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Periodic solutions to non-autonomous second-order systems

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ABSTRACT

We establish the existence of periodic solution of a class of non-autonomous second-order systems, $\ddot{x}+\mu x+V(t,x)=0$, where $V(t,x)=(v_1(t,x),\ldots,v_n(t,x))$, if $\lim_{|x|\to\infty}\frac{v_i(t,x)}{|x|}=0$, $i=1,\ldots,n$, uniformly in t and V is bounded below or above for appropriate ranges of μ.

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1. Introduction

We study the existence of solutions of a class of periodic boundary value problems for non-autonomous second-order systems.

$$\begin{cases}\ddot{x}+\mu x+V(t,x)=0, & t\in[0,T]\\ x(0)=x(T), & x'(0)=x'(T),\end{cases} \tag{1.1}$$

where $x=(x_1,\ldots,x_n)$, $V(t,x)=(v_1(t,x),\ldots,v_n(t,x))\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ is periodic of period T in the t variable, and μ is a constant.

The existence of solutions of (1.1) has been studied by many researchers, see Mawhin and Willem [1] and the references therein. The variational method has been mostly used to prove the existence of solutions of (1.1). Fixed point theorems such as Rothe's theorem can also be used to prove the existence of solutions of (1.1). In this paper, we choose the fixed point theorem in cones [2,3] to establish the existence of a solution for (1.1). We believe that the fixed point theorem in cones can be further used to treat other cases of this problem. The fixed point theorem in cones has been employed to establish the existence of positive solution boundary value problems with some superlinear and sublinear assumptions at zero and infinity, see e. g. Erbe and the author [4], Torres [5], the author [6], Graef, Kong and the author [15]. Systems of differential equations can be treated similarly by constructing appropriate product spaces [7–10]. In a recent paper, O'Regan and the author [11] obtained the existence, multiplicity and nonexistence of positive periodic solutions of general second-order systems. Precup [12,13] gave a vector version of the fixed point theorem in cones and applications to systems of equations whose components have different sublinear or superlinear behaviors. In this paper, while we assume that all components are sublinear at infinity, existence results may be established by using the theorems in [12]. It would be interesting to address it in future research.

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Our result complements a few results in Mawhin and Willem [1] and Zhao and Wu [14]. The existence of solutions of (1.1) is considered in [1, Theorem 3.7] with $V(t,x) = \nabla W(t,x) = \left(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_n}\right)$. Among other requirements, $W(t,\cdot)$ in [1, Theorem 3.7] is convex. [14] studied the existence of solutions of the following non-autonomous second-order systems,

$$\begin{cases} \ddot{x} = \nabla F(t,x), & t \in [0,T] \\ x(0) = x(T), & x'(0) = x'(T), \end{cases} \tag{1.2}$$

among other assumptions, $F(t,x)$ in [14] is required to satisfy the following condition,

$$\begin{aligned} &\text{There exist } g_1, g_2 \in L^1([0,T], \mathbb{R}^+) \text{ and } \int_0^T g_1(t)\mathrm{d}t < \frac{12}{T}, \\ &\quad \text{such that } |\nabla F(t,x)|_1 \le g_1(t)|x|_1 + g_2(t), \quad \text{for all } x \in \mathbb{R}^n, \end{aligned} \tag{1.3}$$

where $|\cdot|_1$ is the Euclidean norm. In order that $F(t,x) = -\frac{1}{2}\mu(|x|_1)^2 - V(t,x)(\nabla F(t,x) = -\mu x - \nabla V)$ satisfies (1.3), μ cannot take all negative values in most cases, which is more restrictive than Theorem 1.1(a). In addition, the solutions we obtain here are twice continuously differentiable functions.

Our main result of this paper is Theorem 1.1. Throughout the paper, we will use the notation $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_+^n = \Pi_{i=1}^n \mathbb{R}_+$, and denote by $|x| = \sum_{i=1}^n |x_i|$ the usual norm of $\mathbb{R}_+^n$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Here we use the summation norm for convenience. Theorem 1.1 is true for the Euclidean norm and the norm $|x|_\infty = \max |x_i|$. Since we know that they are equivalent for $\mathbb{R}^n$, if $\lim_{|x|\to\infty} \frac{v_i(t,x)}{|x|} = 0$ for one of the norms, it is also true for others. We understand that a function $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is bounded below (above) if there is a constant $c > 0$ such that each component of h is greater than $-c$ (less than c) for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

Theorem 1.1. *Assume that* $\lim_{|x|\to\infty} \frac{v_i(t,x)}{|x|} = 0$, $i = 1 \ldots n$, *uniformly in* $t \in [0,T]$.

(a) *If* $\mu \in (-\infty, 0)$ *and* $V(t,x)$ *is bounded below, then* (1.1) *has a solution* $x(t) \in C^2([0,T], \mathbb{R}^n)$.
(b) *If* $\mu \in (0, (\frac{\pi}{T})^2)$ *and* $V(t,x)$ *is bounded above, then* (1.1) *has a solution* $x(t) \in C^2([0,T], \mathbb{R}^n)$.

2. Preliminary results

Consider the two scalar periodic boundary value problems

$$\begin{cases} -v'' + kv = e(t), & k \in (0,\infty) \\ v(0) = v(T), & v'(0) = v'(T), \end{cases} \tag{2.4}$$

and

$$\begin{cases} u'' + ku = e(t), & k \in \left(0, \left(\frac{\pi}{T}\right)^2\right) \\ u(0) = u(T), & u'(0) = u'(T), \end{cases} \tag{2.5}$$

where $e(t)$ is a continuous function on $[0,T]$. A direct calculation verifies that positive solutions of (2.4) and (2.5) can be expressed in the following forms

$$v(t) = \int_0^T G_1(t,s)e(s)\mathrm{d}s, \qquad u(t) = \int_0^T G_2(t,s)e(s)\mathrm{d}s$$

respectively, where

$$G_1(t,s) = \begin{cases} \dfrac{\mathrm{e}^{\sqrt{k}(t-s)} + \mathrm{e}^{\sqrt{k}(T-t+s)}}{2\sqrt{k}(\mathrm{e}^{\sqrt{k}T} - 1)}, & 0 \le s \le t \le T, \\ \dfrac{\mathrm{e}^{\sqrt{k}(s-t)} + \mathrm{e}^{\sqrt{k}(T-s+t)}}{2\sqrt{k}(\mathrm{e}^{\sqrt{k}T} - 1)}, & 0 \le t \le s \le T. \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{\sin\sqrt{k}(t-s) + \sin\sqrt{k}(T-t+s)}{2\sqrt{k}(1-\cos\sqrt{k}T)}, & 0 \le s \le t \le T, \\ \dfrac{\sin\sqrt{k}(s-t) + \sin\sqrt{k}(T-s+t)}{2\sqrt{k}(1-\cos\sqrt{k}T)}, & 0 \le t \le s \le T. \end{cases}$$

We will need some estimates on G_1 and G_2 in the following lemma. The estimates can be found in [9,11,5]. It can also be proved by simple trig formulas. We omit its proof.

Lemma 2.1. *Let*

$$m = \min\left\{ \frac{\cot\frac{\sqrt{k}T}{2}}{2\sqrt{k}}, \frac{e^{\frac{\sqrt{k}T}{2}}}{\sqrt{k}(e^{\sqrt{k}T}-1)} \right\} > 0$$

and

$$M = \max\left\{ \frac{1}{2\sqrt{k}\sin\frac{\sqrt{k}T}{2}}, \frac{1+e^{\sqrt{k}T}}{2\sqrt{k}(e^{\sqrt{k}T}-1)} \right\} > 0.$$

Then

$$m \le G_1(t,s), G_2(t,s) \le M \quad \textit{for} \;\; t,s \in [0,T].$$

In general solutions of (1.1) are not necessarily positive. By choosing an appropriate transformation, we will seek positive solutions of a new system. A positive solution of systems of differential equations here simply requires that at least one component of the solution is positive. This transformation has been used in semipositone problems. The two cases that $V(t,x)$ is bounded below or above are dealt with separately. Here we are able to rewrite this periodic boundary value problem into appropriate integral equations according to ranges of the parameter μ. Let $u = x + P$, $P = (p,\dots,p)$, $p > 0$ is a constant. If $V(t,x)$ is bounded below and $\mu < 0$, then we rewrite (1.1) as

$$-\ddot{x} - \mu x = V(t,x) \tag{2.6}$$

and further let $u = x + P$, (2.6) is transformed into

$$-\ddot{u} - \mu u = -\mu P + V(t, u-P) \tag{2.7}$$

where $p > 0$ is chosen so that $-p\mu + v_i(t, u-P) > 1$, $i = 1,\dots,n$ for $(t,u) \in \mathbb{R}\times\mathbb{R}^n$.

If $V(t,x)$ is bounded above and $\mu \in (0, (\frac{\pi}{T})^2)$, then we rewrite (1.1) as

$$\ddot{x} + \mu x = -V(t,x) \tag{2.8}$$

and further let $u = x + P$, (2.8) is transformed into

$$\ddot{u} + \mu u = \mu P - V(t, u-P) \tag{2.9}$$

where $p > 0$ is chosen so that $p\mu - v_i(t, u-P) > 1$, $i = 1,\dots,n$ for $(t,u) \in \mathbb{R}\times\mathbb{R}^n$.

Now solutions of (1.1) can be rewritten as fixed points of operators $\mathcal{T}^i u(t)$ in an appropriate Banach space, where

$$\mathcal{T}^i u(t) = \int_0^T G(t,s) f^i(s,u(s))\mathrm{d}s, \quad i = 1,\dots,n \; 0 \le t \le T, \tag{2.10}$$

and G, f^i take different expressions for (2.7) and (2.9). When $V(t,x)$ is bounded below and $\mu < 0$,

$$G(t,s) = G_1(t,s), \qquad f^i(t,u) = -\mu p + v_i(t, u-P).$$

When $V(t,x)$ is bounded above and $\mu \in (0, (\frac{\pi}{T})^2)$,

$$G(t,s) = G_2(t,s), \qquad f^i(t,u) = \mu p - v_i(t, u-P).$$

In either case, we see that $f^i > 1$ for $(t,u) \in \mathbb{R}\times\mathbb{R}^n$ and $\lim_{|u|\to\infty} \frac{f_i(t,u)}{|u|} = 0$, $i = 1\dots,n$, uniformly in t if the conditions of Theorem 1.1 hold. In the remaining of the paper, we only deal with f^i. We summarize the properties of f^i needed in the following lemma.

Lemma 2.2. *If the conditions of Theorem 1.1 hold. Then, for $i = 1,\dots,n$, $f^i(t,u)$ in (2.10) is continuous on $\mathbb{R}\times\mathbb{R}^n$, $f^i(t,u) > 1$ for $(t,u) \in \mathbb{R}\times\mathbb{R}^n$, $\lim_{|u|\to\infty} \frac{f^i(t,u)}{|u|} = 0$ uniformly in t.*

We shall use a well-known fixed point theorem in a cone to establish the existence of periodic solutions of (1.1). Let E be a Banach space and K be a closed, nonempty subset of E. K is said to be a cone if (*i*) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \ge 0$ and (*ii*) $u, -u \in K$ imply $u = 0$.

Lemma 2.3 ([2,3]). *Let X be a Banach space and K $(\subset X)$ be a cone. Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$\mathcal{T} : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|\mathcal{T}u\| \geq \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|\mathcal{T}u\| \leq \|u\|,\ u \in K \cap \partial\Omega_2$; *or*
(ii) $\|\mathcal{T}u\| \leq \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|\mathcal{T}u\| \geq \|u\|,\ u \in K \cap \partial\Omega_2$.

Then $\mathcal{T}$ *has a fixed point in* $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Consider the Banach space $X = \underbrace{C[0,T] \times \cdots \times C[0,T]}_{n}$, and for $u = (u_1, \ldots, u_n) \in X$, let

$$\|u\| = \sum_{i=1}^{n} \sup_{t\in[0,T]} |u_i(t)|.$$

Let K be the cone given by

$$K = \{u = (u_1, \ldots, u_n) \in X : u_i(t) \geq 0, \quad t \in [0,T],\ i = 1, \ldots, n,$$
$$\text{and } \min_{0\leq t\leq T} \sum_{i=1}^{n} u_i(t) \geq \frac{m}{M}\|u\|\}$$

Also, for $r > 0$, define K_r by

$$K_r = \{u \in K : \|u\| < r\}.$$

Note that $\partial K_r = \{u \in K : \|u\| = r\}$. For simplicity, we denote by $\mathcal{T} : K \to X$ the operator

$$\mathcal{T} = (\mathcal{T}^1, \ldots, \mathcal{T}^n)$$

where $\mathcal{T}^i$ is defined in (2.10). Now we need to verify whether the operator $\mathcal{T}$ satisfies the conditions of Lemma 2.3.

Lemma 2.4. *$\mathcal{T}(K) \subset K$ and $\mathcal{T} : K \to K$ is completely continuous.*

Proof. For $u \in K, i = 1, \ldots, n$, we have

$$\begin{aligned}
\min_{t\in[0,T]} \sum_{i=1}^{n} \mathcal{T}^i u(t) &\geq \sum_{i=1}^{n} \min_{0\leq t\leq T} \mathcal{T}^i x(t) \\
&\geq \sum_{i=1}^{n} m \int_0^T f^i(s, u(s))\mathrm{d}s \\
&= \sum_{i=1}^{n} \frac{m}{M} M \int_0^T f^i(s, u(s))\mathrm{d}s \\
&\geq \sum_{i=1}^{n} \frac{m}{M} \sup_{0\leq t\leq T} \mathcal{T}^i u(t) \\
&\geq \frac{m}{M} \sum_{i=1}^{n} \sup_{0\leq t\leq T} \mathcal{T}^i u(t) = \frac{m}{M}\|\mathcal{T}u\|.
\end{aligned}$$

Thus, $\mathcal{T}(K) \subset K$. It is standard to verify that $\mathcal{T}$ is completely continuous, see e.g. [2]. □

For each $i = 1, \ldots, n$, let $\hat{f}^i : \mathbb{R}_+ \to \mathbb{R}_+$ be the function given by

$$\hat{f}^i(t) = \max_{u\in\mathbb{R}^n_+ \text{ and } |u|\leq t} \{\max_{s\in[0,T]} f^i(s, u)\}.$$

Note that $\hat{f}^i(t)$ is monotone. Lemma 2.5 will simplify our proof. A more general form of the following lemma was proved in [10]. Here we omit its proof.

Lemma 2.5 ([10]). *If* $\lim_{|u|\to\infty} \frac{\max_{s\in[0,T]} f^i(s,u)}{|u|} = 0$, *then* $\lim_{t\to\infty} \frac{\hat{f}^i(t)}{t} = 0$.

3. Proof of Theorem 1.1

In view of Lemma 2.2, $f^i(t, u) > 1$ for $(t, u) \in [0,T] \times \mathbb{R}^n$. It then follows that there is a sufficiently small $r_1 > 0$ such that

$$f^i(t, u) \geq \eta|u|$$

for $u = (u_1, \ldots, u_n) \in \mathbb{R}^n_+$ and $|u| \leq r_1$, where $\eta > 0$ is chosen so that

$$Tm\eta\frac{m}{M} > 1.$$

If $u = (u_1, \ldots, u_n) \in \partial K_{r_1}$, then

$$\begin{aligned}\|\mathcal{T}u\| &\geq \max_{0\leq t\leq T} T^i u(t)\\ &\geq m\int_0^T f^i(s, u(s))\mathrm{d}s\\ &\geq m\int_0^T \eta\sum_{i=1}^n u_i(s)\mathrm{d}s\\ &\geq Tm\eta\frac{m}{M}\|u\|\\ &\geq \|u\|.\end{aligned}$$

On the other hand, by the construction of f, we know that $\lim_{|u|\to\infty}\frac{f^i(t,u)}{|u|} = 0$ uniformly in t. Because of the uniformity in t, it follows that $\lim_{|u|\to\infty}\frac{\max_{t\in[0,T]} f^i(t,u)}{|u|} = 0$. Thus Lemma 2.5 implies that $\lim_{t\to\infty}\frac{\hat{f}^i(t)}{t} = 0$, $i = 1, \ldots, n$. Therefore there is an $r_2 > 2r_1$ such that

$$\hat{f}^i(r_2) \leq \varepsilon r_2, \quad i = 1, \ldots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$MnT\varepsilon < 1.$$

Thus, for $u \in \partial K_{r_2}$, we have

$$\begin{aligned}\|\mathcal{T}u\| &= \sum_{i=1}^n \max_{0\leq t\leq T} T^i u(t)\\ &\leq M\sum_{i=1}^n\int_0^T f^i(s, u(s))\mathrm{d}s\\ &\leq M\sum_{i=1}^n\int_0^T \max_{\tau\in[0,T]} f^i(\tau, u(s))\mathrm{d}s\\ &\leq M\sum_{i=1}^n\int_0^T \hat{f}^i(r_2)\mathrm{d}s\\ &\leq MnT\varepsilon\|u\|\\ &\leq \|u\|.\end{aligned}$$

By Lemma 2.3, it follows that $\mathcal{T}_\lambda$ has a fixed point u in $K_{r_2} \setminus \bar{K}_{r_1}$. Then $x = u - P$ is the desired solution of (1.1). $\square$

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