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Positive radial solutions for quasilinear systems in an annulus

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Abstract

We show that either superlinearity or sublinearity assumptions can guarantee the existence of positive radial solutions for quasilinear systems involving the p-Laplacian.

Keywords: Elliptic system; p-Laplacian; Existence; Cone

1. Introduction

In this paper we consider the existence of positive radial solutions for the boundary value problem for the quasilinear system

$$\begin{cases} \operatorname{div}(|\nabla u_1|^{p-2}\nabla u_1) + f^1(u_1, \dots, u_n) = 0, \\ \dots \\ \operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) + f^n(u_1, \dots, u_n) = 0, \\ u_i = 0 \text{ on } |x| = R_1 \text{ and } |x| = R_2, \quad i = 1, \dots, n, \end{cases} \tag{1}$$

in the domain $0 < R_1 < |x| < R_2 < \infty$, $x \in \mathbb{R}^N$, $N \geqslant 2$, where $p > 1$.

When $p = 2$ and $n = 1$, (1) has been studied in [1,2,7,8] and others. In particular, when f is nonnegative and continuous, [1,7] have established the existence of positive radial solutions of the problem under the assumption that f is superlinear, i.e., $f_0 = \lim_{u\to 0} f(u)/u = 0$ and $f_\infty = \lim_{u\to\infty} f(u)/u = \infty$. On the other hand, the author [9] established the existence of positive radial solutions of the problem under the assumption that f is sublinear, i.e., $f_0 = \lim_{u\to 0} f(u)/u = \infty$ and $f_\infty = \lim_{u\to\infty} f(u)/u = 0$. In a recent paper [10], the

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author proved that appropriate combinations of superlinearity and sublinearity of f at zero and infinity can guarantee the existence, multiplicity and nonexistence of positive radial solutions of the problem when $n=1$.

In this short paper, we show that (1) has at least one positive solution if (1) is superlinear or sublinear. For this purpose, we introduce notation $\mathbf{f}_0$ and $\mathbf{f}_\infty$, to characterize superlinearity and sublinearity for (1). They are natural extensions of f_0 and f_∞ defined above for the scalar equation.

Our arguments are based on the fixed point index. Many authors have used the fixed point index for the existence of positive solutions of differential equations, see e.g. [3–5,9–11]. Variational methods have been frequently used for Hamiltonian systems and gradient systems. However, there is apparently no possibility of using variational methods for the n-dimensional quasilinear elliptic system (1), and one has to use topological methods.

We now turn to general assumptions for this paper. Let $\varphi(t)=|t|^{p-2}t$, $p>1$, $\mathbb{R}=(-\infty,\infty)$, $\mathbb{R}_+=[0,\infty)$ and

$$\mathbb{R}_+^n=\underbrace{\mathbb{R}_+\times\cdots\times\mathbb{R}_+}_{n}.$$

Also, for $\mathbf{u}=(u_1,\ldots,u_n)\in\mathbb{R}_+^n$, let $\|\mathbf{u}\|=\sum_{i=1}^n|u_i|$.

(H1) $f^i:\mathbb{R}_+^n\to\mathbb{R}_+$ is continuous, $i=1,\ldots,n$.

In order to state our results, let $\mathbf{u}=(u_1,\ldots,u_n)\in\mathbb{R}_+^n$ and $\mathbf{f}(\mathbf{u})=(f^1(\mathbf{u}),\ldots,f^n(\mathbf{u}))$, then we introduce the notation $f_0^i=\lim_{\|\mathbf{u}\|\to 0}f^i(\mathbf{u})/\varphi(\|\mathbf{u}\|)$, $f_\infty^i=\lim_{\|\mathbf{u}\|\to\infty}f^i(\mathbf{u})/\varphi(\|\mathbf{u}\|)$, $i=1,\ldots,n$, $\mathbf{f}_0=\sum_{i=1}^n f_0^i$, $\mathbf{f}_\infty=\sum_{i=1}^n f_\infty^i$.

Our main results are:

Theorem 1.1. *Assume* (H1) *holds.*

(a) *If* $\mathbf{f}_0=0$ *and* $\mathbf{f}_\infty=\infty$, *then* (1) *has a positive radial solution.*
(b) *If* $\mathbf{f}_0=\infty$ *and* $\mathbf{f}_\infty=0$, *then* (1) *has a positive radial solution.*

2. Preliminaries

A radial solution of (1) can be considered as a solution of the system

$$\begin{cases}(r^{N-1}\varphi(u_1'(r)))'+r^{N-1}f^1(u_1,\ldots,u_n)=0,\\ \ldots\\ (r^{N-1}\varphi(u_n'(r)))'+r^{N-1}f^n(u_1,\ldots,u_n)=0,\\ u_i(R_1)=u_i(R_2)=0,\quad i=1,\ldots,n.\end{cases}\tag{2}$$

We shall treat classical solutions of (2), namely vector-valued functions $\mathbf{u}=(u_1(r),\ldots,u_n(r))\in C^1([R_1,R_2],\mathbb{R}^n)$ with $\varphi(u_i')\in C^1(R_1,R_2)$, $i=1,\ldots,n$, which satisfies (2) for $r\in(R_1,R_2)$. A solution $\mathbf{u}(r)=(u_1(r),\ldots,u_n(r))$ is positive if $u_i(r)\geqslant 0$, $i=1,\ldots,n$, for all $r\in(R_1,R_2)$ and there is at least one nontrivial component of $\mathbf{u}$. In fact, we shall show that such a nontrivial component of $\mathbf{u}$ is positive on (R_1,R_2).

Applying the change of variables, $r = (R_2 - R_1)t + R_1$, we can transform (2) into the form

$$\begin{cases} (q(t)\varphi(\zeta u_1'))' + h_1(t) f^1(\mathbf{u}) = 0, \\ \dots \\ (q(t)\varphi(\zeta u_n'))' + h_n(t) f^n(\mathbf{u}) = 0, \\ \mathbf{u}(0) = \mathbf{u}(1) = 0, \end{cases} \tag{3}$$

$0 < t < 1$, where $\mathbf{u}(t) = (u_1(t), \dots, u_n(t))$,

$$q(t) = ((R_2 - R_1)t + R_1)^{N-1}, \quad \zeta = \frac{1}{R_2 - R_1}$$

and $h_i(t) = (R_2 - R_1)((R_2 - R_1)t + R_1)^{N-1}, \; i = 1, \dots, n$. It is clear that $q(t) \in C[0, 1]$ with $q > 0$ and is nondecreasing for $t \in [0, 1]$.

For (3) we shall prove Theorem 2.1, which immediately implies that Theorem 1.1 is true.

Theorem 2.1. *Assume* (H1) *holds.*

(a) *If* $\mathbf{f}_0 = 0$ *and* $\mathbf{f}_\infty = \infty$, *then* (3) *has a positive solution.*
(b) *If* $\mathbf{f}_0 = \infty$ *and* $\mathbf{f}_\infty = 0$, *then* (3) *has a positive solution.*

The following well-known result of the fixed point index is crucial in our arguments.

Lemma 2.2 (*Guo and Lakshmikanthan [5], Krasnoselskii [6]*). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|x\| = r\}$.*

(i) *If $\|Tx\| \geqslant \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.*
(ii) *If $\|Tx\| \leqslant \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.*

In order to apply Lemma 2.2 to (3), let X be the Banach space

$$\underbrace{C[0, 1] \times \cdots \times C[0, 1]}_{n}$$

and, for $\mathbf{u} = (u_1, \dots, u_n) \in X$, $\|\mathbf{u}\| = \sum_{i=1}^n \sup_{t \in [0,1]} |u_i(t)|$. For $\mathbf{u} \in X$ or $\mathbb{R}_+^n$, $\|\mathbf{u}\|$ denotes the norm of $\mathbf{u}$ in X or $\mathbb{R}_+^n$, respectively. Define K be a cone in X by

$$K = \left\{ \mathbf{u} = (u_1, \dots, u_n) \in X : u_i(t) \geqslant 0, \; t \in [0, 1], \; i = 1, \dots, n, \right.$$
$$\left. \text{and} \quad \min_{1/4 \leqslant t \leqslant 3/4} \sum_{i=1}^n u_i(t) \geqslant \frac{1}{4} \|\mathbf{u}\| \right\}.$$

Also, define, for r a positive number, Ω_r by $\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| < r\}$. Note that $\partial \Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| = r\}$.

Let $\mathbf{T} : K \to X$ be a map with components $(T^1, \ldots, T^n)$. We define T^i, $i = 1, \ldots, n$, by

$$T^i\mathbf{u}(t) = \begin{cases} \int_0^t \frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i} h_i(\tau) f^i(\mathbf{u}(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s, & 0 \leqslant t \leqslant \sigma_i, \\ \int_t^1 \frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s} h_i(\tau) f^i(\mathbf{u}(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s, & \sigma_i \leqslant t \leqslant 1, \end{cases} \tag{4}$$

where $\sigma_i \in (0, 1)$ is a solution of the equation $\Theta^i\mathbf{u}(t) = 0$, $0 \leqslant t \leqslant 1$, where the map $\Theta^i : K \to C[0, 1]$ is defined by

$$\Theta^i\mathbf{u}(t) = \int_0^t \frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_s^t h_i(\tau) f^i(\mathbf{u}(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s - \int_t^1 \frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_t^s h_i(\tau) f^i(\mathbf{u}(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s.$$

By virtue of Lemma 2.3, the operator $\mathbf{T}$ is well defined.

Lemma 2.3 (*Wang [10,11]*). *Assume* (H1) *holds. Then, for any* $\mathbf{u} \in K$ *and* $i = 1, \ldots, n$, $\Theta^i\mathbf{u}(t)=0$ *has at least one solution in* $(0, 1)$. *In addition, if* $\sigma_i^1 < \sigma_i^2 \in (0, 1)$, $i=1, \ldots, n$, *are two solutions of* $\Theta^i\mathbf{u}(t) = 0$, *then* $h_i(t) f^i(\mathbf{u}(t)) \equiv 0$ *for* $t \in [\sigma_i^1, \sigma_i^2]$ *and any* $\sigma_i \in [\sigma_i^1, \sigma_i^2]$ *is also a solution of* $\Theta^i\mathbf{u}(t) = 0$. *Furthermore*, $T^i\mathbf{u}(t)$, $i = 1, \ldots, n$, *is independent of the choice of* $\sigma_i \in [\sigma_i^1, \sigma_i^2]$.

The following lemma is a standard result due to the concavity of $u(t)$ on $[0, 1]$ (see e.g. [5,10,11] for proofs).

Lemma 2.4. *Let* $u \in C^1[0, 1]$ *with* $u(t) \geqslant 0$ *for* $t \in [0, 1]$. *Assume* $q(t) \in C[0, 1]$ *with* $q > 0$ *and is nondecreasing for* $t \in [0, 1]$. *If* $q(t)\varphi(\zeta u')$ *is nonincreasing on* $[0, 1]$, *then* $u(t) \geqslant \min\{t, 1 - t\}\sup_{t\in[0,1]}|u(t)|$, $t \in [0, 1]$. *In particular*, $\min_{1/4 \leqslant t \leqslant 3/4} u(t) \geqslant \frac{1}{4}\sup_{t\in[0,1]} |u(t)|$.

We remark that, according to Lemma 2.4, any nontrivial component of nonnegative solutions of (3) is positive on $(0, 1)$.

Now it is easy to see that $\mathbf{T}(K) \subset K$ and $\mathbf{T} : K \to K$ is compact and continuous (see [11]), and that (3) is equivalent to the fixed point equation $\mathbf{Tu} = \mathbf{u}$ in K.

Note that for $t > 0$, $\varphi(t) = t^{p-1}$, $p > 1$ and $\varphi^{-1}(t) = t^{1/(p-1)}$.

Lemma 2.5. *For all* $\sigma, x \in (0, \infty)$, $\varphi^{-1}(\sigma\varphi(x)) = \varphi^{-1}(\sigma)x$.

For $i = 1, \ldots, n$, and $t \in [\frac{1}{4}, \frac{3}{4}]$, let

$$\gamma_i(t) = \frac{1}{8}\left[\int_{1/4}^t \frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_s^t h_i(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \int_t^{3/4} \frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_t^s h_i(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right].$$

It follows that

$$\Gamma = \min\{\gamma_i(t) : \tfrac{1}{4} \leqslant t \leqslant \tfrac{3}{4},\ i = 1, \ldots, n\} > 0.$$

Lemma 2.6. *Assume* (H1) *holds. Let* $\mathbf{u}=(u_1(t),\ldots,u_n(t))\in K$ *and* $\eta>0$. *If there exists a component* f^i *of* $\mathbf{f}$ *such that* $f^i(\mathbf{u}(t))\geqslant\varphi(\eta\sum_{i=1}^n u_i(t))$ *for* $t\in[\frac{1}{4},\frac{3}{4}]$, *then* $\|\mathbf{Tu}\|\geqslant\Gamma\eta\|\mathbf{u}\|$.

Proof. Note, from the definition of $\mathbf{Tu}$, that $T^i\mathbf{u}(\sigma_i)$ is the maximum value of $T^i\mathbf{u}$ on [0,1]. If $\sigma_i\in[\frac{1}{4},\frac{3}{4}]$, we consider

$$
\begin{aligned}
2\|\mathbf{Tu}\| &\geqslant 2\sup_{t\in[0,1]}|T^i\mathbf{u}(t)|\\
&\geqslant \int_{1/4}^{\sigma_i}\frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i}h_i(\tau)f^i(\mathbf{u}(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad+\int_{\sigma_i}^{3/4}\frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s}h_i(\tau)f^i(\mathbf{u}(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\geqslant \int_{1/4}^{\sigma_i}\frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i}h_i(\tau)\varphi\left(\eta\sum_{i=1}^n u_i(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad+\int_{\sigma_i}^{3/4}\frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s}h_i(\tau)\varphi\left(\eta\sum_{i=1}^n u_i(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s\\
&\geqslant \int_{1/4}^{\sigma_i}\frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i}h_i(\tau)\varphi\left(\eta\frac{1}{4}\|\mathbf{u}\|\right)\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad+\int_{\sigma_i}^{3/4}\frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s}h_i(\tau)\varphi\left(\eta\frac{1}{4}\|\mathbf{u}\|\right)\mathrm{d}\tau\right)\mathrm{d}s.
\end{aligned}
$$

Now, because of Lemma 2.5, we have

$$
\begin{aligned}
\|\mathbf{Tu}\| &\geqslant \frac{\eta\|\mathbf{u}\|\frac{1}{4}}{2}\left[\int_{1/4}^{\sigma_i}\frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i}h_i(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right.\\
&\quad\left.+\int_{\sigma_i}^{3/4}\frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s}h_i(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right]\\
&\geqslant \Gamma\eta\|\mathbf{u}\|.
\end{aligned}
$$

For $\sigma_i>\frac{3}{4}$, it is easy to see

$$
\|\mathbf{Tu}\|\geqslant\int_{1/4}^{3/4}\frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{3/4}h_i(\tau)f^i(\mathbf{u}(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s.
$$

On the other hand, we have

$$
\|\mathbf{Tu}\|\geqslant\int_{1/4}^{3/4}\frac{1}{\zeta}\,\varphi^{-1}\left(\frac{1}{q(s)}\int_{1/4}^{s}h_i(\tau)f^i(\mathbf{u}(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s.
$$

if $\sigma_i<\frac{1}{4}$. Therefore, similar arguments show that $\|\mathbf{Tu}\|\geqslant\Gamma\eta\|\mathbf{u}\|$ if $\sigma_i>\frac{3}{4}$ or $\sigma_i<\frac{1}{4}$. □

For each $i=1,\ldots,n$, define a new function $\hat{f}^i(t):\mathbb{R}_+\to\mathbb{R}_+$ by $\hat{f}^i(t)=\max\{f^i(\mathbf{u}):\mathbf{u}\in\mathbb{R}_+^n$ and $\|\mathbf{u}\|\leqslant t\}$. Note that $\hat{f}_0^i=\lim_{t\to 0}\hat{f}^i(t)/\varphi(t)$ and $\hat{f}_\infty^i=\lim_{t\to\infty}\hat{f}^i(t)/\varphi(t)$.

Lemma 2.7 (*Wang [11]*). *Assume* (H1) *holds. Then* $\hat{f}_0^i = f_0^i$ *and* $\hat{f}_\infty^i = f_\infty^i$, $i = 1, \ldots, n$.

Lemma 2.8. *Assume* (H1) *holds and let* $r > 0$. *If there exists an* $\varepsilon > 0$ *such that* $\hat{f}^i(r) \leqslant \varphi(\varepsilon)$ *breakvarphi*(r), $i = 1, \ldots, n$, *then* $\|\mathbf{Tu}\| \leqslant \varepsilon\hat{C}\|\mathbf{u}\|$ *for* $\mathbf{u} \in \partial\Omega_r$, *where the constant*

$$\hat{C} = \frac{1}{\zeta} \sum_{i=1}^{n} \varphi^{-1}\left(\frac{1}{q(0)} \int_0^1 h_i(\tau)\,\mathrm{d}\tau\right).$$

Proof. From the definition of T, for $\mathbf{u} \in \partial\Omega_r$, we have

$$\begin{aligned}
\|\mathbf{Tu}\| &= \sum_{i=1}^{n} \sup_{t\in[0,1]} |T^i\mathbf{u}(t)| \\
&\leqslant \frac{1}{\zeta} \sum_{i=1}^{n} \varphi^{-1}\left[\frac{1}{q(0)} \int_0^1 h_i(\tau) f^i(\mathbf{u}(\tau))\,\mathrm{d}\tau\right] \\
&\leqslant \frac{1}{\zeta} \sum_{i=1}^{n} \varphi^{-1}\left[\frac{1}{q(0)} \int_0^1 h_i(\tau)\,\mathrm{d}\tau \hat{f}^i(r)\right] \\
&\leqslant \frac{1}{\zeta} \sum_{i=1}^{n} \varphi^{-1}\left[\frac{1}{q(0)} \int_0^1 h_i(\tau)\,\mathrm{d}\tau \varphi(\varepsilon r)\right].
\end{aligned}$$

Then Lemma 2.5 implies that

$$\|\mathbf{Tu}\| \leqslant \varepsilon r \frac{1}{\zeta} \sum_{i=1}^{n} \varphi^{-1}\left(\frac{1}{q(0)} \int_0^1 h_i(\tau)\,\mathrm{d}\tau\right) = \varepsilon\hat{C}\|\mathbf{u}\|. \qquad \square$$

3. Proof of Theorem 2.1

Proof. *Part* (a): $\mathbf{f}_0 = 0$ implies that $f_0^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.7 that $\hat{f}_0^i = 0$, $i = 1, \ldots, n$. Therefore, we can choose $r_1 > 0$ so that $\hat{f}^i(r_1) \leqslant \varphi(\varepsilon)\varphi(r_1)$, $i = 1, \ldots, n$, where the constant $\varepsilon > 0$ satisfies $\varepsilon\hat{C} < 1$, and $\hat{C}$ is the positive constant defined in Lemma 2.8. We have by Lemma 2.8 that $\|\mathbf{Tu}\| \leqslant \varepsilon\hat{C}\|\mathbf{u}\| < \|\mathbf{u}\|$ for $\mathbf{u} \in \partial\Omega_{r_1}$. Now, since $\mathbf{f}_\infty = \infty$, there exists a component f^i of $\mathbf{f}$ such that $f_\infty^i = \infty$. Therefore, there is an $\hat{H} > 0$ such that $f^i(\mathbf{u}) \geqslant \varphi(\eta)\varphi(\|\mathbf{u}\|)$ for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \geqslant \hat{H}$, where $\eta > 0$ is chosen so that $\Gamma\eta > 1$. Let $r_2 = \max\{2r_1, 4\hat{H}\}$. If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_2}$, then $\min_{1/4 \leqslant t \leqslant 3/4} \sum_{i=1}^n u_i(t) \geqslant \frac{1}{4}\|\mathbf{u}\| = \frac{1}{4} r_2 \geqslant \hat{H}$, which implies that $f^i(\mathbf{u}(t)) \geqslant \varphi(\eta)\varphi$ $(\sum_{i=1}^n u_i(t)) = \varphi(\eta\sum_{i=1}^n u_i(t))$ for $t \in [\frac{1}{4}, \frac{3}{4}]$. It follows from Lemma 2.6 that $\|\mathbf{Tu}\| \geqslant \Gamma\eta$ $\|\mathbf{u}\| > \|\mathbf{u}\|$ for $\mathbf{u} \in \partial\Omega_{r_2}$. By Lemma 2.2, $i(\mathbf{T}, \Omega_{r_1}, K) = 1$ and $i(\mathbf{T}, \Omega_{r_2}, K) = 0$. It follows from the additivity of the fixed point index that $i(\mathbf{T}, \Omega_{r_2}\backslash\bar{\Omega}_{r_1}, K) = -1$. Thus, $i(\mathbf{T}, \Omega_{r_2}\backslash\bar{\Omega}_{r_1}, K) \neq 0$, which implies $\mathbf{T}$ has a fixed point $\mathbf{u} \in \Omega_{r_2}\backslash\bar{\Omega}_{r_1}$ by the existence property of the fixed point index. The fixed point $\mathbf{u} \in \Omega_{r_2}\backslash\bar{\Omega}_{r_1}$ is the desired positive solution of (3).

Part (b): If $\mathbf{f}_0 = \infty$, there exists a component f^i such that $f_0^i = \infty$. Therefore, there is an $r_1 > 0$ such that $f^i(\mathbf{u}) \geqslant \varphi(\eta)\varphi(\|\mathbf{u}\|)$ for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \leqslant r_1$, where $\eta > 0$ is chosen so that $\Gamma\eta > 1$. If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_1}$, then $f^i(\mathbf{u}(t)) \geqslant \varphi(\eta)\varphi(\sum_{i=1}^n u_i(t)) = \varphi(\eta \sum_{i=1}^n u_i(t))$ for $t \in [0, 1]$. Lemma 2.6 implies that $\|\mathbf{Tu}\| \geqslant \Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\|$ for $\mathbf{u} \in \partial\Omega_{r_1}$. We now determine Ω_{r_2}. $\mathbf{f}_\infty = 0$ implies that $f_\infty^i = 0, i = 1, \ldots, n$. It follows from Lemma 2.7 that $\hat{f}_\infty^i = 0, i = 1, \ldots, n$. Therefore there is an $r_2 > 2r_1$ such that $\hat{f}^i(r_2) \leqslant \varphi(\varepsilon)\varphi(r_2),\ i = 1, \ldots, n$, where the constant $\varepsilon > 0$ satisfies $\varepsilon\hat{C} < 1$, and $\hat{C}$ is the positive constant defined in Lemma 2.8. Thus, we have by Lemma 2.8 that $\|\mathbf{Tu}\| \leqslant \varepsilon\hat{C}\|\mathbf{u}\| < \|\mathbf{u}\|$ for $\mathbf{u} \in \partial\Omega_{r_2}$. By Lemma 2.2, $i(\mathbf{T}, \Omega_{r_1}, K) = 0$ and $i(\mathbf{T}, \Omega_{r_2}, K) = 1$. It follows from the additivity of the fixed point index that $i(\mathbf{T}, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $\mathbf{T}$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (3). □

References

[1] C. Bandle, C.V. Coffman, M. Marcus, Nonlinear elliptic problems in annular domains, J. Differential Equations 69 (1987) 322–345.

[2] C. Bandle, M. Kwong, Semilinear elliptic problems in annular domains, Z. Angew. Math. Phys. 40 (1989) 245–257.

[3] D. Dunninger, H. Wang, Existence and multiplicity of positive radial solutions for elliptic systems, Nonlinear Anal. 29 (1997) 1051–1060.

[4] D.G. de Figueiredo, P. Lions, R.D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. Pures Appl. 61 (1982) 41–63.

[5] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, Orlando, FL, 1988.

[6] M. Krasnoselskii, Positive Solutions of Operator Equations, Noordhoff, Groningen, 1964.

[7] S.S. Lin, On the existence of positive radial solutions for semilinear elliptic equations in annular domains, J. Differential Equations 81 (1989) 221–233.

[8] W.-M. Ni, R.D. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$, Comm. Pure Appl. Math. 38 (1985) 67–108.

[9] H. Wang, On the existence of positive solutions for semilinear elliptic equations in the annulus, J. Differential Equations 109 (1994) 1–7.

[10] H. Wang, On the structure of positive radial solutions for quasilinear equations in annular domains, Adv. Differential Equations 8 (2003) 111–128.

[11] H. Wang, On the number of positive solutions of nonlinear systems, J. Math. Anal. Appl. 281 (2003) 287–306.