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AN EXISTENCE THEOREM FOR QUASILINEAR SYSTEMS

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Abstract This paper deals with the existence of positive radial solutions for the quasilinear system $\operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i) + \lambda f^i(u_1,\dots,u_n) = 0$, $|x| < 1$, $u_i(x) = 0$, on $|x| = 1$, $i = 1,\dots,n$, $p > 1$, $\lambda > 0$, $x \in \mathbb{R}^N$. The f^i, $i = 1,\dots,n$, are continuous and non-negative functions. Let $\boldsymbol{u} = (u_1,\dots,u_n)$, $\|\boldsymbol{u}\| = \sum_{i=1}^n |u_i|$,

$$f_0^i = \lim_{\|\boldsymbol{u}\|\to 0} \frac{f^i(\boldsymbol{u})}{\|\boldsymbol{u}\|^{p-1}},$$

$i = 1,\dots,n$, $\boldsymbol{f} = (f^1,\dots,f^n)$, $\boldsymbol{f}_0 = \sum_{i=1}^n f_0^i$. We prove that the problem has a positive solution for sufficiently small $\lambda > 0$ if $\boldsymbol{f}_0 = \infty$. Our methods employ a fixed-point theorem in a cone.

Keywords: p-Laplacian; elliptic system; existence; fixed-point theorem; cone

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1. Introduction

In this paper we consider the existence and non-existence of positive radial solutions for the quasilinear elliptic system

$$\left.\begin{aligned} \operatorname{div}(|\nabla u_1|^{p-2}\nabla u_1) + \lambda f^1(u_1,\dots,u_n) &= 0 \text{ in } B,\\ &\vdots\\ \operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) + \lambda f^n(u_1,\dots,u_n) &= 0 \text{ in } B,\\ u_i = 0 \text{ on } \partial B, \quad i = 1,\dots,n&, \end{aligned}\right\} \tag{1.1}$$

where $p > 1$, $B = \{x \in \mathbb{R}^N : |x| < 1,\ N \geqslant 2\}$ and $\lambda > 0$ is a parameter.

When $p = 2$, (1.1) becomes

$$\left.\begin{aligned} \Delta u_1 + \lambda f^1(u_1,\dots,u_n) &= 0 \text{ in } B,\\ &\vdots\\ \Delta u_n + \lambda f^n(u_1,\dots,u_n) &= 0 \text{ in } B,\\ u_i = 0 \text{ on } \partial B, \quad i = 1,\dots,n&. \end{aligned}\right\} \tag{1.2}$$

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When $n = 1$ and $p = 2$, (1.1) becomes

$$\left.\begin{aligned} \Delta u + \lambda f(u) = 0 \text{ in } B, \\ u = 0 \text{ on } \partial B. \end{aligned}\right\} \tag{1.3}$$

System (1.3) has been the subject of extensive investigation over the past several decades. Lions [**5**] discussed the existence and non-existence of positive solutions of (1.3) in a general bounded regular domain in $\mathbb{R}^N$. The results of [**5**] are also interpreted in terms of bifurcation diagrams.

Joseph and Lundgren [**4**] determined the number of solutions for (1.3) in the case $f(u) = \mathrm{e}^u$ and $f(u) = (1 + \alpha u)^\beta$ for $\alpha, \beta > 0$. If $0 < \beta < 1$, it is understood that $f(u) = (1 + \alpha u)^\beta$ (or u^β) is sublinear. If we define

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u},$$

then $f_0 = \infty$ for $f(u) = (1 + \alpha u)^\beta$ (or u^β), $0 < \beta < 1$. Note that

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} = \infty$$

for $f(u) = \mathrm{e}^u$. Also $f_0 = \infty$ can apply to the case in which $f(0) = 0$ (for instance, $f(u) = \sqrt{u}$). For n-dimensional system (1.1), we define $\boldsymbol{f}_0$ in (1.4), which is a natural extension of f_0. As in the scalar case, $\boldsymbol{f}_0 = \infty$ can also apply to $\boldsymbol{f}(0) = 0$, and thus zero is a trivial solution in this case. We shall prove that (1.1) has a positive solution for sufficiently small $\lambda > 0$ if $\boldsymbol{f}_0 = \infty$, regardless of the behaviour of $\boldsymbol{f}$ at ∞.

Our arguments are based on the fixed-point index. Many authors have used the fixed-point index to prove the existence of positive solutions of differential equations (see, for example, [**1**, **3**, **6**–**8**]). Variational methods have frequently been used for Hamiltonian systems and gradient systems. However, there is apparently no possibility of using variational methods for the n-dimensional quasilinear elliptic system (1.1), and one has to use topological methods.

We now turn to general assumptions made in this paper. Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$ and

$$\mathbb{R}_+^n = \underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{n}.$$

Also, for $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$, let $\|\boldsymbol{u}\| = \sum_{i=1}^n |u_i|$. We make the following assumption.

(H1) $f^i : \mathbb{R}_+^n \to \mathbb{R}_+$ is continuous, $i = 1, \dots, n$.

In order to state our results we introduce the notation

$$\boldsymbol{f}(\boldsymbol{u}) = (f^1(\boldsymbol{u}), \dots, f^n(\boldsymbol{u})) = (f^1(u_1, \dots, u_n), \dots, f^n(u_1, \dots, u_n)),$$

$$f_0^i = \lim_{\|\boldsymbol{u}\| \to 0} \frac{f^i(\boldsymbol{u})}{\|\boldsymbol{u}\|^{p-1}},$$

where $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$,

$$\boldsymbol{f}_0 = \sum_{i=1}^{n} f_0^i. \tag{1.4}$$

Our main result is the following theorem.

Theorem 1.1. *Assume that (H1) holds. If $\boldsymbol{f}_0 = \infty$, then (1.1) has a positive radial solution for sufficient small $\lambda > 0$.*

For the ordinary differential equation case ($N = 1$), Wang [**8**] proved that the existence, multiplicity and non-existence of positive solutions of (1.1) can be determined by appropriate combinations of superlinearity and sublinearity of $\boldsymbol{f}(u)$ at zero and infinity.

2. Preliminaries

Let $\varphi(t) = |t|^{p-2}t$; then, for $t > 0$, $\varphi(t) = t^{p-1}$ and $\varphi^{-1}(t) = t^{1/(p-1)}$. It is easy to see that $\varphi^{-1}(\sigma\varphi(t)) = \varphi^{-1}(\sigma)t$ for $t > 0$ and $\sigma > 0$.

A radial solution of (1.1) can be considered as a solution of the system

$$\left.\begin{aligned} (r^{N-1}\varphi(u_1'(r)))' + \lambda r^{N-1} f^1(\boldsymbol{u}) = 0, \quad 0 < r < 1,\\ \vdots \qquad\qquad \\ (r^{N-1}\varphi(u_n'(r)))' + \lambda r^{N-1} f^n(\boldsymbol{u}) = 0, \quad 0 < r < 1,\\ \boldsymbol{u}'(0) = \boldsymbol{u}(1) = 0, \quad i = 1, \dots, n. \end{aligned}\right\} \tag{2.1}$$

We will deal with classical solutions of (2.1), namely a vector-valued function $\boldsymbol{u} = (u_1(r), \dots, u_n(r))$ with $u_i \in C^1[0,1]$, and $\varphi(u_i') \in C^1(0,1)$, $i = 1, \dots, n$, which satisfies (2.1). A solution $\boldsymbol{u}(r) = (u_1(r), \dots, u_n(r))$ is positive if $u_i(r) \geqslant 0$, $i = 1, \dots, n$, for all $r \in (0,1)$ and there is at least one non-trivial component of $\boldsymbol{u}$. In fact, it is easy to prove that such a non-trivial component of $\boldsymbol{u}$ is positive on $(0,1)$.

The following well-known result of the fixed-point index is crucial in our arguments.

Lemma 2.1 (see [2, 3]). *Let E be a Banach space equipped with a norm $\|\cdot\|_*$ and let K be a cone in E. For $r > 0$, define $K_r = \{u \in K : \|x\|_* < r\}$, and $\partial K_r = \{u \in K : \|x\|_* = r\}$, which is the relative boundary of K_r with respect to K. Assume that $T : \bar{K}_r \to K$ is completely continuous.*

(i) *If there exists a $x_0 \in K \setminus \{0\}$ such that*

$$x - Tx \neq tx_0, \quad \text{for all } x \in \partial K_r \text{ and } t \geqslant 0,$$

then

$$i(T, K_r, K) = 0.$$

(ii) *If $\|Tx\|_* \leqslant \|x\|_*$ for $x \in \partial K_r$ and $Tx \neq x$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.1 to (2.1), let X be the Banach space

$$\underbrace{C[0,1]\times\cdots\times C[0,1]}_{n}$$

and, for $\boldsymbol{u}=(u_1,\dots,u_n)\in X$, define its norm by

$$\|\boldsymbol{u}\|_*=\sum_{i=1}^{n}\sup_{t\in[0,1]}|u_i(t)|.$$

Define K to be a cone in X by

$$K=\{(u_1,\dots,u_n)\in X: u_i(t)\geqslant 0,\ t\in[0,1],\ i=1,\dots,n\}.$$

Also, define Ω_r, for r a positive number, by

$$\Omega_r=\{\boldsymbol{u}\in K:\|\boldsymbol{u}\|_*<r\}.$$

Note that $\partial\Omega_r=\{\boldsymbol{u}\in K:\|\boldsymbol{u}\|_*=r\}$.

Let $\boldsymbol{T}_\lambda: K\to X$ be a map with components $(T^1_\lambda,\dots,T^n_\lambda)$. We define T^i_λ, $i=1,\dots,n$, by

$$T^i_\lambda\boldsymbol{u}(r)=\int_r^1\varphi^{-1}\left(\frac{1}{s^{N-1}}\int_0^s\tau^{N-1}\lambda f^i(\boldsymbol{u}(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s,\quad r\in[0,1]. \tag{2.2}$$

It is straightforward to verify that (2.1) is equivalent to the fixed-point equation

$$\boldsymbol{T}_\lambda\boldsymbol{u}=\boldsymbol{u}\quad\text{in } K.$$

Lemma 2.2. *Assume that (H1) holds. Then $\boldsymbol{T}_\lambda(K)\subset K$ and $\boldsymbol{T}_\lambda: K\to K$ is compact and continuous.*

Proof. The proof of the lemma is standard, and is omitted. □

Lemma 2.3. *Assume that (H1) holds. If $\boldsymbol{u}\in\partial\Omega_r$, $r>0$, then*

$$\|\boldsymbol{T}_\lambda\boldsymbol{u}\|_*\leqslant n\varphi^{-1}(\lambda)\varphi^{-1}(\hat{M}_r),$$

where

$$\hat{M}_r=1+\max\{f^i(\boldsymbol{u}):\boldsymbol{u}\in\mathbb{R}^n_+\ \text{and}\ \|\boldsymbol{u}\|\leqslant r,\ i=1,\dots,n\}>0.$$

Proof. From the definition of T_λ, for $\boldsymbol{u}\in\partial\Omega_r$, we have

$$\begin{aligned}\|\boldsymbol{T}_\lambda\boldsymbol{u}\|_*&=\sum_{i=1}^{n}\sup_{t\in[0,1]}|T^i_\lambda\boldsymbol{u}(t)|\\&\leqslant\sum_{i=1}^{n}\int_0^1\varphi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s\tau^{N-1}\lambda\hat{M}_r\,\mathrm{d}\tau\right]\mathrm{d}s\\&=\sum_{i=1}^{n}\int_0^1\varphi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s\tau^{N-1}\,\mathrm{d}\tau\varphi(\varphi^{-1}(\lambda\hat{M}_r))\right]\mathrm{d}s\\&\leqslant n\varphi^{-1}[\varphi(\varphi^{-1}(\lambda\hat{M}_r))].\end{aligned}$$

Then the fact that $\varphi^{-1}(\varphi(t)) = t$ implies that

$$\begin{aligned}\|\boldsymbol{T}_\lambda \boldsymbol{u}\|_* &\leqslant n\varphi^{-1}(\lambda \hat{M}_r)\\ &= n\varphi^{-1}(\lambda)\varphi^{-1}(\hat{M}_r).\end{aligned}$$

□

3. Proof of Theorem 1.1

Proof. Fix a number $r_2 > 0$. Lemma 2.3 implies that there exists a $\lambda_0 > 0$ such that

$$\|\boldsymbol{T}_\lambda \boldsymbol{u}\|_* < \|\boldsymbol{u}\|_*, \quad \text{for } \boldsymbol{u} \in \partial\Omega_{r_2},\ 0 < \lambda < \lambda_0.$$

Since $\boldsymbol{f}_0 = \infty$, there exists a component f^i such that $f_0^i = \infty$. Therefore, there is an $0 < r_1 < r_2$ such that

$$f^i(\boldsymbol{u}) \geqslant \varphi(\eta)\varphi(\|\boldsymbol{u}\|) \tag{3.1}$$

for $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$ and $\|\boldsymbol{u}\| \leqslant r_1$, where $\eta > 0$ is chosen so that

$$\frac{\eta\varphi^{-1}(\lambda)}{2}\varphi^{-1}\left(\frac{1}{N4^N}\right) \geqslant 1. \tag{3.2}$$

If $\boldsymbol{u} - \boldsymbol{T}_\lambda \boldsymbol{u} = 0$ for some $\boldsymbol{u} \in \partial U_{r_1}$, we find the desired solution of (1.1). Therefore, we assume that

$$\boldsymbol{u} - \boldsymbol{T}_\lambda \boldsymbol{u} \neq 0, \qquad \text{for all } \boldsymbol{u} \in \partial U_{r_1}. \tag{3.3}$$

We now claim that

$$\boldsymbol{u} - \boldsymbol{T}_\lambda \boldsymbol{u} \neq t\boldsymbol{v}, \quad \text{for all } \boldsymbol{u} \in \partial\Omega_{r_1} \text{ and } t \geqslant 0, \tag{3.4}$$

where $\boldsymbol{v} = (\theta(r), \dots, \theta(r))$, and $\theta \in C[0,1]$ such that $0 \leqslant \theta(r) \leqslant 1$ on $[0,1]$, $\theta(r) \equiv 1$ on $[0, \frac{1}{4}]$ and $\theta(r) \equiv 0$ on $[\frac{1}{2}, 1]$. Thus, $\boldsymbol{v} \in K \setminus \{0\}$. If there exists $\boldsymbol{u}^* = (u_1^*, \dots, u_n^*) \in \partial\Omega_{r_1}$ and $t_0 \geqslant 0$ such that $\boldsymbol{u}^* - \boldsymbol{T}_\lambda \boldsymbol{u}^* = t_0 \boldsymbol{v}$, we will show that this leads to a contradiction. Since (3.3) is true, we have $t_0 > 0$. Since $\boldsymbol{T}_\lambda(K) \subset K$, we find that $u_i^*(r) \geqslant t_0\theta(r)$ for all $r \in [0,1]$. Let

$$t^* = \sup\{t : u_i^*(r) \geqslant t\theta(r) \text{ for all } r \in [0,1]\}.$$

It follows that $t_0 \leqslant t^* < \infty$ and $u_i^*(r) \geqslant t^*\theta(r)$ for all $r \in [0,1]$. Now, for $r \in [0,1]$, we have

$$\begin{aligned}u_i^*(r) &= \boldsymbol{T}_\lambda^i \boldsymbol{u}^*(r) + t_0\theta(r)\\ &= \int_r^1 \varphi^{-1}\left(\frac{1}{s^{N-1}}\int_0^s \tau^{N-1}\lambda f^i(\boldsymbol{u}^*(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s + t_0\theta(r).\end{aligned}$$

Note that

$$\sum_{j=1}^n u_j^*(r) \leqslant r_1 \quad \text{for } r \in [0,1].$$

Inequality (3.1) implies that, for $r \in [0, \frac{1}{2}]$,

$$\begin{aligned}
u_i^*(r) &\geqslant \int_{1/2}^{1} \varphi^{-1}\biggl(\frac{1}{s^{N-1}} \int_0^s \tau^{N-1} \lambda\varphi(\eta)\varphi\biggl(\sum_{j=1}^{n} u_j^*(\tau)\biggr)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s + t_0\theta(r) \\
&\geqslant \int_{1/2}^{1} \varphi^{-1}\biggl(\int_0^s \tau^{N-1} \lambda\varphi(\eta)\varphi(u_i^*(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s + t_0\theta(r) \\
&\geqslant \tfrac{1}{2}\varphi^{-1}\biggl(\int_0^{1/4} \tau^{N-1} \lambda\varphi(\eta)\varphi(t^*\theta(\tau))\,\mathrm{d}\tau\biggr) + t_0\theta(r) \\
&= \tfrac{1}{2}\varphi^{-1}\biggl(\int_0^{1/4} \tau^{N-1}\,\mathrm{d}\tau\varphi(\varphi^{-1}(\lambda))\varphi(\eta)\varphi(t^*)\biggr) + t_0\theta(r) \\
&= \tfrac{1}{2}\varphi^{-1}\biggl(\frac{1}{N4^N}\varphi(\varphi^{-1}(\lambda)\eta t^*)\biggr) + t_0\theta(r).
\end{aligned}$$

Now, in view of the fact that $\varphi^{-1}(\sigma\varphi(t)) = \varphi^{-1}(\sigma)t$, we have, for $r \in [0, \frac{1}{2}]$,

$$\begin{aligned}
u_i^*(r) &\geqslant t^* \frac{\eta\varphi^{-1}(\lambda)}{2}\varphi^{-1}\biggl(\frac{1}{N4^N}\biggr) + t_0\theta(r) \\
&\geqslant t^* + t_0\theta(r) \\
&\geqslant (t^* + t_0)\theta(r),
\end{aligned}$$

and hence

$$u_i^*(r) \geqslant (t^* + t_0)\theta(r), \quad r \in [0, 1],$$

which is a contradiction to the definition of t^*. Thus, in view of Lemma 2.1,

$$\begin{aligned}
i(\boldsymbol{T}_\lambda, \Omega_{r_1}, K) &= 0. \\
i(\boldsymbol{T}_\lambda, \Omega_{r_2}, K) &= 1.
\end{aligned}$$

It follows from the additivity of the fixed-point index that $i(\boldsymbol{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $\boldsymbol{T}_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (1.1). □

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