense in X.

Now consider the (nonlinear) map $\Phi : W \to \mathbb{E}$ defined according to

$$\Phi(\varphi,\ \psi) = (u,v) \Longleftrightarrow \begin{cases} -\Delta u + u = f(\varphi,\ \psi), & \text{in } B_R, \\ -\Delta v + v = g(\varphi,\ \psi), & \text{in } B_R, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial B_R. \end{cases} \tag{4.1}$$

Lemma 4.1 *([3, Lemma 3.2]). The map Φ is well defined and, in fact, $\Phi : W \to W$. Moreover, Φ is continuous and, if $U \subset W$ is a bounded set, then $\Phi(U)$ has compact closure.*

4.2. Equivalent operator equation in wedges

Let us consider the problem

$$\begin{cases} -\Delta u + u = f(u,v), & \text{in } B_R, \\ -\Delta v + v = g(u,v), & \text{in } B_R, \\ u > 0, \quad v > 0, & \text{in } B_R, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial B_R. \end{cases} \tag{4.2}$$

Let

$$y := u - \alpha, \qquad z := v - \beta. \tag{4.3}$$

Then (4.2) can be rewritten as

$$\begin{cases} -\Delta y + y = f(y+\alpha,\ z+\beta) - \alpha, & \text{in } B_R, \\ -\Delta z + z = g(y+\alpha,\ z+\beta) - \beta, & \text{in } B_R, \\ y > -\alpha,\quad z > -\beta, & \text{in } B_R, \\ \partial_\nu y = \partial_\nu z = 0, & \text{on } \partial B_R. \end{cases} \tag{4.4}$$

Define

$$h(s,t) = \begin{cases} f(s+\alpha,\ t+\beta) - \alpha, & s \ge -\alpha \text{ and } t \ge -\beta, \\ f(0,\ t+\beta) - \alpha, & s < -\alpha \text{ and } t \ge -\beta, \\ f(s+\alpha,\ 0) - \alpha, & s \ge -\alpha \text{ and } t < -\beta, \\ f(0,\ 0) - \alpha, & s < -\alpha \text{ and } t < -\beta, \end{cases} \tag{4.5}$$

and

$$k(s,t) = \begin{cases} g(s+\alpha,\ t+\beta) - \beta, & s \ge -\alpha \text{ and } t \ge -\beta, \\ g(0,\ t+\beta) - \beta, & s < -\alpha \text{ and } t \ge -\beta, \\ g(s+\alpha,\ 0) - \beta, & s \ge -\alpha \text{ and } t < -\beta, \\ g(0,\ 0) - \beta, & s < -\alpha \text{ and } t < -\beta. \end{cases} \tag{4.6}$$

Then, it follows from (A0) that

$$\begin{pmatrix} h(y,z) \\ k(y,z) \end{pmatrix} = \begin{pmatrix} f_u(\alpha,\ \beta) & f_v(\alpha,\ \beta) \\ g_u(\alpha,\ \beta) & g_v(\alpha,\ \beta) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + o(\sqrt{y^2+z^2}), \quad \text{as } (y,z) \to (0,0).$$

Moreover, (4.4) is equivalent to

$$\begin{cases} -\Delta y + y = h(y,z), & \text{in } B_R, \\ -\Delta z + z = k(y,z), & \text{in } B_R, \\ y > -\alpha, z > -\beta, & \text{in } B_R, \\ \partial_\nu y = \partial_\nu z = 0, & \text{on } \partial B_R. \end{cases} \tag{4.7}$$

To study the nonconstant radial positive solutions of (4.2), let us consider the auxiliary problem

$$\begin{cases} -\Delta y + y = \lambda h(y,z), & \text{in } B_R, \\ -\Delta z + z = \lambda k(y,z), & \text{in } B_R, \\ \partial_\nu y = \partial_\nu z = 0, & \text{on } \partial B_R, \end{cases} \tag{4.8}$$

where $\lambda > 0$ is a parameter.

As prescribed by Definition 3.1 we observe that

$$W_0 = \{(y,z) \in \mathbb{E} \mid y, z \text{ are both bounded and nondecreasing}\}$$

and

$$S_0 = \{(y, z) \in \mathbb{E} \mid y, z \text{ are both constants}\}.$$

Notice that W_0 is not closed in $\mathbb{E}$, and that

$$\overline{W}_0 = \{(y, z) \in \mathbb{E} \mid y, z \text{ are both nondecreasing}\}.$$

Define a nonlinear map $\Psi : \mathbb{R}^+ \times W_0 \to \mathbb{E}$,

$$\Psi(\lambda, (\varphi,\ \psi)) = (y, z) \iff \begin{cases} -\Delta y + y = \lambda h(\varphi,\ \psi) & \text{in } B_R, \\ -\Delta z + z = \lambda k(\varphi,\ \psi) & \text{in } B_R, \\ \partial_\nu y = \partial_\nu z = 0, & \text{on } \partial B_R. \end{cases}$$

As an immediate consequence of Lemma 4.1, we have

Lemma 4.2. *The map Ψ is well defined and, in fact, $\Psi : \mathbb{R}^+ \times W_0 \to W_0$. Moreover, Ψ is continuous and, if $U \subset \mathbb{R}^+ \times W_0$ is a bounded set, then $\Psi(U)$ has compact closure.*

4.3. Local bifurcation at $(\lambda_{*,2}, (0, 0))$ in $\mathbb{E}$

Recall that $\lambda_{*,2} = \frac{\lambda_2^r}{\gamma_2^P(1)}$.

In what follows, we use the terminology of Rabinowitz [19]. Let S_k^+ denote the set of functions in X which have exactly $k-1$ interior nodal (i.e. non-degenerate) zeros in $(0, R)$ and are positive near $r = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^- \cup S_k^+$.

Lemma 4.3. *Assume that (A1) holds. Let $(\lambda, (y, z)) \in [0, 1] \times (S_k^\nu \times S_k^\nu)$ be a solution of (4.8). Then*

$$y(r) > -\alpha, \qquad z(r) > -\beta, \qquad r \in [0, R].$$

Proof. Suppose on the contrary that there exist $x_0, x_1 \in [0, R]$ such that one of the following cases occur:

(1) $y(r) > -\alpha$, $r \in [0, R]$; $z(x_0) = \min\limits_{r\in[0,R]} z(r) = -\beta$;
(2) $z(r) > -\beta$, $r \in [0, R]$; $y(x_1) = \min\limits_{r\in[0,R]} y(r) = -\alpha$;
(3) $y(x_1) = \min\limits_{r\in[0,R]} y(r) = -\alpha$; $z(x_0) = \min\limits_{r\in[0,R]} z(r) = -\beta$.

If Case (1) occurs, then there exists $r_0 \in [0, R]$ such that either

$$z(r_0) = 0,\ z(r) < 0 \text{ for } r \in [x_0, r_0),\ z'(r) > 0 \text{ for } r \in (x_0, r_0]; \tag{4.9}$$

or

$$z(r_0) = 0,\ z(r) < 0 \text{ for } r \in (r_0, x_0],\ z'(r) < 0 \text{ for } r \in [r_0, x_0). \tag{4.10}$$

We only deal with the case (4.9), the case (4.10) can be treated by the similar way.

Since h and k are monotone increasing in both y and z, we have, for $\lambda \in [0, 1]$, that

$$-\Delta(-\beta) + (-\beta) \le \lambda k(-\alpha, -\beta).$$

Combining this with

$$-\Delta z + z = \lambda k(y, z)$$

implies

$$-\Delta(z+\beta) + (z+\beta) \geq \lambda(k(y,z) - k(-\alpha, -\beta)) \geq 0.$$

Denote

$$w := z + \beta,$$

then

$$w'' + \frac{N-1}{r} w' - w \leq 0.$$

It follows from [10, Theorem 3.5] or [18, Theorem 3 in Chapter 1] that, w can not achieve a non-positive minimum in the interval (x_0, r_0) unless it is constant. From (4.9), it follows that

$$\inf_{[x_0, r_0]} w(r) = \min\{w(x_0),\ w(r_0)\} = w(x_0) = 0.$$

This together with $w'(x_0) = 0$ imply that

$$w(r) \equiv 0, \quad r \in [x_0, r_0].$$

However, this contradicts the fact that $w'(r) > 0$, $r \in (x_0, r_0)$.

Using the same method with obvious changes, we may get the desired contradiction in Case (2) and Case (3).

Therefore, we always have that $y(r) > -\alpha$, $z(r) > -\beta$, $r \in [0, R]$. $\square$

To show that (4.8) with $\lambda = 1$ has a $S_2^- \times S_2^-$-solution, let us consider the auxiliary problem

$$\begin{cases} \begin{pmatrix} -\Delta y + y \\ -\Delta z + z \end{pmatrix} = \lambda \begin{pmatrix} f_u(\alpha,\ \beta) & f_v(\alpha,\ \beta) \\ g_u(\alpha,\ \beta) & g_v(\alpha,\ \beta) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \lambda \begin{pmatrix} \xi(y,z) \\ \zeta(y,z) \end{pmatrix}, \\ \partial_\nu y = \partial_\nu z = 0, \qquad \text{on } \partial B_R \end{cases} \tag{4.11}$$

as a bifurcation problem from the trivial solution $(y, z) \equiv (0, 0)$.

Notice that for a given radial function $h \in X$, there exists a unique radial function $v := Th \in X$ which solves the Neumann problem

$$\begin{cases} -\Delta v + v = h, & \text{in } B_R, \\ \partial_\nu v = 0, & \text{on } \partial B_R. \end{cases}$$

Moreover, the operator $T : X \to X$ is compact, and it follows from (A0) that

$$\begin{aligned} \frac{\|T\zeta(y,z)\|_X}{\|(y,z)\|_{\mathbb{E}}} &\leq \frac{\|T\|_{X\to X}}{\|(y,z)\|_{\mathbb{E}}} (\|D\zeta(y,z)\|_\infty + \|\zeta(y,z)\|_\infty) \\ &\leq \|T\|_{X\to X} \Big(|\frac{\partial \zeta}{\partial y}| \frac{|\nabla y|}{\|(y,z)\|_{\mathbb{E}}} + |\frac{\partial \zeta}{\partial z}| \frac{|\nabla z|}{\|(y,z)\|_{\mathbb{E}}} + \frac{\|\zeta(y,z)\|_\infty}{\|(y,z)\|_{\mathbb{E}}} \Big) \\ &\to 0, \quad \text{as } \|(y,z)\|_{\mathbb{E}} \to 0. \end{aligned}$$

Here we have used the facts that $\frac{|\nabla y|}{\|(y,z)\|_{\mathbb{E}}}$ and $\frac{|\nabla z|}{\|(y,z)\|_{\mathbb{E}}}$ are bounded and

$$\frac{\partial\zeta}{\partial y} \to 0, \quad \frac{\partial\zeta}{\partial z} \to 0, \quad \text{as } \|(y,z)\|_{\mathbb{E}} \to 0.$$

Similarly, $\frac{\|T\xi(y,z)\|_X}{\|(y,z)\|_{\mathbb{E}}} \to 0$ as $\|(y,z)\|_{\mathbb{E}} \to 0$.

Let us define

$$F(\lambda,(y,z)) := \begin{pmatrix} \Delta y - y \\ \Delta z - z \end{pmatrix} + \lambda \begin{pmatrix} f_u(\alpha,\ \beta) & f_v(\alpha,\ \beta) \\ g_u(\alpha,\ \beta) & g_v(\alpha,\ \beta) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \lambda \begin{pmatrix} \xi(y,z) \\ \zeta(y,z) \end{pmatrix},$$

and

$$\begin{aligned} F_{(y,z)}(\lambda,(y,z)) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} \Delta\psi_1 - \psi_1 \\ \Delta\psi_2 - \psi_2 \end{pmatrix} + \lambda \begin{pmatrix} f_u(\alpha,\ \beta) & f_v(\alpha,\ \beta) \\ g_u(\alpha,\ \beta) & g_v(\alpha,\ \beta) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ + \lambda \begin{pmatrix} \xi_y(y,z) & \xi_z(y,z) \\ \zeta_y(y,z) & \zeta_z(y,z) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \end{aligned}$$

then

$$F_{(y,z)}(\lambda,(0,0)) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \Delta\psi_1 - \psi_1 \\ \Delta\psi_2 - \psi_2 \end{pmatrix} + \lambda \begin{pmatrix} f_u(\alpha,\ \beta) & f_v(\alpha,\ \beta) \\ g_u(\alpha,\ \beta) & g_v(\alpha,\ \beta) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

and so

$$N\Big(F_{(y,z)}\big(\frac{\lambda_2^r}{\gamma_2^P(1)},(0,0)\big)\Big) = \begin{pmatrix} x_1\varphi_2^r \\ x_2\varphi_2^r \end{pmatrix},$$

where φ_2^r is the nondecreasing eigenfunction corresponding to λ_2^r and

$$M_P \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \gamma_2^P(1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

By the well-known Perron–Frobenius Theorem, $x_1 \geq 0$, $x_2 \geq 0$ and $x_1^2 + x_2^2 \neq 0$. It is not difficult to prove that

$$R\Big(F_{(y,z)}\big(\frac{\lambda_2^r}{\gamma_2^P(1)},(0,0)\big)\Big) \subseteq \Big\{(u,v) \in [C(B_R)]^2 : \int_{B_R} (ux_1 + vx_2)\varphi_2^r = 0\Big\}.$$

Define

$$F_{(y,z),\lambda}(\lambda,(y,z)) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} f_u(\alpha,\ \beta) & f_v(\alpha,\ \beta) \\ g_u(\alpha,\ \beta) & g_v(\alpha,\ \beta) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \begin{pmatrix} \xi_y(y,z) & \xi_z(y,z) \\ \zeta_y(y,z) & \zeta_z(y,z) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

then

$$F_{(y,z),\lambda}(\lambda,(0,0)) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} f_u(\alpha,\ \beta) & f_v(\alpha,\ \beta) \\ g_u(\alpha,\ \beta) & g_v(\alpha,\ \beta) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

It follows from the fact

$$-\Delta\varphi_2^r + \varphi_2^r = \lambda_2^r\varphi_2^r \ \text{ in } B_R, \quad \partial_\nu\varphi_2^r = 0 \ \text{ on } \partial B_R$$

that

$$\int_{B_R} |\nabla\varphi_2^r|^2 = (\lambda_2^r - 1)\int_{B_R} (\varphi_2^r)^2.$$

Since $\lambda_2^r - 1 > 0$, it follows that

$$\int_{B_R} (\varphi_2^r)^2 > 0.$$

This implies

$$F_{(y,z),\lambda}(\lambda,(0,0))\begin{pmatrix} x_1\varphi_2^r \\ x_2\varphi_2^r \end{pmatrix} \notin R\Big(F_{(y,z)}\big(\frac{\lambda_2^r}{\gamma_2^P(1)},(0,0)\big)\Big).$$

So we may apply the Crandall–Rabinowitz Theorem [5] (or see Lemma 3.1) on bifurcation from a simple eigenvalue. Thus bifurcation occurs at $Q := \big(\lambda_{*,2},(0,0)\big)$, and there exists a ball $\mathfrak{B}_\rho(Q)$ with the center Q and the radius ρ in $\mathbb{R}\times\mathbb{E}$:

$$\mathfrak{B}_\rho(Q) := \{(\lambda,(y,z)) \in \mathbb{R}\times\mathbb{E} : \ |\lambda - \lambda_{*,2}| + \|(y,z)\|_{\mathbb{E}} < \rho\},$$

an interval $J = (-\epsilon,\epsilon)$, continuously differentiable functions

$$\psi_j : \mathbb{R} \to \big\{u \in X : \int_{B_R} (ux_1 + ux_2)\varphi_2^r = 0\big\}, \qquad j = 1,2,$$

and $\mu : \mathbb{R} \to \mathbb{R}$, such that $\psi_j(0) = 0$, $\mu(0) = 0$,

$$F^{-1}\big((0,0)\big) \cap \mathfrak{B}_\rho(Q) = \big\{\big(\lambda(s),(y(s),z(s))\big) : |s| < \epsilon\big\} \cup \big\{(\lambda,(0,0)) : (\lambda,(0,0)) \in \mathfrak{B}_\rho(Q)\big\}, \tag{4.12}$$

and

$$(y(s),z(s)) = s(x_1\varphi_2^r + \psi_1(s),\ x_2\varphi_2^r + \psi_2(s)),\ \lambda(s) = \lambda_{*,2} + \mu(s), \ \text{ for } -\epsilon < s < \epsilon.$$

Denote

$$\Gamma_\epsilon := \{(\lambda_{*,2} + \mu(s),\ s(x_1\varphi_2^r + \psi_1(s),\ x_2\varphi_2^r + \psi_2(s))) : 0 \le s < \epsilon\}. \tag{4.13}$$

Lemma 4.4.

(1) *Assume (A5)(i) holds. Then there exists a constant $s_0 > 0$, such that*

$$\mu(s) > 0, \qquad \textit{for } 0 < s \le s_0.$$

(2) *Assume (A5)(ii) holds. Then there exists a constant $s_0 > 0$, such that*

$$\mu(s) < 0, \qquad \textit{for } 0 < s \le s_0.$$

Proof. For $0 \le s < \epsilon$, let

$$\lambda(s) = \lambda_{*,2} + \mu(s), \quad (y(s), z(s)) = s(x_1\varphi_2^r + \psi_1(s),\ x_2\varphi_2^r + \psi_2(s)),$$

where $\mu : \mathbb{R} \to \mathbb{R}$ and

$$\psi_j : \mathbb{R} \to X, \quad j = 1, 2$$

are smooth functions satisfying

$$\int_{B_R} (x_1\psi_1 + x_2\psi_2)\varphi_2^r = 0.$$

Since

$$(-\Delta + I)\begin{pmatrix} y \\ z \end{pmatrix} = \lambda \begin{pmatrix} h(y,z) \\ k(y,z) \end{pmatrix},$$

then

$$\begin{aligned}&\begin{pmatrix} -s\Delta(x_1\varphi_2^r + \psi_1(s)) + s(x_1\varphi_2^r + \psi_1(s)) \\ -s\Delta(x_2\varphi_2^r + \psi_2(s)) + s(x_2\varphi_2^r + \psi_2(s)) \end{pmatrix} \\ &= (\lambda_{*,2} + \mu(s))\begin{pmatrix} f_u(\alpha,\ \beta) & f_v(\alpha,\ \beta) \\ g_u(\alpha,\ \beta) & g_v(\alpha,\ \beta) \end{pmatrix}\begin{pmatrix} s(x_1\varphi_2^r + \psi_1(s)) \\ s(x_2\varphi_2^r + \psi_2(s)) \end{pmatrix} + (\lambda_{*,2} + \mu(s))\begin{pmatrix} \xi(y,z) \\ \zeta(y,z) \end{pmatrix},\end{aligned}$$

and also

$$\begin{aligned}&\begin{pmatrix} -\Delta(x_1\varphi_2^r + \psi_1(s)) + x_1\varphi_2^r + \psi_1(s) \\ -\Delta(x_2\varphi_2^r + \psi_2(s)) + x_2\varphi_2^r + \psi_2(s) \end{pmatrix} \\ &= (\lambda_{*,2} + \mu(s))\begin{pmatrix} f_u(\alpha,\ \beta) & f_v(\alpha,\ \beta) \\ g_u(\alpha,\ \beta) & g_v(\alpha,\ \beta) \end{pmatrix}\begin{pmatrix} x_1\varphi_2^r + \psi_1(s) \\ x_2\varphi_2^r + \psi_2(s) \end{pmatrix} + \frac{(\lambda_{*,2} + \mu(s))}{s}\begin{pmatrix} \xi(y,z) \\ \zeta(y,z) \end{pmatrix}.\end{aligned}$$

Multiplying both sides by φ_2^r and integrating over B_R, it deduces

$$\begin{aligned}&\lambda_2^r\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\int_{B_R} (\varphi_2^r)^2 \\ &= (\lambda_{*,2} + \mu(s))\begin{pmatrix} f_u(\alpha,\ \beta) & f_v(\alpha,\ \beta) \\ g_u(\alpha,\ \beta) & g_v(\alpha,\ \beta) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\int_{B_R} (\varphi_2^r)^2 + \frac{(\lambda_{*,2} + \mu(s))}{s}\begin{pmatrix} \int_{B_R} \varphi_2^r\xi(y,z) \\ \int_{B_R} \varphi_2^r\zeta(y,z) \end{pmatrix}.\end{aligned}$$

From Lemma 2.5, we have that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mu(s)\gamma_2^P(1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\int_{B_R} (\varphi_2^r)^2 + \frac{(\lambda_{*,2} + \mu(s))}{s}\begin{pmatrix} \int_{B_R} \varphi_2^r\xi(y,z) \\ \int_{B_R} \varphi_2^r\zeta(y,z) \end{pmatrix}.$$

Multiplying both sides by the eigenvector (x_1, x_2), and we get that

$$0 = \mu(s)\gamma_2^P(1)(x_1^2 + x_2^2)\int_{B_R} (\varphi_2^r)^2 + \frac{(\lambda_{*,2} + \mu(s))}{s}\Big(x_1\int_{B_R} \varphi_2^r\xi(y,z) + x_2\int_{B_R} \varphi_2^r\zeta(y,z)\Big)$$

and

$$\mu(s) = -\frac{\lambda_{*,2}\big(x_1\int_{B_R}\varphi_2^r\frac{\xi(y,z)}{s} + x_2\int_{B_R}\varphi_2^r\frac{\zeta(y,z)}{s}\big)}{\gamma_2^P(1)(x_1^2+x_2^2)\int_{B_R}(\varphi_2^r)^2 + x_1\int_{B_R}\varphi_2^r\frac{\xi(y,z)}{s} + x_2\int_{B_R}\varphi_2^r\frac{\zeta(y,z)}{s}}.$$

Combining this with the fact

$$\lim_{s\to 0}\frac{\xi(y(s),z(s))}{s} = 0, \quad \lim_{s\to 0}\frac{\zeta(y(s),z(s))}{s} = 0$$

and using Condition (A5)(i), it deduces

$$\mu(s) > 0 \quad \text{if } s > 0 \text{ is small enough;} \tag{4.14}$$

using Condition (A5)(ii), it deduces

$$\mu(s) < 0 \quad \text{if } s > 0 \text{ is small enough.} \qquad \square \tag{4.15}$$

4.4. Unbounded connected component containing $(\lambda_{*,2},(0,0))$ in the wedge W_0

Lemma 4.5.

(1) *Assume (A3) and (A5)(i) hold. Then there exists a positive constant δ, such that*

$$i_{W_0}(\Psi(\lambda,(\cdot,\cdot)),(0,0)) \neq 0, \qquad \lambda\in(\lambda_{*,2}-\delta,\ \lambda_{*,2}), \tag{4.16}$$

$$i_{W_0}(\Psi(\lambda,(\cdot,\cdot)),(0,0)) = 0, \qquad \lambda\in(\lambda_{*,2},\ \lambda_{*,2}+\delta). \tag{4.17}$$

(2) *Assume (A3) and (A5)(ii) hold. Then there exists a positive constant δ, such that*

$$i_{W_0}(\Psi(\lambda,(\cdot,\cdot)),(0,0)) = 0, \qquad \lambda\in(\lambda_{*,2},\ \lambda_{*,2}+\delta),$$
$$i_{W_0}(\Psi(\lambda,(\cdot,\cdot)),(0,0)) \neq 0, \qquad \lambda\in(\lambda_{*,2}-\delta,\ \lambda_{*,2}).$$

Proof. We only prove (1) is valid. The proof of (2) can be treated by the similar way.

By Crandall–Rabinowitz local bifurcation theorem and (4.14), there exists a constant $\delta_0 > 0$ and a ball $\mathcal{B}_d = \{(y,z)\in\mathbb{E}\,|\,||(y,z)||_{\mathbb{E}} < d\}$, such that

$$\Gamma_\epsilon \cap \big([\lambda_{*,2}-\delta_0,\lambda_{*,2}+\delta_0]\times\mathcal{B}_d\big) = \Gamma_\epsilon \cap \big([\lambda_{*,2},\lambda_{*,2}+\delta_0]\times\mathcal{B}_d\big)$$

and

$$\Gamma_\epsilon \cap \big([\lambda_{*,2},\lambda_{*,2}+\delta_0]\times\mathcal{B}_d\big) = \{(\lambda_{*,2}+\mu(s),(y(s),z(s)) : s\in[0,s_1]\}$$

for some $s_1\in(0,\infty)$. Moreover, $\mu(s) > 0$ for $s\in(0,s_1]$.

Since $\gamma_2^P(\lambda_{*,2}) = \lambda_2^r$ and $\gamma_2^P(\lambda)$ is increasing, it follows that there exists a constant $\delta_1\in(0,\delta_0)$, such that

$$\lambda_1^r < \gamma_2^P(\lambda) < \lambda_2^r, \qquad \lambda\in(\lambda_{*,2}-\delta_1,\lambda_{*,2}).$$

Since $0<\lambda_{*,2}<1$ and $\gamma_1^P(\lambda_{*,2}) \neq \lambda_1^r$, it follows that there exists a constant $\delta_2\in(0,\delta_0)$, such that

$$\lambda_1^r \neq \gamma_1^P(\lambda), \quad \lambda\in(\lambda_{*,2}-\delta_2,\lambda_{*,2}+\delta_2).$$

Take $\delta := \min\{\delta_1, \delta_2\}$. Then it follows from the fact $\gamma_1^P(\lambda) \le \gamma_2^P(\lambda)$ that

$$\sigma(\lambda M_P) \cap \sigma(-\Delta + I) = \emptyset, \qquad \lambda \in (\lambda_{*,2} - \delta, \lambda_{*,2}).$$

Now, from Lemma 3.6, we may deduce that

$$\deg(I - (-\Delta + I)^{-1}\lambda M_P, \mathcal{B}_\varrho, 0) \neq 0, \qquad \lambda \in (\lambda_{*,2} - \delta, \lambda_{*,2}),$$

where ϱ is a small positive constant.

For each fixed $\lambda \in (0, \infty)$, let us define a linear map $L_\lambda : W_0 \to \mathbb{E}$ by

$$L_\lambda(y,\ z) = (y, z),$$

where (y, z) is the unique solution of (3.3). Then

$$L_\lambda := (-\Delta + I)^{-1}\lambda M_P : W_0 \to W_0.$$

Since $I - L_\lambda$ is invertible for $\lambda \in (\lambda_{*,2} - \delta, \lambda_{*,2})$, by applying the similar argument to get [3, Lemma 4.6], we may deduce that L_λ has no property α. Hence it deduces from Lemma 3.3 that

$$i_{W_0}(\Psi(\lambda, (\cdot,\cdot)), (0,0)) = i_{\mathbb{E}}(L_\lambda, (0,0)) = (-1)^{m(\gamma_1^P(\lambda)) + m(\gamma_2^P(\lambda))} \neq 0, \qquad \lambda \in (\lambda_{*,2} - \delta, \lambda_{*,2}).$$

Therefore, (4.16) is valid.

Equation (4.17) is an immediate consequence of [3, Lemma 4.6] and Lemma 3.2(i). □

As an immediate consequence of the Lemma 4.5 and Index Jump Principle (Lemma 3.5), we get the following

Lemma 4.6.

(a) *Assume that (A5)(i) holds. Then there exists $\tilde{s} > 0$, such that*

$$\{(\lambda_{*,2} + \mu(s), s(x_1\varphi_2^r + \psi_1(s), x_2\varphi_2^r + \psi_2(s))) : 0 < s < \tilde{s}\} \subset (\lambda_{*,2}, \lambda_{*,2} + \delta) \times W_0.$$

(b) *Assume that (A5)(ii) holds. Then there exists $\tilde{s} > 0$, such that*

$$\{(\lambda_{*,2} + \mu(s), s(x_1\varphi_2^r + \psi_1(s), x_2\varphi_2^r + \psi_2(s))) : 0 < s < \tilde{s}\} \subset (\lambda_{*,2} - \delta, \lambda_{*,2}) \times W_0. \qquad \square$$

In the rest of this paper, we only deal with the case that (A5)(i) holds. The other case can be treated by the same way.

Combining Lemma 4.4 and Lemma 4.6 and using Sard Theorem, it follows that for arbitrary $n \in \mathbb{N}$, there exists $s_n > 0$ such that $|\mu(s_n) - 0| < \frac{1}{n}$, and $(y(s_n), z(s_n))$ is an isolated solution of $(y, z) = \Psi(\lambda(s_n), (y, z))$ in W_0, and $\mu'(s_n) \neq 0$. Furthermore, it follows from the homotopy invariance of index in wedge that for some $\hat{s} \in (0, \tilde{s})$, we may assume that

$$i_{W_0}(\Psi(\lambda(\hat{s}), (\cdot,\cdot)), (y(\hat{s}), z(\hat{s}))) \neq 0,$$

and

$$\mu'(\hat{s}) > 0.$$

The case $\mu'(\hat{s}) < 0$ can be treated by the similar method with obvious changes.

Now, we take two small positive constants $\hat{\epsilon}$ and $\hat{\delta}$, such that

(P0) $(y,z)\neq\Psi(\lambda,(y,z))$ for $(y,z)\in W_0$, $0<\|(y,z)-(y(\hat{s}),z(\hat{s}))\|<\hat{\epsilon}$ and $\lambda=\lambda(\hat{s})$;
(P1) $(y,z)\neq\Psi(\lambda,(y,z))$ for $(y,z)\in W_0$, $\|(y,z)-(y(\hat{s}),z(\hat{s}))\|=\hat{\epsilon}$ and $|\lambda-\lambda(\hat{s})|\leq\hat{\delta}$;
(P2) $\mu(s)\neq 0$ and $\mu'(s)\neq 0$ for $s\in\big\{s:\ \|(y(s),z(s))-(y(\hat{s}),z(\hat{s}))\|\leq 2\hat{\epsilon},\ |\lambda-\lambda(\hat{s})|\leq 2\hat{\delta}\big\}$;
(P3) $\{(\lambda,(y,z))\in\mathbb{R}^+\times W_0:\ \|(y,z)-(y(\hat{s}),z(\hat{s}))\|\leq 2\hat{\epsilon},\ |\lambda-\lambda(\hat{s})|\leq 2\hat{\delta}\}\subset\mathfrak{B}_\rho(Q)$.

Now, we are in the position to apply Lemma 3.4. Let T denote the component of

$$\{(\lambda,(y,z))\in\mathbb{R}^+\times W_0:(y,z)=\Psi(\lambda,(y,z))\}\setminus\left\{(\lambda,(y,z)):\begin{array}{l}\|(y,z)-(y(\hat{s}),z(\hat{s}))\|\leq\hat{\epsilon},\\ \lambda(\hat{s})-\hat{\delta}<\lambda<\lambda(\hat{s})\end{array}\right\}$$

containing $(\lambda(\hat{s}),(y(\hat{s}),z(\hat{s})))$. Then

(i) T is unbounded, or
(ii) $\inf\{\lambda:\ (\lambda,(y,z))\in T\}=0$, or
(iii) $T\cap\{(\lambda,(y,z))\ :\ \lambda=\lambda(\hat{s})-\hat{\delta},\ \|(y,z)-(y(\hat{s}),z(\hat{s}))\|<\hat{\varepsilon}\}\neq\emptyset$.

Notice that (ii) can not occur since $(0,(0,0))$ is not a bifurcation point.

Suppose on the contrary that (iii) occurs. Then there exists

$$\big(\eta_1,(w_1,z_1)\big)\in T\cap\big\{(\lambda,(y,z)):\ \lambda=\lambda(\hat{s})-\hat{\delta},\ \|(y,z)-(y(\hat{s}),z(\hat{s}))\|<\hat{\varepsilon}\big\}.$$

If

$$\big(\eta_1,(y_1,z_1)\big)\in\Gamma_\epsilon,$$

then T will contain a *loop* $\mathcal{O}$ with $(\lambda(\hat{s}),(y(\hat{s}),z(\hat{s})))\in\mathcal{O}$, which contradicts (4.12). If

$$\big(\eta_1,(y_1,z_1)\big)\notin\Gamma_\epsilon,$$

then we get a desired contradiction from (4.12) again.

So, we may assume that (iii) does not occur for $\hat{\epsilon}$ and $\hat{\delta}$.

Thus, T must be unbounded.

Lemma 4.7. $T\subset\big(\mathbb{R}^+\times(W_0\setminus(S_1^+\times S_1^+))\big)$.

Proof. Assume on the contrary that there exists $(\tilde{\lambda},(\tilde{y},\tilde{z}))\in T\cap\big(\mathbb{R}^+\times(S_1^+\times S_1^+)\big)$. Then there exists $(\hat{\eta},(\hat{y},\hat{z}))\in T\setminus\big\{\big(\frac{\lambda_2^r}{\gamma_2^P(1)},(0,0)\big)\big\}$ such that one of the following cases must occur

(i) $\hat{y}(0)=0$, $\hat{z}(0)>0$;
(ii) $\hat{y}(0)>0$, $\hat{z}(0)=0$;
(iii) $\hat{y}(0)=0$, $\hat{z}(0)=0$.

If (i) holds, then it follows from

$$-\hat{y}''(r)-\frac{N-1}{r}\hat{y}'(r)+\hat{y}(r)=\hat{\eta}h(\hat{y}(r),\hat{z}(r))\tag{4.18}$$

and the definition of h that

$$\hat{y}''(0) < 0,$$

and subsequently $\hat{y}$ is concave down near $r = 0$. However, this contradicts the fact that $\hat{y}$ is nondecreasing in $[0, R]$.

If (ii) holds, then we have from

$$-\hat{z}''(r) - \frac{N-1}{r}\hat{z}'(r) + \hat{z}(r) = \hat{\eta}k(\hat{y}(r), \hat{z}(r)) \tag{4.19}$$

and the definition of k that

$$\hat{z}''(0) < 0,$$

and subsequently $\hat{z}$ is concave down near $r = 0$. However, this contradicts the fact that $\hat{z}$ is nondecreasing in $[0, R]$.

If (iii) holds, then

$$\hat{y}(0) = \hat{y}'(0) = \hat{z}(0) = \hat{z}'(0) = 0.$$

This together with (4.18) and (4.19) imply that

$$\hat{y}(r) = \hat{z}(r) = 0, \quad r \in [0, R],$$

and accordingly,

$$\hat{\eta} = \frac{\lambda_2^r}{\gamma_2^P(1)}.$$

However, this contradicts the fact $(\hat{\eta}, (\hat{y}, \hat{z})) \in T \setminus \{(\frac{\lambda_2^r}{\gamma_2^P(1)}, (0, 0))\}$. $\square$

4.5. Proof of the main results

In view of Lemma 4.3, it is clear that any solution of (4.8) of the form $(1, (y, z))$ with $(y, z) \in S_2^- \times S_2^-$ yields a solution (y, z) of (4.4). We shall show T crosses the hyperplane $\{1\} \times \mathbb{E}$, i.e.

$$T \cap (\{1\} \times \mathbb{E}) \neq \emptyset. \tag{4.20}$$

Let $n_0 \in \mathbb{N}$ be such that

$$\frac{1}{n_0} < \lambda_{*,2}.$$

For $n \geq n_0$, let $(\eta_n, (y_n, z_n)) \in T$ satisfy

$$\eta_n + \|(y_n, z_n)\|_{\mathbb{E}} \to \infty.$$

It is easy to check that

$$\eta_n > 0, \quad n \geq n_0. \tag{4.21}$$

From (A3), it follows that $\lambda_{*,2} < 1$, i.e.

$$\frac{\lambda_2^r}{\gamma_2^P(1)} < 1. \tag{4.22}$$

Assume on the contrary that $T \cap (\{1\} \times \mathbb{E}) = \emptyset$. Then

$$T \subset (0,1) \times \mathbb{E},$$

and accordingly,

$$0 < \eta_n < 1.$$

Thus

$$\|(y_n, z_n)\|_{\mathbb{E}} \to \infty, \quad n \to \infty, \tag{4.23}$$

which together with Lemma 2.3 and (A4) imply that

$$\|(y_n, z_n)\|_\infty \to \infty, \quad n \to \infty. \tag{4.24}$$

This means that T is unbounded in $C[0,R] \times C[0,R]$.

We may assume that $\eta_n \to \bar{\eta} \in [0,1]$ as $n \to \infty$. Let

$$\hat{y}_n := \frac{y_n}{\|(y_n, z_n)\|_\infty}, \quad \hat{z}_n := \frac{z_n}{\|(y_n, z_n)\|_\infty}.$$

Then $\|(\hat{y}_n, \hat{z}_n)\|_\infty = 1$. From (A4) and (4.8), we can get that

$$\begin{cases} -\hat{y}_n'' - \dfrac{N-1}{r}\hat{y}_n' + \hat{y}_n = \eta_n\Big[h_1\hat{y}_n + h_2\hat{z}_n + \dfrac{\hat{\xi}(y_n, z_n)}{\|(y_n, z_n)\|_\infty}\Big], \ r \in (0,R), \\ -\hat{z}_n'' - \dfrac{N-1}{r}\hat{z}_n' + \hat{z}_n = \eta_n\Big[k_1\hat{y}_n + k_2\hat{z}_n + \dfrac{\hat{\zeta}(y_n, z_n)}{\|(y_n, z_n)\|_\infty}\Big], \ r \in (0,R), \\ \hat{y}_n'(0) = \hat{y}_n'(R) = 0, \\ \hat{z}_n'(0) = \hat{z}_n'(R) = 0. \end{cases} \tag{4.25}$$

After taking subsequence if necessary, we may assume that

$$(\eta_n, (\hat{y}_n, \hat{z}_n)) \to (\bar{\eta}, (y^*, z^*)), \quad \text{in } \mathbb{R}^+ \times \mathbb{E}. \tag{4.26}$$

Here

$$\|(y^*, z^*)\|_\infty = 1. \tag{4.27}$$

Let $\tau(1,n)$ and $t(1,n)$ denote the zeros of y_n and z_n, respectively. Then, after taking a subsequence if necessary,

$$\lim_{n\to\infty} \tau(1,n) =: \tau(1,\infty), \quad \lim_{n\to\infty} t(1,n) =: t(1,\infty). \tag{4.28}$$

Denote

$$J_1 := (0, \tau(1,\infty)), \quad J_2 := (\tau(1,\infty), R), \quad I_1 := (0, t(1,\infty)), \quad I_2 := (t(1,\infty), R).$$

Claim. *We claim that*

$$J_1 = \emptyset = I_1 \tag{4.29}$$

and

$$\lim_{n\to\infty} y_n(r) = \lim_{n\to\infty} z_n(r) = +\infty \quad \textit{uniformly in } [\epsilon', R], \tag{4.30}$$

where $\epsilon' > 0$ is small constant.

In fact, suppose on the contrary that

$$J_1 \neq \emptyset, \quad I_1 \neq \emptyset.$$

Then we have from Lemma 4.3 that

$$-\alpha < y_n(r) < 0, \quad r \in (0, \tau(1,n)).$$

Then, for any $r \in (0, \tau(1,n))$, it follows from (4.24) and

$$-\hat{y}_n'' - \frac{N-1}{r}\hat{y}_n' + \hat{y}_n = \eta_n \frac{h(y_n, z_n)}{\|(y_n, z_n)\|_\infty}, \quad r \in (0, \tau(1,n)),$$

that

$$\begin{cases} -{y^*}'' - \frac{N-1}{r}{y^*}' + y^* = 0, \quad r \in J_1, \\ y^*(\tau) = 0 = {y^*}'(\tau), \end{cases}$$

for some $\tau \in J_1$. This implies that

$$y^*(r) = 0, \quad r \in J_1,$$

and so $y^*(r) = 0$, $r \in (0, R)$. Similarly, it is easy to prove that $z^*(r) = 0$, $r \in (0, R)$. However, this contradicts (4.27).

Therefore, the Claim is true.

In the following, we shall use some idea from the proof of [13, Lemma 3.2] and the proof of main results of [6,14] to show (4.20) is valid.

Let $(y_n)^-$ and $(z_n)^-$ be the negative part of y_n and z_n, respectively. Then it follows from Lemma 4.3 that $0 \le (y_n)^- < \alpha$ and $0 \le (z_n)^- < \beta$ since $\eta_n \in (0, 1)$, and consequently,

$$(\hat{y}_n)^- \to 0, \quad (\hat{z}_n)^- \to 0, \quad \text{as } n \to \infty.$$

Combining this with the Claim and using (4.25), it concludes that

$$\begin{cases} -{y^*}'' - \dfrac{N-1}{r}{y^*}' + y^* = \bar{\eta}\big(h_1(y^*)^+ + h_2(z^*)^+\big), \text{ a.e. } r \in (0, R), \\ -{z^*}'' - \dfrac{N-1}{r}{z^*}' + z^* = \bar{\eta}\big(k_1(y^*)^+ + k_2(z^*)^+\big), \text{ a.e. } r \in (0, R), \\ {y^*}'(0) = {y^*}'(R) = 0, \\ {z^*}'(0) = {z^*}'(R) = 0, \end{cases} \tag{4.31}$$

where $(y^*)^+$ and $(z^*)^+$ are the positive part of y^* and z^*, respectively. It is easy to show that

$$y^*(r) \geq 0, \quad z^*(r) \geq 0, \qquad r \in [0, R].$$

Now, we only need to show that

$$y^*(r) > 0, \quad z^*(r) > 0, \qquad r \in [0, R]. \tag{4.32}$$

Suppose on the contrary that one of the sets

$$\{r \in [0, R] : y^*(r) = 0\}, \qquad \{r \in [0, R] : z^*(r) = 0\}$$

is nonempty. Then one of the following cases must occur

(i) $y^*(0) = 0$, $z^*(0) > 0$;
(ii) $y^*(0) > 0$, $z^*(0) = 0$;
(iii) $y^*(0) = 0$, $z^*(0) = 0$.

If (i) holds, then it follows from

$$-y^{*\prime\prime}(r) - \frac{N-1}{r} y^{*\prime}(r) + y^*(r) = \bar{\eta}\big(h_1(y^*)^+ + h_2(z^*)^+\big) \tag{4.33}$$

and $h_2 > 0$ that

$$y^{*\prime\prime}(0) < 0,$$

and subsequently y^* is concave down near $r = 0$. However, this contradicts the fact that y^* is nondecreasing in $[0, R]$.

If (ii) holds, then we have from

$$-z^{*\prime\prime}(r) - \frac{N-1}{r} z^{*\prime}(r) + z^*(r) = \bar{\eta}\big(k_1(y^*)^+ + k_2(z^*)^+\big) \tag{4.34}$$

and $k_1 > 0$ that

$$z^{*\prime\prime}(0) < 0,$$

and subsequently z^* is concave down near $r = 0$. However, this contradicts the fact that z^* is nondecreasing in $[0, R]$.

If (iii) holds, then

$$y^*(0) = y^{*\prime}(0) = z^*(0) = z^{*\prime}(0) = 0. \tag{4.35}$$

This together with (4.33) and (4.34) imply that

$$y^*(r) = z^*(r) = 0, \quad r \in [0, R],$$

and accordingly,

$$\bar{\eta} = \frac{\lambda_2^r}{\gamma_2^P(1)}.$$

However, this contradicts the fact $||(y^*, z^*)||_\infty = 1$.

Therefore, (4.32) is valid.

However, (4.32) contradicts Lemma 4.7. Thus, (4.20) is valid. This completes the proof of Theorem 1.1. □

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