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Positivity

Positive Periodic Solutions of Systems of Second Order Ordinary Differential Equations

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Abstract. We establish the existence, multiplicity and nonexistence of positive periodic solutions of systems of second order ordinary differential equations.

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1. Introduction

In this paper we consider the existence, multiplicity and nonexistence of positive solutions for the periodic boundary value problem

$$\begin{aligned} &\mathbf{x}'' + m^2\mathbf{x} = \lambda \mathbf{G}(t)\mathbf{F}(\mathbf{x}) \quad 0 \leqslant t \leqslant 2\pi, \\ &\mathbf{x}(0) = \mathbf{x}(2\pi), \qquad \mathbf{x}'(0) = \mathbf{x}'(2\pi), \end{aligned} \tag{1.1}$$

where $m \in (0, 1/2)$ is a constant, $\mathbf{x} = [x_1, \ldots, x_n]^T$, $\mathbf{G}(t) = \text{diag}[g_1(t), g_2(t), \ldots, g_n(t)]$, $\mathbf{F}(\mathbf{x}) = [f^1(\mathbf{x}), f^2(\mathbf{x}), \ldots, f^n(\mathbf{x})]^T$, and $\lambda > 0$ is a positive parameter.

The existence of positive solutions for the periodic scalar boundary value problem have been studied in many papers, see Jiang-Chu-O'Regan-Agarwal [3] and Torres [5] and the references therein. It was shown that the scalar periodic boundary value problem has one positive solution provided f is superlinear or sublinear at zero and infinity. Periodic boundary value problems are closely related to the Dirichlet/Neumann boundary value problems, which have been extensively investigated in the literature (see, e.g., Agarwal-O'Regan-Wong [1] and Wang [7]). Our arguments as in Jiang-Chu-O'Regan-Agarwal [3] and Torres [5] are based on the fixed point index in a cone. We are able to establish several criteria for the existence, multiplicity and nonexistence of positive solutions of (1.1). The same approach also was used in Wang [6, 7] to prove analogous results for the existence, multiplicity and nonexistence of positive solutions of boundary value problems solutions for other boundary data.

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^n = \Pi_{i=1}^n \mathbb{R}_+$, and for any $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$, $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$. Our assumptions for this paper are:

(H1) $f^i : \mathbb{R}_+^n \to [0, \infty)$ is continuous, $i = 1, \ldots, n$.

(H2) $g_i(t) : [0, 2\pi] \to [0, \infty)$ is continuous and $\int_0^{2\pi} g_i(t)dt > 0, i = 1, \ldots, n$.

(H3) $f^i(\mathbf{u}) > 0$ for $\|\mathbf{u}\| > 0, i = 1, \ldots, n$.

In order to state our results we introduce the notation

$$f_0^i = \lim_{\|\mathbf{u}\| \to 0} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|}, \qquad f_\infty^i = \lim_{\|\mathbf{u}\| \to \infty} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|}, \quad \mathbf{u} \in \mathbb{R}_+^n, \ i = 1, \ldots, n$$
$$\mathbf{F}_0 = \max_{i=1,\ldots,n} \{f_0^i\}, \qquad \mathbf{F}_\infty = \max_{i=1,\ldots,n} \{f_\infty^i\}. \tag{1.2}$$

Our main results are:

THEOREM 1.1. *Assume* (H1)–(H2) *hold.*

(a). If $\mathbf{F}_0 = 0$ and $\mathbf{F}_\infty = \infty$, then for all $\lambda > 0$ (1.1) has a positive solution.

(b). If $\mathbf{F}_0 = \infty$ and $\mathbf{F}_\infty = 0$, then for all $\lambda > 0$ (1.1) has a positive solution.

THEOREM 1.2. *Assume* (H1)–(H3) *hold.*

(a). If $\mathbf{F}_0 = 0$ or $\mathbf{F}_\infty = 0$, then there exists a $\lambda_0 > 0$ such that (1.1) has a positive solution for $\lambda > \lambda_0$.

(b). If $\mathbf{F}_0 = \infty$ or $\mathbf{F}_\infty = \infty$, then there exists a $\lambda_0 > 0$ such that (1.1) has a positive solution for $0 < \lambda < \lambda_0$.

(c). If $\mathbf{F}_0 = \mathbf{F}_\infty = 0$, then there exists a $\lambda_0 > 0$ such that (1.1) has two positive solutions for $\lambda > \lambda_0$.

(d). If $\mathbf{F}_0 = \mathbf{F}_\infty = \infty$, then there exists a $\lambda_0 > 0$ such that (1.1) has two positive solutions for $0 < \lambda < \lambda_0$.

(e). If $\mathbf{F}_0 < \infty$ and $\mathbf{F}_\infty < \infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ (1.1) has no positive solution.

(f). If $\mathbf{F}_0 > 0$ and $\mathbf{F}_\infty > 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ (1.1) has no positive solution.

This paper is organized in the following ways. In Section 2, we transform (1.1) into a fixed point problem. Furthermore, we establish several inequalities. In Section 3, we apply the fixed point index to show the existence, multiplicity and nonexistence of positive solutions of (1.1) based on the inequalities.

2. Preliminaries

We recall some concepts and conclusions on the fixed point index in a cone in Guo-Lakshmikantham [2] and Krasnoselskii [4]. Let E be a Banach space and K be a closed, nonempty subset of E. K is said to be a cone if (i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geqslant 0$ and (ii) $u, -u \in K$ imply $u = 0$. Assume Ω is a bounded open subset in E with the boundary $\partial\Omega$, and let $T : K \cap \overline{\Omega} \to K$ be completely continuous such that $Tx \neq x$ for $x \in \partial\Omega \cap K$, then the fixed point index $i(T, K \cap \Omega, K)$ is defined. If $i(T, K \cap \Omega, K) \neq 0$, then T has a fixed point in $K \cap \Omega$. The following well-known result of the fixed point index is crucial in our arguments.

LEMMA 2.1. (Guo-Lakshmikantham [2] and Krasnoselskii [4]). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{v \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{v \in K : \|x\| = r\}$.*

(i) If $\|Tx\| \geqslant \|x\|$ for $x \in \partial K_r$, then

$$i(T, K_r, K) = 0.$$

(ii) If $\|Tx\| \leqslant \|x\|$ for $x \in \partial K_r$, then

$$i(T, K_r, K) = 1.$$

Following Jiang-Chu-O'Regan-Agarwal [3], we consider the function

$$G(t,s) = \begin{cases} \frac{\sin m(t-s) + \sin m(2\pi - t + s)}{2m(1-\cos 2m\pi)} & 0 \leqslant s \leqslant t \leqslant 2\pi, \\ \frac{\sin m(s-t) + \sin m(2\pi - s + t)}{2m(1-\cos 2m\pi)} & 0 \leqslant t \leqslant s \leqslant 2\pi. \end{cases} \tag{2.3}$$

Let $\hat{G}(x) = (\sin(mx) + \sin m(2\pi - x))/(2m(1 - \cos 2m\pi))$, $x \in [0, 2\pi]$. It is easy to check that $\hat{G}$ is increasing on $[0, \pi]$ and decreasing on $[\pi, 2\pi]$, and $G(t, s) = \hat{G}(|t - s|)$. Thus

$$\frac{\sin 2m\pi}{2m(1 - \cos 2m\pi)} = \hat{G}(0) \leqslant G(t,s) \leqslant \hat{G}(\pi) = \frac{\sin m\pi}{m(1 - \cos 2m\pi)}$$

for $s, t \in [0, 2\pi]$.

Let X be the Banach space $\underbrace{C[0, 2\pi] \times \cdots \times C[0, 2\pi]}_{n}$ and for $\mathbf{u} = (u_1, \ldots, u_n) \in X$,

$$\|\mathbf{u}\| = \sum_{i=1}^{n} \sup_{t \in [0, 2\pi]} |u_i(t)|.$$

For $\mathbf{u} \in X$ or $\mathbb{R}_+^n$, $\|\mathbf{u}\|$ denotes the norm of $\mathbf{u}$ in X or $\mathbb{R}_+^n$, respectively. Let K be the cone given by

$$K = \{\mathbf{u} = (u_1, \ldots, u_n) \in X : u_i(t) \geqslant 0, \quad t \in [0, 2\pi], \ \ i = 1, \ldots, n, \\ \text{and} \min_{0 \leqslant t \leqslant 2\pi} \sum_{i=1}^{n} u_i(t) \geqslant \sigma \|\mathbf{u}\|\},$$

where $\sigma = \cos m\pi > 0$. Also, define, for r a positive number, Ω_r by

$$\Omega_r = \{\mathbf{u} \in K \ : \ \|\mathbf{u}\| < r\}.$$

Note that $\partial\Omega_r = \{\mathbf{u} \in K \ : \ \|\mathbf{u}\| = r\}$.

Let $\mathbf{T}_\lambda \colon K \to X$ be a map with components $(T_\lambda^1, \ldots, T_\lambda^n)$. We define T_λ^i, $i = 1, \ldots, n$, by

$$T_\lambda^i \mathbf{u}(t) = \int_0^{2\pi} \lambda G(t, s) g_i(s) f^i(\mathbf{u}(s)) ds, \quad 0 \leqslant t \leqslant 2\pi. \tag{2.4}$$

LEMMA 2.2. *Assume* (H1)–(H2) hold. Then $\mathbf{T}_\lambda(K) \subset K$ and $\mathbf{T}_\lambda \colon K \to K$ is compact and continuous.

Proof. Let $\mathbf{u} \in K$, then, for $i = 1, \ldots, n$

$$\begin{aligned} \min_{0 \leqslant t \leqslant 2\pi} T_\lambda^i \mathbf{u}(t) &\geqslant \hat{G}(0)\lambda \int_0^{2\pi} g_i(s) f^i(\mathbf{u}(s)) ds \\ &= \sigma \hat{G}(\pi)\lambda \int_0^{2\pi} g_i(s) f^i(\mathbf{u}(s)) ds \\ &\geqslant \sigma \sup_{0 \leqslant t \leqslant 2\pi} T_\lambda^i \mathbf{u}(t). \end{aligned}$$

Thus, $\mathbf{T}_\lambda(K) \subset K$ (note $\min_{[0,2\pi]} \sum_{i=1}^n |T_\lambda^i u(t)| \geqslant \sum_{i=1}^n \min_{[0,2\pi]} |T_\lambda^i u(t)|$). It is easy to verify that T_λ is compact and continuous. □

Notice that $\mathbf{u} \in K$ is a positive fixed point of $\mathbf{T}_\lambda$ if only if $\mathbf{u}$ is a positive solution of (1.1).

Let

$$\Gamma = \hat{G}(0)\sigma \min_{i=1,\ldots,n} \int_0^{2\pi} g_i(s) ds > 0.$$

LEMMA 2.3. *Assume* (H1)–(H2) hold. Let $\mathbf{u} = (u_1(t), \ldots, u_n(t)) \in K$ and $\eta > 0$. If there exists a component f^i of $\boldsymbol{f}$ such that

$$f^i(\mathbf{u}(t)) \geqslant \eta \sum_{i=1}^{n} u_i(t) \quad \text{for } t \in [0, 2\pi]$$

then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda \Gamma \eta \|\mathbf{u}\|.$$

Proof. From the definition of $\mathbf{T}_\lambda u$ it follows that

$$\begin{aligned}
\|\mathbf{T}_\lambda \mathbf{u}\| &\geqslant \max_{0 \leqslant t \leqslant 2\pi} T_\lambda^i \mathbf{u}(t) \\
&\geqslant \lambda \hat{G}(0) \int_0^{2\pi} g_i(s) f^i(\mathbf{u}(s)) ds \\
&\geqslant \lambda \hat{G}(0) \int_0^{2\pi} g_i(s) \eta \sum_{i=1}^{n} u_i(s) ds \\
&\geqslant \lambda \hat{G}(0) \sigma \int_0^{2\pi} g_i(s) ds \eta \|\mathbf{u}\| \\
&= \lambda \Gamma \eta \|\mathbf{u}\|.
\end{aligned}$$

□

For each $i = 1, \ldots, n$, let $\hat{f}^i(t) \colon \mathbb{R}_+ \to \mathbb{R}_+$ be the function given by

$$\hat{f}^i(t) = \max\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n \text{ and } \|\mathbf{u}\| \leqslant \text{t}\}.$$

Let $\hat{f}_0^i = \lim_{t \to 0} \frac{\hat{f}^i(t)}{t}$ and $\hat{f}_\infty^i = \lim_{t \to \infty} \frac{\hat{f}^i(t)}{t}$.

LEMMA 2.4. (Wang [7]). *Assume* (H1) holds. Then

$$\hat{f}_0^i = f_0^i \quad \text{and} \quad \hat{f}_\infty^i = f_\infty^i, \quad i = 1, \ldots, n.$$

LEMMA 2.5. *Assume* (H1)–(H2) hold and let $r > 0$. If there exits an $\varepsilon > 0$ such that

$$\hat{f}^i(r) \leqslant \varepsilon r, \quad i = 1, \ldots, n,$$

then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \lambda \hat{C} \varepsilon \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_r,$$

where the constant $\hat{C} = \hat{G}(\pi) \sum_{i=1}^{n} \int_0^{2\pi} g_i(s) ds$.

Proof. From the definition of $\mathbf{T}_\lambda$, we have for $\mathbf{u} \in \partial\Omega_r$,

$$\begin{aligned}
\|\mathbf{T}_\lambda \mathbf{u}\| &= \sum_{i=1}^{n} \max_{0 \leqslant t \leqslant 2\pi} T_\lambda^i \mathbf{u}(t) \\
&\leqslant \sum_{i=1}^{n} \lambda \hat{G}(\pi) \int_0^{2\pi} g_i(s) f^i(\mathbf{u}(s)) ds \\
&\leqslant \sum_{i=1}^{n} \lambda \hat{G}(\pi) \int_0^{2\pi} g_i(s) \hat{f}^i(r) ds \\
&\leqslant \sum_{i=1}^{n} \lambda \hat{G}(\pi) \int_0^{2\pi} g_i(s) ds \varepsilon \|\mathbf{u}\| \\
&= \lambda \hat{C} \varepsilon \|\mathbf{u}\|.
\end{aligned}$$

□

The following two lemmas are weak forms of Lemmas 2.3 and 2.5.

LEMMA 2.6. *Assume* (H1)–(H3) hold. If $\mathbf{u} \in \partial\Omega_r$, $r > 0$, then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda \hat{m}_r \Gamma \frac{1}{\sigma},$$

where $\hat{m}_r = \min\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n \text{ and } \sigma r \leqslant \|\mathbf{u}\| \leqslant r, i = 1, \ldots, n\} > 0$.

Proof. Since $f^i(\mathbf{u}(t)) \geqslant \hat{m}_r$ for $t \in [0, 2\pi], i = 1, \ldots, n$ (we just note that $r = \|u\| \geqslant \sup_{[0,2\pi]} \sum_{i=1}^n |u_i(t)| \geqslant \inf_{[0,2\pi]} \sum_{i=1}^n |u_i(t)| \geqslant \sigma r$ if $u \in \partial\Omega_r$), a slight modification of the proof in Lemma 2.3 yields the result. □

LEMMA 2.7. *Assume* (H1)–(H3) hold. If $\mathbf{u} \in \partial\Omega_r$, $r > 0$, then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \lambda \hat{M}_r \hat{C},$$

where $\hat{M}_r = \max\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n \text{ and } \|\mathbf{u}\| \leqslant r, i = 1, \ldots, n\} > 0$ and $\hat{C}$ is the positive constant defined in Lemma 2.5.

Proof. Since $f^i(\mathbf{u}(t)) \leqslant \hat{M}_r$ for $t \in [0, 2\pi]$, $i = 1, \ldots, n$, a slight modification of the proof in Lemma 2.5 guarantees the result. □

3. Proof of Theorem 1.1

Proof. Part (a). $\mathbf{F}_0 = 0$ implies that $f_0^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.4 that $\hat{f}_0^i = 0$, $i = 1, \ldots, n$. Therefore, we can choose $r_1 > 0$ so that $\hat{f}^i(r_1) \leqslant \varepsilon r_1$, $i = 1, \ldots, n$, where the constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.5. We have by Lemma 2.5 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \lambda \varepsilon \hat{C} \|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial \Omega_{r_1}.$$

Now, since $\mathbf{F}_\infty = \infty$, there exists a component f^i of $\mathbf{F}$ such that $f_\infty^i = \infty$. Therefore, there is an $\hat{H} > 0$ such that

$$f^i(\mathbf{u}) \geqslant \eta \|\mathbf{u}\|$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \geqslant \hat{H}$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Let $r_2 = \max\{2r_1, \frac{1}{\sigma}\hat{H}\}$. If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial \Omega_{r_2}$, then

$$\min_{0 \leqslant t \leqslant 2\pi} \sum_{i=1}^n u_i(t) \geqslant \sigma \|\mathbf{u}\| = \sigma r_2 \geqslant \hat{H},$$

which implies that

$$f^i(\mathbf{u}(t)) \geqslant \eta \sum_{i=1}^n u_i(t) \quad \text{for } t \in [0, 2\pi].$$

It follows from Lemma 2.3 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda \Gamma \eta \|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial \Omega_{r_2}.$$

By Lemma 2.1,

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0.$$

It follows from the additivity of the fixed point index that

$$i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = -1.$$

Thus, $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) \neq 0$, which implies $\mathbf{T}_\lambda$ has a fixed point $\mathbf{u} \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ by the existence property of the fixed point index. The fixed point $\mathbf{u} \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ is the desired positive solution of (1.1).

Part (b). If $\mathbf{F}_0 = \infty$, there exists a component f^i such that $f_0^i = \infty$. Therefore, there is an $r_1 > 0$ such that

$$f^i(\mathbf{u}) \geqslant \eta \|\mathbf{u}\|$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \leqslant r_1$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial \Omega_{r_1}$, then

$$f^i(\mathbf{u}(t)) \geqslant \eta \sum_{i=1}^{n} u_i(t) \text{ for } t \in [0, 1].$$

Lemma 2.3 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda \Gamma \eta \|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial \Omega_{r_1}.$$

We now determine Ω_{r_2}. $\mathbf{F}_\infty = 0$ implies that $f_\infty^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.4 that $\hat{f}_\infty^i = 0$, $i = 1, \ldots, n$. Therefore there is an $r_2 > 2r_1$ such that

$$\hat{f}^i(r_2) \leqslant \varepsilon r_2, \;\; i = 1, \ldots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.5. Thus, we have by Lemma 2.5 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \lambda \varepsilon \hat{C} \|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial \Omega_{r_2}.$$

By Lemma 2.1,

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1.$$

It follows from the additivity of the fixed point index that $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (1.1). □

4. Proof of Theorem 1.2

Proof. Part (a). Fix a number $r_1 > 0$. Lemma 2.6 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{\mathrm{r}_1}, \ \ \lambda > \lambda_0.$$

If $\mathbf{F}_0 = 0$, then $f_0^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.4 that

$$\hat{f}_0^i = 0, \quad i = 1, \ldots, n.$$

Therefore, we can choose $0 < r_2 < r_1$ so that

$$\hat{f}^i(r_2) \leqslant \varepsilon r_2, \quad i = 1, \ldots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.5. We have by Lemma 2.5 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \lambda \varepsilon \hat{C} \|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{\mathrm{r}_2}.$$

If $\mathbf{F}_\infty = 0$, then $f_\infty^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.4 that $\hat{f}_\infty^i = 0$, $i = 1, \ldots, n$. Therefore there is an $r_3 > 2r_1$ such that

$$\hat{f}^i(r_3) \leqslant \varepsilon r_3, \quad i = 1, \ldots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.5. Thus, we have by Lemma 2.5 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \lambda \varepsilon \hat{C} \|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{\mathrm{r}_3}.$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0, \qquad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1 \quad \textit{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 1.$$

Thus $i(\mathbf{T}_\lambda, \Omega_{r_1}\backslash\bar{\Omega}_{r_2}, K) = -1$ and $i(\mathbf{T}_\lambda, \Omega_{r_3}\backslash\bar{\Omega}_{r_1}, K) = 1$. Hence, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1}\backslash\bar{\Omega}_{r_2}$ or $\Omega_{r_3}\backslash\bar{\Omega}_{r_1}$ according to $\mathbf{F}_0 = 0$ or $\mathbf{F}_\infty = 0$, respectively. Consequently, (1.1) has a positive solution for $\lambda > \lambda_0$.

Part (b). Fix a number $r_1 > 0$. Lemma 2.7 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_1}, \;\; 0 < \lambda < \lambda_0.$$

If $\mathbf{F}_0 = \infty$, there exists a component f^i of $\mathbf{F}$ such that $f_0^i = \infty$. Therefore, there is a positive number $r_2 < r_1$ such that

$$f^i(\mathbf{u}) \geqslant \eta\|\mathbf{u}\|$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \leqslant r_2$, where $\eta > 0$ is chosen so that

$$\lambda\Gamma\eta > 1.$$

Then

$$f^i(\mathbf{u}(t)) \geqslant \eta \sum_{i=1}^{n} u_i(t),$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_2}$, $t \in [0, 1]$. Lemma 2.3 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

If $\mathbf{F}_\infty = \infty$, there exists a component f^i of $\mathbf{F}$ such that $f_\infty^i = \infty$. Therefore, there is an $\hat{H} > 0$ such that

$$f^i(\mathbf{u}) \geqslant \eta\|\mathbf{u}\|$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \geqslant \hat{H}$, where $\eta > 0$ is chosen so that

$$\lambda\Gamma\eta > 1.$$

Let $r_3 = \max\{2r_1, \frac{\hat{H}}{\sigma}\}$. If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_3}$, then

$$\min_{0 \leqslant t \leqslant 2\pi} \sum_{i=1}^{n} u_i(t) \geqslant \sigma\|\mathbf{u}\| = \sigma r_3 \geqslant \hat{H},$$

which implies that

$$f^i(\mathbf{u}(t)) \geqslant \eta \sum_{i=1}^{n} u_i(t) \quad \text{for } t \in [0, 2\pi].$$

It follows from Lemma 2.3 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \lambda \Gamma \eta \|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1, \qquad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 0,$$

and hence, $i(\mathbf{T}_\lambda, \Omega_{r_1} \backslash \bar{\Omega}_{r_2}, K) = 1$ and $i(\mathbf{T}_\lambda, \Omega_{r_3} \backslash \bar{\Omega}_{r_1}, K) = -1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \backslash \bar{\Omega}_{r_2}$ or $\Omega_{r_3} \backslash \bar{\Omega}_{r_1}$ according to $f_0 = \infty$ or $f_\infty = \infty$, respectively. Consequently, (1.1) has a positive solution for $0 < \lambda < \lambda_0$.

Part (c). Fix two numbers $0 < r_3 < r_4$. Lemma 2.6 implies that there exists a $\lambda_0 > 0$ such that we have for $\lambda > \lambda_0$,

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_i} \ \ (i = 3, 4).$$

Since $\mathbf{F}_0 = 0$ and $\mathbf{F}_\infty = 0$, it follows from the proof of Theorem 1.2 (a) that we can choose $0 < r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_i} \ \ (i = 1, 2).$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1, \qquad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1,$$

and

$$i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 0, \qquad i(\mathbf{T}_\lambda, \Omega_{r_4}, K) = 0$$

and hence, $i(\mathbf{T}_\lambda, \Omega_{r_3} \backslash \bar{\Omega}_{r_1}, K) = -1$ and $i(\mathbf{T}_\lambda, \Omega_{r_2} \backslash \bar{\Omega}_{r_4}, K) = 1$. Thus, $\mathbf{T}_\lambda$ has two fixed points $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ such that $\mathbf{u}_1(t) \in \Omega_{r_3} \backslash \bar{\Omega}_{r_1}$ and $\mathbf{u}_2(t) \in \Omega_{r_2} \backslash \bar{\Omega}_{r_4}$, which are the desired distinct positive solutions of (1.1) for $\lambda > \lambda_0$ satisfying

$$r_1 < \|\mathbf{u}_1\| < r_3 < r_4 < \|\mathbf{u}_2\| < r_2.$$

Part (d). Fix two numbers $0 < r_3 < r_4$. Lemma 2.7 implies that there exists a $\lambda_0 > 0$ such that we have, for $0 < \lambda < \lambda_0$,

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_i} \ \ (i = 3, 4).$$

Since $\mathbf{F}_0 = \infty$ and $\mathbf{F}_\infty = \infty$, it follows from the proof of Theorem 1.2 (b) that we can choose $0 < r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_i} \ \ (i = 1, 2).$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0, \qquad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0,$$

and

$$i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 1, \qquad i(\mathbf{T}_\lambda, \Omega_{r_4}, K) = 1$$

and hence, $i(\mathbf{T}_\lambda, \Omega_{r_3} \backslash \bar{\Omega}_{r_1}, K) = 1$ and $i(\mathbf{T}_\lambda, \Omega_{r_2} \backslash \bar{\Omega}_{r_4}, K) = -1$. Thus, $\mathbf{T}_\lambda$ has two fixed points $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ such that $\mathbf{u}_1(t) \in \Omega_{r_3} \backslash \bar{\Omega}_{r_1}$ and $\mathbf{u}_2(t) \in \Omega_{r_2} \backslash \bar{\Omega}_{r_4}$, which are the desired distinct positive solutions of (1.1) for $\lambda < \lambda_0$ satisfying

$$r_1 < \|\mathbf{u}_1\| < r_3 < r_4 < \|\mathbf{u}_2\| < r_2.$$

Part (e). Since $\mathbf{F}_0 < \infty$ and $\mathbf{F}_\infty < \infty$, then $f_0^i < \infty$ and $f_\infty^i < \infty$, $i = 1, \ldots, n$. It is easy to show (see Wang [7]) that there exists an $\varepsilon > 0$ such that

$$f^i(\mathbf{u}) \leqslant \varepsilon \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \mathbb{R}_+^n \ \ (i = 1, \ldots, n).$$

Assume $\mathbf{v}(t)$ is a positive solution of (1.1). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0$, where

$$\lambda_0 = \frac{1}{\sum_{i=1}^n \hat{G}(\pi) \int_0^{2\pi} g_i(s) ds \varepsilon}.$$

In fact, for $0 < \lambda < \lambda_0$, since $\mathbf{T}_\lambda \mathbf{v}(t) = \mathbf{v}(t)$ for $t \in [0, 1]$, we find

$$\begin{aligned}
\|\mathbf{v}\| &= \|\mathbf{T}_\lambda \mathbf{v}\| \\
&= \sum_{i=1}^{n} \max_{0 \leqslant t \leqslant 2\pi} T_\lambda^i \mathbf{v}(t) \\
&\leqslant \sum_{i=1}^{n} \lambda \hat{G}(\pi) \int_0^{2\pi} g_i(s) f^i(\mathbf{v}(s)) ds \\
&\leqslant \sum_{i=1}^{n} \lambda \hat{G}(\pi) \int_0^{2\pi} g_i(s) ds \varepsilon \|\mathbf{u}\| \\
&< \|\mathbf{v}\|,
\end{aligned}$$

which is a contradiction.

Part (f). Since $\mathbf{F}_0 > 0$ and $\mathbf{F}_\infty > 0$, there exist two components f^i and f^j of $\mathbf{F}$ such that $f_0^i > 0$ and $f_\infty^j > 0$. It is easy to show (see Wang [7]) that there exist positive numbers η, r_1 such that

$$f^i(\mathbf{u}) \geqslant \eta \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \mathbb{R}_+^n, \quad \|\mathbf{u}\| \leqslant r_1 \tag{4.5}$$

and

$$f^j(\mathbf{u}) \geqslant \eta \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \mathbb{R}_+^n, \ \|\mathbf{u}\| \geqslant \sigma r_1. \tag{4.6}$$

Assume $\mathbf{v}(t) = (v_1, \ldots, v_n)$ is a positive solution of (1.1). We will show that this leads to a contradiction for $\lambda > \lambda_0 = \frac{1}{\Gamma\eta}$. In fact, if $\|\mathbf{v}\| \leqslant r_1$, (4.5) implies that

$$f^i(\mathbf{v}(t)) \geqslant \eta \sum_{i=1}^{n} v_i(t) \quad \text{for } t \in [0, 1].$$

On the other hand, if $\|\mathbf{v}\| > r_1$, then

$$\min_{0 \leqslant t \leqslant 2\pi} \sum_{i=1}^{n} v_i(t) \geqslant \sigma \|\mathbf{v}\| > \sigma r_1,$$

which, together with (4.6), implies that

$$f^j(\mathbf{v}(t)) \geqslant \eta \sum_{i=1}^{n} v_i(t) \quad \text{for } t \in [0, 2\pi].$$

Since $\mathbf{T}_\lambda \mathbf{v}(t) = \mathbf{v}(t)$ for $t \in [0, 1]$, it follows from Lemma 2.3 that, for $\lambda > \lambda_0$,

$$\begin{aligned}\|\mathbf{v}\| &= \|\mathbf{T}_\lambda \mathbf{v}\| \\ &\geqslant \lambda \Gamma \eta \|\mathbf{v}\| \\ &> \|\mathbf{v}\|,\end{aligned}$$

which is a contradiction. □

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