MULTIPLE POSITIVE RADIAL SOLUTIONS FOR QUASILINEAR EQUATIONS IN ANNULAR DOMAINS

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We study the number of positive radial solutions of elliptic equations when nonlinearity has zeros. We show that the problem has k positive solutions if the nonlinearity has k zeros. Similar results are also true for elliptic systems.

1. Introduction

In this paper we consider the multiplicity of positive radial solutions for the quasilinear equations

$$\begin{gathered}\operatorname{div}(A(|\nabla u|)\nabla u)+\lambda b(|x|)f(u)=0 \quad \text{in } D,\\ u=0 \quad \text{for } x\in\partial D,\end{gathered} \tag{1.1}$$

$$\begin{gathered}\operatorname{div}(A(|\nabla u|)\nabla u)+\lambda b_1(|x|)g_1(u,v)=0 \quad \text{in } D,\\ \operatorname{div}(A(|\nabla v|)\nabla v)+\lambda b_2(|x|)g_2(u,v)=0 \quad \text{in } D,\\ u=v=0 \quad \text{for } x\in\partial D,\end{gathered} \tag{1.2}$$

where $D=\{x : x\in\mathbb{R}^n,\ n\geq 2,\ 0<R_1<|x|<R_2<\infty\}$.

The function A originates from a variety of practical applications, for instance, the degenerate m-Laplace operator, namely, $A(|p|)=|p|^{m-2}$, $m>1$. When $A\equiv 1$, we recall that (1.1) reduces to the classical semilinear elliptic equation

$$\begin{gathered}\Delta u+\lambda b(|x|)f(u)=0 \quad \text{in } D,\\ u=0 \quad \text{for } x\in\partial D.\end{gathered} \tag{1.3}$$

In the recent paper [5], the author discussed the problem under assumption (H1) on the function A, which covers the two important cases $A\equiv 1$ and $A(|p|)=|p|^{m-2}$, $m>1$,

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that is, the degenerate m-Laplace operator. It was proved that appropriate combinations of superlinearity and sublinearity of the quotient $f(u)/A(u)$ at zero and infinity guarantee the existence, multiplicity, and nonexistence of positive radial solutions of (1.1) and (1.2).

The purpose of this paper is to study the number of positive radial solutions of (1.1) if f has zeros. We will show that (1.1) has at least k positive radial solutions if f has k zeros. A similar result is also true for (1.2). For the scalar equation (1.1) and the case $A \equiv 1$, previous work on this problem has been done by Hess [2]. We also obtain a similar multiplicity result for elliptic systems. Our arguments are based on a fixed point theorem in a cone due to Krasnoselskii.

2. Multiplicity results

Let $\mathbb{R} = (-\infty,\infty)$, $\varphi(t) = A(|t|)t$. We make the following assumptions.

(H1) φ is an odd increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ and there exist two increasing homeomorphisms ψ_1 and ψ_2 of $(0,\infty)$ onto $(0,\infty)$ such that

$$\psi_1(\sigma)\varphi(t) \le \varphi(\sigma t) \le \psi_2(\sigma)\varphi(t), \quad \forall \sigma, t > 0. \tag{2.1}$$

(H2) $b : [R_1,R_2] \to [0,\infty)$ is continuous and $b \not\equiv 0$ on any subinterval of $[R_1,R_2]$.

(H3) $f : [0,\infty) \to [0,\infty)$ is continuous.

(H4) There exist k numbers $a_k > a_{k-1} > \cdots > a_1 > 0$ such that $a_i > 4a_{i-1}$, $f(a_i) = 0$ for $i = 1,\dots,k$ and $f(u) > 0$ for $a_{i-1} < u < a_i$, $i = 1,\dots,k$, where $a_0 = 0$.

Our multiplicity result for (1.1) is as follows.

Theorem 2.1. *Assume* (H1)–(H4) *hold. Then there exists λ_0 such that for $\lambda > \lambda_0$ (1.1) has k positive solutions, $u_1,u_2,\dots,u_k$, such that*

$$a_{i-1} < \sup_{t\in[0,1]} u_i(t) \le a_i, \quad i = 1,\dots,k. \tag{2.2}$$

We assume the following additional conditions for (1.2).

(H5) $b_i : [R_1,R_2] \to [0,\infty)$ is continuous and $b_i \not\equiv 0$ on any subinterval of $[R_1,R_2]$, $i = 1,2$.

(H6) $g_i : [0,\infty) \to [0,\infty)$ is continuous, $i = 1,2$.

(H7) There exist k numbers $a_k > a_{k-1} > \cdots > a_1 > 0$ such that $a_i > 4a_{i-1}$, $g_1(u,v) = 0$, and $g_2(u,v) = 0$ for $u+v = a_i$, $i = 1,\dots,k$, and $g_1(u,v) > 0$ and $g_2(u,v) > 0$ for $a_{i-1} < u+v < a_i$, $i = 1,\dots,k$, where $a_0 = 0$.

Our multiplicity result for (1.2) is as follows.

Theorem 2.2. *Assume* (H1) *and* (H5)–(H7) *hold. Then there exists λ_0 such that for $\lambda > \lambda_0$ (1.2) has k positive solutions, $(u_1v_1),(u_2,v_2),\dots,(u_k,v_k)$, such that*

$$a_{i-1} < \sup_{t\in[0,1]} \big(u_i(t)+v_i(t)\big) \le a_i, \quad i = 1,\dots,k. \tag{2.3}$$

3. Preliminaries

A radial solution of (1.1) can be considered as a solution of the equation

$$\begin{gathered}(r^{n-1}\varphi(u'(r)))' + \lambda r^{n-1}b(r)f(u(r)) = 0 \quad \text{in } R_1 < r < R_2, \\ u(R_1) = u(R_2) = 0.\end{gathered} \tag{3.1}$$

We will treat classical solutions of (3.1), namely, functions u of class C^1 on $[R_1,R_2]$ with $\varphi(u') \in C^1(R_1,R_2)$, which satisfies (3.1). A solution u is positive if $u(r) > 0$ for all $r \in (R_1,R_2)$.

Applying the change of variables, $r = (R_2 - R_1)t + R_1$, we can transform (3.1) into the form

$$\begin{gathered}(q(t)\varphi(pu'))' + \lambda h(t)f(u) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0,\end{gathered} \tag{3.2}$$

where

$$\begin{gathered}q(t) = ((R_2 - R_1)t + R_1)^{n-1}, \qquad p = \frac{1}{R_2 - R_1}, \\ h(t) = (R_2 - R_1)((R_2 - R_1)t + R_1)^{n-1}b((R_2 - R_1)t + R_1).\end{gathered} \tag{3.3}$$

We will prove there are k positive solutions for (3.2), which immediately implies that Theorem 2.1 is true.

The following well-known result of the fixed point index is crucial in our arguments.

Lemma 3.1 [1, 3]. *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|x\| = r\}$.*

(i) *If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then*

$$i(T,K_r,K) = 0. \tag{3.4}$$

(ii) *If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then*

$$i(T,K_r,K) = 1. \tag{3.5}$$

In order to apply Lemma 3.1 to (3.2), let X be the Banach space $C[0,1]$ with $\|u\| = \sup_{t\in[0,1]} |u(t)|$, $u \in X$.

Define K to be a cone in X by

$$K = \left\{u \in X : u(t) \geq 0, \min_{1/4\leq t\leq 3/4} u(t) \geq \frac{1}{4}\|u\|\right\}. \tag{3.6}$$

Also, define, for r a positive number, Ω_r by

$$\Omega_r = \{u \in K : \|u\| < r\}. \tag{3.7}$$

Note that $\partial\Omega_r = \{u \in K : \|u\| = r\}$.

For $i = 1,\dots,k$, let f_i satisfy

$$f_i(u) = \begin{cases} f(u), & 0 \le u \le a_i, \\ 0, & a_i \le u, \end{cases} \tag{3.8}$$

and let the map $T_\lambda^i : K \to X$ be defined by

$$T_\lambda^i u(t) = \begin{cases} \displaystyle\int_0^t p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma \lambda h(\tau) f_i(u(\tau))\,d\tau\right) ds, & 0 \le t \le \sigma, \\ \displaystyle\int_t^1 p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\int_\sigma^s \lambda h(\tau) f_i(u(\tau))\,d\tau\right) ds, & \sigma \le t \le 1, \end{cases} \tag{3.9}$$

where $\sigma \in (0,1)$ is a solution of the equation

$$\Theta_i u(t) = 0, \quad 0 \le t \le 1, \tag{3.10}$$

where the map $\Theta_i : K \to C[0,1]$ is defined by

$$\begin{aligned} \Theta_i u(t) = &\int_0^t \varphi^{-1}\left(\frac{1}{q(s)}\int_s^t \lambda h(\tau) f_i(u(\tau))\,d\tau\right) ds \\ &- \int_t^1 \varphi^{-1}\left(\frac{1}{q(s)}\int_t^s \lambda h(\tau) f_i(u(\tau))\,d\tau\right) ds, \quad 0 \le t \le 1. \end{aligned} \tag{3.11}$$

By virtue of Lemma 3.2, the operator T_λ^i is well defined.

Lemma 3.2 [4, 5]. *Assume* (H1)–(H3) *hold. Then, for any $u \in K$, $\Theta_i u(t) = 0$, $i = 1,\dots,k$, has at least one solution in* $(0,1)$. *In addition, if $\sigma^1 < \sigma^2 \in (0,1)$ are two solutions of $\Theta_i u(t) = 0$, then $h(t) f_i(u(t)) \equiv 0$ for $t \in [\sigma^1,\sigma^2]$ and any $\sigma \in [\sigma^1,\sigma^2]$ is also a solution of $\Theta_i u(t) = 0$. Furthermore, $T_\lambda^i u(t)$ is independent of the choice of $\sigma \in [\sigma^1,\sigma^2]$.*

Lemma 3.3 follows from the concavity of u.

Lemma 3.3 [4, 5]. *Assume* (H1)-(H2) *hold. Let u and $v \in X$ with $u(t) \ge 0$ and $v(t) \le 0$ for $t \in [0,1]$. If $(q(t)\varphi(pu'))' = v$, then*

$$u(t) \ge \min\{t, 1-t\}\|u\|, \quad t \in [0,1]. \tag{3.12}$$

In particular,

$$\min_{1/4 \le t \le 3/4} u(t) \ge \frac{1}{4}\|u\|. \tag{3.13}$$

We remark that, according to Lemma 3.3, any nonnegative solution of (3.2) is positive unless it is identical to zero.

Lemma 3.4. *Assume* (H1)–(H3) *hold. If* $u \in K$ *such that* $T_\lambda^i u = u$ *in* K, *then* u *is a solution of (3.2) such that*

$$\sup_{t\in[0,1]} u(t) \le a_i. \tag{3.14}$$

Proof. It is easy to see that u satisfies the following problem:

$$\begin{gathered}(q(t)\varphi(pu'))' + \lambda h(t) f_i(u) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0.\end{gathered} \tag{3.15}$$

Let $t_0 \in (0,1)$ such that $u(t_0) = \sup_{t\in[0,1]} u(t)$. It follows that $u'(t_0) = 0$. If $u(t_0) > a_i$, then there exist two numbers $0 \le t_1 < t_0 < t_2 \le 1$ such that $u(t) > a_i$ for $t \in (t_1,t_2)$ and $u(t_1) = u(t_2) = a_i$. Since $f_i(u) = 0$ for $u \ge a_i$, we have

$$(q(t)\varphi(pu'(t)))' = 0 \quad \text{for } t \in [t_1,t_2]. \tag{3.16}$$

Thus, $\varphi(pu'(t))$ is constant on $[t_1,t_2]$. Since $u'(t_0) = 0$, it follows that $u'(t) = 0$ for $t \in [t_1,t_2]$. Consequently, $u(t)$ is constant on $[t_1,t_2]$. This is a contradiction. Therefore, $\sup_{t\in[0,1]} u(t) \le a_i$. On the other hand, since $f(u) \equiv f_i(u)$ for $0 \le u \le a_i$, u is a solution of (3.2). □

Lemma 3.5 [4, 5]. *Assume* (H1)–(H3) *hold. Then* $\Theta_i : K \to C[0,1]$, $i = 1,\dots,k$, *is compact and continuous.*

Lemma 3.6 [4, 5]. *Assume* (H1)–(H3) *hold. Then* $T_\lambda(K) \subset K$ *and* $T_\lambda^i : K \to K$, $i = 1,\dots,k$, *are compact and continuous.*

Lemma 3.7 [4, 5]. *Assume* (H1) *holds. Then for all* $\sigma, t \in (0,\infty)$,

$$\psi_2^{-1}(\sigma)t \le \varphi^{-1}(\sigma\varphi(t)) \le \psi_1^{-1}(\sigma)t. \tag{3.17}$$

Set

$$\gamma(t) = \frac{1}{2}\left[\int_{1/4}^{t} p^{-1}\psi_2^{-1}\left(\frac{1}{q(s)}\int_s^t h(\tau)d\tau\right)ds + \int_t^{3/4} p^{-1}\psi_2^{-1}\left(\frac{1}{q(s)}\int_t^s h(\tau)d\tau\right)ds\right], \tag{3.18}$$

where $t \in [1/4,3/4]$. It follows from (H1)-(H2) that

$$\Gamma = \inf\left\{\gamma(t) : \frac{1}{4} \le t \le \frac{3}{4}\right\} > 0. \tag{3.19}$$

Lemma 3.8. *Assume* (H1)–(H4) *hold. For* $i = 1,\dots,k$, *let* $r > 0$ *such that* $[r/4,r] \subset (a_{j-1},a_j)$ *for some* $1 \le j \le i$. *Then*

$$\|T_\lambda^i u\| \ge \Gamma\psi_2^{-1}(\lambda)\varphi^{-1}(\omega_r^i) \quad \text{for } u \in \partial\Omega_r, \tag{3.20}$$

where $\omega_r^i = \min_{1/4r\le t\le r}\{f_i(t)\} > 0$.

Proof. Note, from the definition of $T_\lambda^i u$, that $T_\lambda^i u(\sigma)$ is the maximum value of $T_\lambda^i u$ on $[0,1]$. If $\sigma \in [1/4,3/4]$, we have

$$\begin{aligned} \|T_\lambda^i u\| \geq \frac{1}{2}\bigg[&\int_{1/4}^{\sigma} p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma}\lambda h(\tau)f_i(u(\tau))\,d\tau\right)ds \\ &+\int_{\sigma}^{3/4} p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma}^{s}\lambda h(\tau)f_i(u(\tau))\,d\tau\right)ds\bigg]. \end{aligned} \tag{3.21}$$

Since $f_i(u(t)) \geq \omega_r^i = \varphi(\varphi^{-1}(\omega_r^i))$ for $t \in [1/4,3/4]$, we find, by condition (H1),

$$\begin{aligned} \|T_\lambda^i u\| &\geq \frac{1}{2}\bigg[\int_{1/4}^{\sigma} p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma} h(\tau)d\tau\psi_2(\psi_2^{-1}(\lambda))\varphi(\varphi^{-1}(\omega_r^i))\right)ds \\ &\qquad +\int_{\sigma}^{3/4} p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma}^{s} h(\tau)d\tau\psi_2(\psi_2^{-1}(\lambda))\varphi(\varphi^{-1}(\omega_r^i))\right)ds\bigg] \\ &\geq \frac{1}{2}\bigg[\int_{1/4}^{\sigma} p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma} h(\tau)d\tau\varphi(\psi_2^{-1}(\lambda)\varphi^{-1}(\omega_r^i))\right)ds \\ &\qquad +\int_{\sigma}^{3/4} p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma}^{s} h(\tau)d\tau\varphi(\psi_2^{-1}(\lambda)\varphi^{-1}(\omega_r^i))\right)ds\bigg]. \end{aligned} \tag{3.22}$$

Now, because of Lemma 3.7, we have

$$\begin{aligned} \|T_\lambda^i u\| &\geq \frac{\psi_2^{-1}(\lambda)\varphi^{-1}(\omega_r^i)}{2}\bigg[\int_{1/4}^{\sigma} p^{-1}\psi_2^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma} h(\tau)d\tau\right)ds \\ &\qquad +\int_{\sigma}^{3/4} p^{-1}\psi_2^{-1}\left(\frac{1}{q(s)}\int_{\sigma}^{s} h(\tau)d\tau\right)ds\bigg] \\ &\geq \Gamma\psi_2^{-1}(\lambda)\varphi^{-1}(\omega_r^i). \end{aligned} \tag{3.23}$$

For $\sigma > 3/4$, it is easy to see

$$\|T_\lambda^i u\| \geq \int_{1/4}^{3/4} p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\lambda\int_s^{3/4} h(\tau)f_i(u(\tau))\,d\tau\right)ds. \tag{3.24}$$

On the other hand, we have

$$\|T_\lambda^i u\| \geq \int_{1/4}^{3/4} p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\lambda\int_{1/4}^{s} h(\tau)f_i(u(\tau))\,d\tau\right)ds \quad \text{for } \sigma < \frac{1}{4}. \tag{3.25}$$

Therefore, the same arguments show that

$$\|T_\lambda^i u\| \geq \Gamma\psi_2^{-1}(\lambda)\varphi^{-1}(\omega_r^i) \quad \text{if } \sigma > \frac{3}{4} \text{ or } \sigma < \frac{1}{4}. \tag{3.26}$$

□

4. Proof of Theorem 2.1

For each $i = 1,\ldots,k$, in view of condition (H4), there is an $r_i < a_i$ such that $r_i > 4a_{i-1}$. It follows that $[r_i/4, r_i] \subset (a_{i-1}, a_i)$. By Lemma 3.8, we infer that there exists a $\lambda_i > 0$ such that

$$||T_\lambda^i u|| > \|u\| \quad \text{for } u \in \partial\Omega_{r_i},\ \lambda > \lambda_i. \tag{4.1}$$

On the other hand, since $f_i(u)$ is bounded, there is an $R_i > r_i$ such that

$$||T_\lambda^i u|| < \|u\| \quad \text{for } u \in \partial\Omega_{R_i},\ \lambda > \lambda_i. \tag{4.2}$$

It follows from Lemma 3.1 that

$$i(T_\lambda^i, \Omega_{r_i}, K) = 0, \qquad i(T_\lambda^i, \Omega_{R_i}, K) = 1, \tag{4.3}$$

and hence $i(T_\lambda^i, \Omega_{R_i} \setminus \bar{\Omega}_{r_i}, K) = 1$. Thus, T_λ^i has a fixed point u_i in $\Omega_{R_i} \setminus \bar{\Omega}_{r_i}$. Lemma 3.4 implies that the fixed point u_i is a solution of (3.2) such that $a_i < \|u_i\| \le a_i$. Consequently, (3.2) has k positive solutions, $u_1, u_2, \ldots, u_k$, such that

$$0 = a_0 < ||u_1|| \le a_1 < ||u_2|| \le a_2 < \cdots \le a_{k-1} < ||u_k|| \le a_k \quad \text{for } \lambda > \lambda_0, \tag{4.4}$$

where $\lambda_0 = \max_{i=1,\ldots,n}\{\lambda_i\}$.

5. Elliptic systems

With the same transformation for (1.1), we can transform (1.2) to the following system:

$$\begin{gathered} (q(t)\varphi(pu'))' + \lambda h_1(t) g_1(u,v) = 0, \quad 0 < t < 1, \\ (q(t)\varphi(pv'))' + \lambda h_2(t) g_2(u,v) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{gathered} \tag{5.1}$$

where

$$h_i(t) = (R_2 - R_1)\big((R_2 - R_1)t + R_1\big)^{n-1} b_i\big((R_2 - R_1)t + R_1\big), \quad i = 1, 2, \tag{5.2}$$

$q(t)$ and p are the same as in (3.2).

In this section, let $\mathbb{X}$ be the Banach space $C[0,1] \times C[0,1]$ with

$$||(u,v)|| = \sup_{t\in[0,1]} |u(t)| + \sup_{t\in[0,1]} |u(t)|, \quad (u,v) \in \mathbb{X}. \tag{5.3}$$

Define $\mathbb{K}$ to be a cone in $\mathbb{X}$ by

$$\mathbb{K} = \left\{(u,v) \in \mathbb{X} : u(t), v(t) \ge 0,\ \min_{1/4\le t\le 3/4} (u(t) + v(t)) \ge \frac{1}{4}(\|u\| + \|v\|)\right\}, \tag{5.4}$$

where $\|u\| = \sup_{t\in[0,1]} u(t)$, $u \in C[0,1]$.

Also, define, for r a positive number, $\mathbb{U}_r$ by

$$\mathbb{U}_r = \{(u,v) \in \mathbb{K} : \|(u,v)\| < r\}. \tag{5.5}$$

Note that $\partial\mathbb{U}_r = \{(u,v) \in \mathbb{K} : \|(u,v)\| = r\}$.

For $i = 1,\dots,k$, $j = 1,2$, let g_j^i satisfy

$$g_j^i(u,v) = \begin{cases} g_j(u,v), & 0 \le u+v \le a_i, \\ 0, & a_i \le u+v, \end{cases} \tag{5.6}$$

and let the map $T^i = (T_1^i, T_2^i) : \mathbb{K} \to \mathbb{X}$ be defined by

$$T_j^i(u,v)(t) = \begin{cases} \displaystyle\int_0^t p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_j} \lambda h_j(\tau) g_j^i(u(\tau),v(\tau))\,d\tau\right) ds, & 0 \le t \le \sigma_j, \\ \displaystyle\int_t^1 p^{-1}\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_j}^{s} \lambda h_j(\tau) g_j^i(u(\tau),v(\tau))\,d\tau\right) ds, & \sigma_j \le t \le 1, \end{cases} \tag{5.7}$$

where $\sigma_j \in (0,1)$ is a solution of the equation

$$\Theta_j^i(u,v)(t) = 0, \quad 0 \le t \le 1, \tag{5.8}$$

and the map $\Theta_j^i : \mathbb{K} \to C[0,1]$ is defined by

$$\begin{aligned} \Theta_j^i(u,v)(t) = {} & \int_0^t \varphi^{-1}\left(\frac{1}{q(s)}\int_s^t \lambda h_j(\tau) g_j^i(u(\tau),v(\tau))\,d\tau\right) ds \\ & - \int_t^1 \varphi^{-1}\left(\frac{1}{q(s)}\int_t^s \lambda h_j(\tau) g_j^i(u(\tau),v(\tau))\,d\tau\right) ds, \quad 0 \le t \le 1. \end{aligned} \tag{5.9}$$

Lemma 5.1 can be proved in a similar manner as in [4, 5].

Lemma 5.1. *Assume* (H1), (H5), *and* (H6) *hold. Then for $i = 1,\dots,k$, T^i is well defined, $T^i(\mathbb{K}) \subset \mathbb{K}$ and $T^i : \mathbb{K} \to \mathbb{K}$ are compact and continuous.*

Lemma 5.2. *Assume* (H1) *and* (H5)–(H7) *hold. If $(u,v) \in \mathbb{K}$ such that $T^i(u,v) = (u,v)$ in $\mathbb{K}$, then (u,v) is a solution of (5.1) such that*

$$\sup_{t\in[0,1]} (u(t)+v(t)) \le a_i. \tag{5.10}$$

Proof. It is easy to see that (u,v) satisfies the following problem:

$$\begin{gathered} \big(q(t)\varphi(pu')\big)' + \lambda h_1(t) g_1^i(u,v) = 0, \quad 0 < t < 1, \\ \big(q(t)\varphi(pv')\big)' + \lambda h_2(t) g_2^i(u,v) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0. \end{gathered} \tag{5.11}$$

Let $t_0 \in (0,1)$ such that $u(t_0)+v(t_0)=\sup_{t\in[0,1]}(u(t)+v(t))$. It follows that $u'(t_0)+v'(t_0)=0$. If $u(t_0)+v(t_0) > a_i$, then there exist two numbers $0 \le t_1 < t_0 < t_2 \le 1$ such that $u(t)+v(t) > a_i$ for $t \in (t_1,t_2)$ and

$$u(t_1)+v(t_1) = u(t_2)+v(t_2) = a_i. \tag{5.12}$$

Since $g_j^i(u,v) = 0$, $j = 1,2$, for $u+v \ge a_i$, we have

$$\begin{aligned} &(q(t)\varphi(pu'(t)))' = 0 \quad \text{for } t \in [t_1,t_2],\\ &(q(t)\varphi(pv'(t)))' = 0 \quad \text{for } t \in [t_1,t_2]. \end{aligned} \tag{5.13}$$

Thus, $\varphi(pu'(t))$ and $\varphi(pu'(t))$ are constant on $[t_1,t_2]$, and so are $u'(t)$ and $v'(t)$. Since $u'(t_0)+v'(t_0) = 0$, it follows that $(u(t)+v(t))' = 0$ for $t \in [t_1,t_2]$. Consequently, $u(t)+v(t)$ is constant on $[t_1,t_2]$. This is a contradiction. Therefore, $\sup_{t\in[0,1]}(u(t)+v(t)) \le a_i$. On the other hand, since $g_j^i(u,v) \equiv g_j(u,v)$ for $0 \le u+v \le a_i$, $j = 1,2$, (u,v) is a solution of (1.2). □

Set

$$\gamma_j(t) = \frac{1}{2}\left[\int_{1/4}^{t} p^{-1}\psi_2^{-1}\left(\frac{1}{q(s)}\int_s^t h_j(\tau)d\tau\right)ds + \int_t^{3/4} p^{-1}\psi_2^{-1}\left(\frac{1}{q(s)}\int_t^s h_j(\tau)d\tau\right)ds\right], \tag{5.14}$$

where $t \in [1/4,3/4]$, $j = 1,2$. It follows from (H1) and (H5) that

$$\hat{\Gamma} = \inf\left\{\gamma_j(t) : \frac{1}{4} \le t \le \frac{3}{4},\ j = 1,2\right\} > 0. \tag{5.15}$$

The following lemma can be proved in the same manner as in Lemma 3.8.

Lemma 5.3. *Assume* (H1) *and* (H5)–(H7) *hold. For $i = 1,\dots,k$, let $r > 0$ such that $[r/4,r] \subset (a_{m-1},a_m)$ for some $1 \le m \le i$. Then*

$$||T^i(u,v)|| \ge \hat{\Gamma}\psi_2^{-1}(\lambda)\varphi^{-1}(\omega_r^i) \quad \text{for } (u,v) \in \partial\mathbb{U}_r, \tag{5.16}$$

where $\omega_r^i = \min_{1/4r\le t+s\le r}\{g_j(t,s),\ j = 1,2\} > 0$.

6. Proof of Theorem 2.2

For each $i = 1,\dots,k$, in view of condition (H7), there is an $r_i < a_i$ such that $r_i > 4a_{i-1}$. It follows that $[r_i/4,r_i] \subset (a_{i-1},a_i)$. By Lemma 5.3, we infer that there exists a $\lambda_i > 0$ such that

$$||T^i(u,v)|| > ||(u,v)|| \quad \text{for } (u,v) \in \partial\mathbb{U}_{r_i},\ \lambda > \lambda_i. \tag{6.1}$$

On the other hand, since $f_j^i(u,v)$, $j = 1,2$, are bounded, there is an $R_i > r_i$ such that

$$||T^i(u,v)|| < ||(u,v)|| \quad \text{for } (u,v) \in \partial\mathbb{U}_{R_i},\ \lambda > \lambda_i. \tag{6.2}$$

It follows from Lemma 3.1 that

$$i(T^i, \mathbb{U}_{r_i}, \mathbb{K}) = 0, \qquad i(T^i, \mathbb{U}_{R_i}, \mathbb{K}) = 1, \tag{6.3}$$

and hence, $i(T^i, \mathbb{U}_{R_i} \setminus \bar{\mathbb{U}}_{r_i}, \mathbb{K}) = 1$. Thus, T^i has a fixed point (u_i, v_i) in $\mathbb{U}_{R_i} \setminus \bar{\mathbb{U}}_{r_i}$. Lemma 5.2 implies that the fixed point (u_i, v_i) is a solution of (1.2) such that $a_{i-1} < \sup_{t\in[0,1]}(u_i(t) + v_i(t)) \leq a_i$. Consequently, (1.2) has k positive solutions, $(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)$, such that

$$a_{i-1} < \sup_{t\in[0,1]} \left(u_i(t) + v_i(t)\right) \leq a_i, \quad i = 1, \ldots, k, \text{ for } \lambda > \lambda_0, \tag{6.4}$$

where $\lambda_0 = \max_{i=1,\ldots,n}\{\lambda_i\}$.

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