Result.Math. 48 (2005) 310-325
1422-6383/05/040310-16
DOI 10.1007/s00025-005-0203-z

Results in Mathematics

POSITIVE PERIODIC SOLUTIONS OF SYSTEMS OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

Consider the n-dimensional nonautonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{G}(\mathbf{x}(t)) - \mathbf{B}(t)\mathbf{F}(\mathbf{x}(t-\tau(t)))$ Let $\mathbf{u} = (u_1, ..., u_n)$, $f_0^i = \lim_{\|\mathbf{u}\|\to 0} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|}$, $f_\infty^i = \lim_{\|\mathbf{u}\|\to\infty} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|}$, $i = 1, ..., n$, $\mathbf{F} = (f^1, ..., f^n)$, $\mathbf{F}_0 = \max_{i=1,...,n}\{f_0^i\}$ and $\mathbf{F}_\infty = \max_{i=1,...,n}\{f_\infty^i\}$. Under some quite general conditions, we prove that either $\mathbf{F}_0 = 0$ and $\mathbf{F}_\infty = \infty$, or $\mathbf{F}_0 = \infty$ and $\mathbf{F}_\infty = 0$, guarantee the existence of positive periodic solutions for the system for all $\lambda > 0$. Furthermore, we show that $\mathbf{F}_0 = \mathbf{F}_\infty = 0$, or $\mathbf{F}_0 = \mathbf{F}_\infty = \infty$ guarantee the multiplicity of positive periodic solutions for the system for sufficiently large, or small λ, respectively. We also establish the nonexistence of the system when either $\mathbf{F}_0$ and $\mathbf{F}_\infty > 0$, or $\mathbf{F}_0$ and $\mathbf{F}_\infty < \infty$ for sufficiently large, or small λ, respectively. We shall use fixed point theorems in a cone.

Keywords: positive periodic solutions, existence, fixed point theorem.

1 Introduction

The existence of periodic solutions of the equation of the form

$$x'(t) = a(t)g(x(t)) - \lambda b(t)f(x(t-\tau(t))). \tag{1.1}$$

and its generalizations have attracted much attention in the literature. See, e.g., Chow [1], Hadeler and Tomiuk [5], Kuang [8, 9], Kuang and Smith [10], Tang and Kuang [12]. The equation of the form (1.1) has

2000 Mathematics Subject Classification. Primary 34K13

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been proposed as models for a variety of population dynamics and physiological processes such as production of blood cells, respiration, and cardiac arrhythmias, See, for example, the above references, and [4, 11, 16].

Motivated by multiple-species ecological models, it is natural to explore nonautonomous n-dimensional systems. Nonautonomous systems are more realistic since real-world models often require us to incorporate temporal inhomogeneity in the models. One of the methods of incorporating temporal nonuniformity of the environments in models is to assume that the parameters are periodic with the same period of the time variable. In this paper, we shall study the existence of positive ω-periodic solutions for the nonautonomous n-dimensional system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{G}(\mathbf{x}(t)) - \lambda \mathbf{B}(t)\mathbf{F}(\mathbf{x}(t-\tau(t))), \tag{1.2}$$

where $\mathbf{A}(t) = \mathrm{diag}[a_1(t), a_2(t), \dots, a_n(t)]$, $\mathbf{B}(t) = \mathrm{diag}[b_1(t), b_2(t), \dots, b_n(t)]$, $\mathbf{F}(\mathbf{x}) = [f^1(\mathbf{x}), f^2(\mathbf{x}), \dots, f^n(\mathbf{x})]^T$, $\mathbf{G}(\mathbf{x}) = [g^1(\mathbf{x}), g^2(\mathbf{x}), \dots, g^n(\mathbf{x})]^T$ and $\lambda > 0$ is a positive parameter.

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^n = \Pi_{i=1}^n \mathbb{R}_+$, and for any $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{R}_+^n$, $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$.

In(1.2), we assume that

(H1) $a_i, b_i \in C(\mathbb{R}, [0, \infty))$ are ω-periodic functions such that $\int_0^\omega a_i(t)dt > 0$, $\int_0^\omega b_i(t)dt > 0$, $i = 1, \dots, n$. $\tau \in C(\mathbb{R}, \mathbb{R})$ is an ω-periodic function.

(H2) $f^i : \mathbb{R}_+^n \to [0, \infty)$ is continuous, $g^i : \mathbb{R}_+^n \to [l, L], 0 < l < L < \infty$ is continuous, $i = 1, \dots, n$.

(H3) $f^i(\mathbf{u}) > 0$ for $\|\mathbf{u}\| > 0$, $i = 1, \dots, n$.

In order to state our theorems, we introduce some notation. Let

$$\begin{aligned} f_0^i &= \lim_{\|\mathbf{u}\| \to 0} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|}, \quad f_\infty^i = \lim_{\|\mathbf{u}\| \to \infty} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|}, \mathbf{u} \in \mathbb{R}_+^n,\ i = 1, ..., n \\ \mathbf{F}_0 &= \max_{i=1,\dots,n} \{f_0^i\}, \quad \mathbf{F}_\infty = \max_{i=1,\dots,n} \{f_\infty^i\}. \end{aligned} \tag{1.3}$$

A solution $\mathbf{u}(t) = (u_1(t), ..., u_n(t))$ is positive if, for each $i = 1, ..., n$, $u_i(t) \geq 0$ for all $t \in \mathbb{R}$ and there is at least one component which is positive on $\mathbb{R}$.

Our main results are:

Theorem 1.1 *Assume* (H1)-(H2) hold.

(a). If $\mathbf{F}_0 = 0$ and $\mathbf{F}_\infty = \infty$, then for all $\lambda > 0$ (1.2) has a positive ω-periodicsolution.

(b). If $\mathbf{F}_0 = \infty$ and $\mathbf{F}_\infty = 0$, then for all $\lambda > 0$ (1.2) has a positive ω-periodic solution.

Theorem 1.2 *Assume* (H1)-(H3) hold.

(a). If $\mathbf{F}_0 = 0$ or $\mathbf{F}_\infty = 0$, then there exists a $\lambda_0 > 0$ such that (1.2) has a positive ω-periodic solution for $\lambda > \lambda_0$.

(b). If $\mathbf{F}_0 = \infty$ or $\mathbf{F}_\infty = \infty$, then there exists a $\lambda_0 > 0$ such that (1.2) has a positive ω-periodic solution for $0 < \lambda < \lambda_0$.

(c). If $\mathbf{F}_0 = \mathbf{F}_\infty = 0$, then there exists a $\lambda_0 > 0$ such that (1.2) has two positive ω-periodic solutions for $\lambda > \lambda_0$.

(d). If $\mathbf{F}_0 = \mathbf{F}_\infty = \infty$, then there exists a $\lambda_0 > 0$ such that (1.2) has two positive ω-periodic solutions for $0 < \lambda < \lambda_0$.

(e). If $\mathbf{F}_0 < \infty$ and $\mathbf{F}_\infty < \infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ (1.2) has no positive ω-periodic solution.

(f). If $\mathbf{F}_0 > 0$ and $\mathbf{F}_\infty > 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ (1.2) has no positive ω-periodic solution.

For $n = 1$, the existence, multiplicity and nonexistence of positive ω-periodic solution of (1.2) with a parameter λ was discussed in Wang [14]. Jiang, Wei and Zhang [6] obtained some existence results for the case when when $g^i \equiv 1$, $i = 1, ..., n$. In a recent paper, Wang, Kuang and Fen [15] proved multiplicity and nonexistence results for a similar equation when $g^i \equiv 1$, $i = 1, ..., n$.

This paper is organized in the following ways. In Section 2, we transform (1.2) into a system of integral equations, and then to a fixed point problem of an equivalent operator in a cone. Further, we establish two inequalities which allow us to estimate the operator. In Section 3, we apply the fixed point index to show the existence, multiplicity and nonexistence of positive ω-periodic solutions of (1.2) based on the inequalities.

2 Preliminaries

In this section, we recall some concepts and conclusions on the fixed point index in a cone in [2, 3, 7]. Let E be a Banach space and K be a closed, nonempty subset of E. K is said to be a cone if (i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geq 0$ and (ii) $u, -u \in K$ imply $u = 0$. Assume Ω is a bounded open subset in E with the boundary $\partial\Omega$, and let $T : K \cap \overline{\Omega} \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial\Omega \cap K$, then the fixed point index $i(T, K \cap \Omega, K)$ is defined. If $i(T, K \cap \Omega, K) \neq 0$, then T has a fixed point in $K \cap \Omega$. The following well-known result on the fixed point index is crucial in our arguments.

Lemma 2.1 ([2, 3, 7]). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|x\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|x\| = r\}$.*

(i) If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.

(ii) If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

In order to apply Lemma 2.1 to (1.2), let X be the Banach space defined by

$$X = \{\mathbf{u}(t) \in C(\mathbb{R}, \mathbb{R}^n) : \mathbf{u}(t + \omega) = \mathbf{u}(t), t \in \mathbb{R}\}$$

with a norm $\|\mathbf{u}\| = \sum_{i=1}^{n} \sup_{t \in [0,\omega]} |u_i(t)|$, for $\mathbf{u} = (u_1, ..., u_n) \in X$. For $\mathbf{u} \in X$ or $\mathbb{R}^n_+$, $\|\mathbf{u}\|$ denotes the norm of $\mathbf{u}$ in X or $\mathbb{R}^n_+$, respectively.

Define

$$K = \{\mathbf{u} = (u_1, ..., u_n) \in X : u_i(t) \geq \frac{\sigma_i^L(1 - \sigma_i^l)}{1 - \sigma_i^L} \sup_{t \in [0,\omega]} |u_i(t)|, i = 1, \ldots, n, t \in [0, \omega], \}$$

where $\sigma_i = e^{-\int_0^\omega a_i(t)dt}, i = 1, ..., n$. It is clear K is cone in X.

For $r > 0$, define $\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| < r\}$. It is clear that $\partial\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| = r\}$. Let $\mathbf{T}_\lambda : K \to X$ be a map with components $(T^1_\lambda, ..., T^n_\lambda)$:

$$T^i_\lambda \mathbf{u}(t) = \lambda \int_t^{t+\omega} G_i(t, s) b_i(s) f^i(\mathbf{u}(s - \tau(s)))ds, \quad i = 1, \ldots, n, \tag{2.1}$$

where

$$G_i(t, s) = \frac{e^{-\int_t^s a_i(\theta)g^i(\mathbf{u}(\theta))d\theta}}{1 - e^{-\int_0^\omega a_i(\theta)g^i(\mathbf{u}(\theta))d\theta}}$$

Note that

$$\frac{\sigma_i^L}{1-\sigma_i^L} \le G_i(t,s) \le \frac{1}{1-\sigma_i^l}, \quad t \le s \le t+\omega, i=1,\ldots,n.$$

Lemma 2.2 *Assume* (H1)-(H2) hold. Then $\mathbf{T}_\lambda(K) \subset K$ and $\mathbf{T}_\lambda : K \to K$ is continuous and completely continuous.

PROOF In view of the definition of K, for $\mathbf{u} \in K$, we have, $i = 1,\ldots,n$,

$$\begin{aligned}
(T_\lambda^i \mathbf{u})(t+\omega) &= \lambda \int_{t+\omega}^{t+2\omega} G_i(t+\omega, s) b_i(s) f^i(\mathbf{u}(s-\tau(s)))ds \\
&= \lambda \int_t^{t+\omega} G_i(t+\omega, \theta+\omega) b_i(\theta+\omega) f^i(\mathbf{u}(\theta+\omega-\tau(\theta+\omega)))d\theta \\
&= \lambda \int_t^{t+\omega} G_i(t,s) b_i(s) f^i(\mathbf{u}(s-\tau(s)))ds \\
&= (T_\lambda^i \mathbf{u})(t).
\end{aligned}$$

It is easy to see that $\int_t^{t+\omega} b_i(s) f^i(\mathbf{u}(s-\tau(s)))ds$ is a constant because of the periodicity of $b_i(t) f^i(\mathbf{u}(t-\tau(t)))$. Notice that, for $\mathbf{u} \in K$ and $t \in [0,\omega]$, $i = 1,\ldots,n$,

$$\begin{aligned}
T_\lambda^i \mathbf{u}(t) &\ge \frac{\sigma_i^L}{1-\sigma_i^L} \lambda \int_t^{t+\omega} b_i(s) f^i(\mathbf{u}(s-\tau(s)))ds \\
&= \frac{\sigma_i^L}{1-\sigma_i^L} \lambda \int_0^{\omega} b_i(s) f^i(\mathbf{u}(s-\tau(s)))ds \\
&= \frac{\sigma_i^L(1-\sigma_i^l)}{1-\sigma_i^L} \frac{1}{1-\sigma_i^l} \lambda \int_0^{\omega} b_i(s) f^i(\mathbf{u}(s-\tau(s)))ds \\
&\ge \frac{\sigma_i^L(1-\sigma_i^l)}{1-\sigma_i^L} \sup_{t\in[0,\omega]} |T_\lambda^i \mathbf{u}(t)|.
\end{aligned}$$

Thus $\mathbf{T}_\lambda(K) \subset K$ and it is easy to show that $\mathbf{T}_\lambda : K \to K$ is continuous and completely continuous. □

Lemma 2.3 *Assume that* (H1)-(H2) hold. Then $\mathbf{u} \in K$ is a positive periodic solution of (1.2) if and only if it is a fixed point of $\mathbf{T}_\lambda$ in K.

PROOF If $\mathbf{u} = (u_1, \ldots, u_n) \in K$ and $\mathbf{T}_\lambda \mathbf{u} = \mathbf{u}$, then, for $i = 1, \ldots, n$,

$$\begin{aligned}
u_i'(t) &= \frac{d}{dt}(\lambda \int_t^{t+\omega} G_i(t,s) b_i(s) f^i(\mathbf{u}(s - \tau(s)))ds) \\
&= \lambda G_i(t, t+\omega) b_i(t+\omega) f^i(\mathbf{u}(t + \omega - \tau(t+\omega)) - \lambda G_i(t,t) b_i(t) f^i(\mathbf{u}(t - \tau(t))) \\
&\quad + a_i(t) g^i(\mathbf{u}(t)) T_\lambda^i \mathbf{u}(t) \\
&= \lambda [G_i(t, t+\omega) - G_i(t,t)] b_i(t) f^i(\mathbf{u}(t - \tau(t))) + a_i(t) g^i(\mathbf{u}(t)) T_\lambda^i \mathbf{u}(t) \\
&= a_i(t) g^i(\mathbf{u}(t)) u_i(t) - \lambda b_i(t) f^i(\mathbf{u}(t - \tau(t))).
\end{aligned}$$

Thus $\mathbf{u}$ is a positive ω-periodic solution of (1.2). On the other hand, if $\mathbf{u} = (u_1, \ldots, u_n)$ is a positive ω-periodic function of (1.2), then $\lambda b_i(t) f^i(\mathbf{u}(t - \tau(t))) = a_i(t) g^i(\mathbf{u}(t)) u_i(t) - u_i'(t)$ and

$$\begin{aligned}
T_\lambda^i \mathbf{u}(t) &= \lambda \int_t^{t+\omega} G_i(t,s) b_i(s) f^i(\mathbf{u}(s - \tau(s)))ds \\
&= \int_t^{t+\omega} G_i(t,s)(a_i(s) g^i(\mathbf{u}(s)) u_i(s) - u_i'(s))ds \\
&= \int_t^{t+\omega} G_i(t,s) a_i(s) g^i(\mathbf{u}(s)) u_i(s) ds - \int_t^{t+\omega} G_i(t,s) u_i'(s) ds \\
&= \int_t^{t+\omega} G_i(t,s) a_i(s) g^i(\mathbf{u}(s)) u_i(s) ds - G_i(t,s) u_i(s)|_t^{t+\omega} \\
&\quad - \int_t^{t+\omega} G_i(t,s) a_i(s) g^i(\mathbf{u}(s)) u_i(s) ds \\
&= u_i(t).
\end{aligned}$$

Thus, $\mathbf{T}_\lambda \mathbf{u} = \mathbf{u}$, Furthermore, in view of the proof of Lemma 2.2, we also have $u_i(t) \geq \frac{\sigma_i^L(1-\sigma_i^l)}{1-\sigma_i^L} \sup_{t\in[0,\omega]} u_i(t)$ for $t \in [0, \omega]$. That is, $\mathbf{u}$ is a fixed point of $\mathbf{T}_\lambda$ in K. $\square$

Define $\Gamma = \min_{i=1,\ldots,n}\{\frac{\sigma_i^L}{1-\sigma_i^L} \int_0^\omega b_i(s)ds\} \min_{i=1,\ldots,n}\{\frac{\sigma_i^L(1-\sigma_i^l)}{1-\sigma_i^L}\} > 0$ and we have the following lemma.

Lemma 2.4 *Assume that* (H1)-(H2) hold. For any $\eta > 0$ and $\mathbf{u} = (u_1, \ldots, u_n) \in K$, if there exists a component f^i of $\mathbf{F}$ such that $f^i(\mathbf{u}(t)) \geq \sum_{j=1}^n u_j(t)\eta$ for $t \in [0, \omega]$, then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \lambda \Gamma \eta \|\mathbf{u}\|.$$

PROOF Since $\mathbf{u} \in K$ and $f^i(\mathbf{u}(t)) \geq \sum_{j=1}^n u_j(t)\eta$ for $t \in [0, \omega]$, we have

$$\begin{aligned}
(T_\lambda^i \mathbf{u})(t) &\geq \frac{\sigma_i^L}{1-\sigma_i^L} \lambda \int_0^\omega b_i(s) f^i(\mathbf{u}(s-\tau(s)))ds \\
&\geq \frac{\sigma_i^L}{1-\sigma_i^L} \lambda \int_0^\omega b_i(s) \sum_{j=1}^n u_j(s-\tau(s))\eta ds \\
&\geq \frac{\sigma_i^L}{1-\sigma_i^L} \lambda \int_0^\omega b_i(s)ds \sum_{j=1}^n \frac{\sigma_j^L(1-\sigma_j^l)}{1-\sigma_j^L} \sup_{t\in[0,\omega]} u_j(t)\eta \\
&\geq \min_{i=1,\ldots,n} \{\frac{\sigma_i^L}{1-\sigma_i^L} \int_0^\omega b_i(s)ds\} \min_{i=1,\ldots,n} \{\frac{\sigma_i^L(1-\sigma_i^l)}{1-\sigma_i^L}\} \lambda\eta \|\mathbf{u}\|.
\end{aligned}$$

Thus $\|\mathbf{T}_\lambda \mathbf{u}\| \geq \lambda\Gamma\eta\|\mathbf{u}\|$. □

For each $i = 1, .., n$, let $\hat{f}^i(t) : \mathbb{R}_+ \to \mathbb{R}_+$ be the function given by

$$\hat{f}^i(t) = \max\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n \text{ and } \|\mathbf{u}\| \leq \mathrm{t}\}.$$

Let $\hat{f}_0^i = \lim_{t\to 0} \frac{\hat{f}^i(t)}{t}$ and $\hat{f}_\infty^i = \lim_{t\to\infty} \frac{\hat{f}^i(t)}{t}$.

Lemma 2.5 *([13]) Assume* (H2) holds. Then $\hat{f}_0^i = f_0^i$ and $\hat{f}_\infty^i = f_\infty^i$, $i = 1, ..., n$.

Lemma 2.6 *Assume* (H1)-(H2) hold and let $r > 0$. If there exits an $\varepsilon > 0$ such that

$$\hat{f}^i(r) \leq \varepsilon r, \ \ i = 1, ..., n,$$

then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \lambda \hat{C} \varepsilon \|\mathbf{u}\|, \ \text{ for } \mathbf{u} \in \partial\Omega_r$$

where $\hat{C} = \sum_{i=1}^n \frac{1}{1-\sigma_i^l} \int_0^\omega b_i(s)ds$.

PROOF From the definition of $\mathbf{T}_\lambda$, for $\mathbf{u} \in \partial\Omega_r$, we have

$$\begin{aligned}
\|\mathbf{T}_\lambda \mathbf{u}\| &\leq \sum_{i=1}^n \frac{1}{1-\sigma_i^l} \lambda \int_0^\omega b_i(s) f^i(\mathbf{u}(s-\tau(s)))ds \\
&\leq \sum_{i=1}^n \frac{1}{1-\sigma_i^l} \lambda \int_0^\omega b_i(s) \hat{f}^i(r)ds \\
&\leq \sum_{i=1}^n \frac{1}{1-\sigma_i^l} \int_0^\omega b_i(s)ds \lambda\varepsilon\|\mathbf{u}\| = \lambda\hat{C}\varepsilon\|\mathbf{u}\|.
\end{aligned}$$

□

The following two lemmas are weak forms of Lemmas 2.4 and 2.6.

Lemma 2.7 *Assume* (H1)-(H3) hold. If $\mathbf{u} \in \partial\Omega_r$, $r > 0$, then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \lambda \hat{m}_r \min_{i=1,...,n} \{ \frac{\sigma_i^L}{1-\sigma_i^L} \int_0^\omega b_i(s)ds \}$$

where $\hat{m}_r = \min\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n \text{ and } \sigma \mathrm{r} \leq \|\mathbf{u}\| \leq \mathrm{r},\ \mathrm{i} = 1, ..., \mathrm{n}.\} > 0$, and $\sigma = \min_{i=1,...,n}\{\frac{\sigma_i^L(1-\sigma_i^l)}{1-\sigma_i^L}\}$.

Proof Note $r = \|\mathbf{u}\| = \sum_{i=1}^n \sup_{[0,\omega]} |u_i(t)| \geq \sum_{i=1}^n \inf_{[0,\omega]} |u_i(t)| \geq \sigma \sum_{i=1}^n \sup_{[0,\omega]} |u_i(t)| = \sigma r$. Thus $f^i(\mathbf{u}(t)) \geq \hat{m}_r$ for $\mathrm{t} \in [0, \omega]$, $i = 1, ..., n$. A slight modification of the proof in Lemma 2.4 yields the result.

□

Lemma 2.8 *Assume* (H1)-(H3) hold. If $\mathbf{u} \in \partial\Omega_r$, $r > 0$, then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \lambda \hat{M}_r \hat{C},$$

where $\hat{M}_r = \max\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n \text{ and } \|\mathbf{u}\| \leq \mathrm{r},\ \mathrm{i} = 1, ..., \mathrm{n}\} > 0$ and $\hat{C}$ is the positive constant defined in Lemma 2.6

Proof Since $f^i(\mathbf{u}(t)) \leq \hat{M}_r$ for $\mathrm{t} \in [0, \omega]$, $i = 1, ..., n$, a slight modification of the proof in Lemma 2.6 guarantees the result. □

3 Proof of Theorem 1.1

Proof Part (a). $\mathbf{F}_0 = 0$ implies that $f_0^i = 0$, $i = 1, ..., n$. It follows from Lemma 2.5 that $\hat{f}_0^i = 0$, $i = 1, ..., n$. Therefore, we can choose $r_1 > 0$ so that $\hat{f}^i(r_1) \leq \varepsilon r_1$, $i = 1, ..., n$, where the constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.6. We have by Lemma 2.6 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \lambda \varepsilon \hat{C} \|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_1}.$$

Now, since $\mathbf{F}_\infty = \infty$, there exists a component f^i of $\mathbf{F}$ such that $f_\infty^i = \infty$. Therefore, there is an $\hat{H} > 0$ such that

$$f^i(\mathbf{u}) \geq \eta \|\mathbf{u}\|$$

for $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{R}^n_+$ and $\|\mathbf{u}\| \geq \hat{H}$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Let $r_2 = \max\{2r_1, \frac{1}{\sigma}\hat{H}\}$, where $\sigma = \min_{i=1,...,n}\{\frac{\sigma_i^L(1-\sigma_i^l)}{1-\sigma_i^L}\}$. If $\mathbf{u} = (u_1, ..., u_n) \in \partial\Omega_{r_2}$, then

$$\min_{0\leq t\leq\omega} \sum_{i=1}^{n} u_i(t) \geq \sigma\|\mathbf{u}\| = \sigma r_2 \geq \hat{H},$$

which implies that

$$f^i(\mathbf{u}(t)) \geq \eta \sum_{i=1}^{n} u_i(t) \text{ for t} \in [0, \omega].$$

It follows from Lemma 2.4 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \lambda\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

By Lemma 2.1,

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1 \ \text{ and } \ i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0.$$

It follows from the additivity of the fixed point index that

$$i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = -1.$$

Thus, $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) \neq 0$, which implies $\mathbf{T}_\lambda$ has a fixed point $\mathbf{u} \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ by the existence property of the fixed point index. The fixed point $\mathbf{u} \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ is the desired positive solution of (1.2).

Part (b). If $\mathbf{F}_0 = \infty$, there exists a component f^i such that $f^i_0 = \infty$. Therefore, there is an $r_1 > 0$ such that

$$f^i(\mathbf{u}) \geq \eta\|\mathbf{u}\|$$

for $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{R}^n_+$ and $\|\mathbf{u}\| \leq r_1$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

If $\mathbf{u} = (u_1, ..., u_n) \in \partial\Omega_{r_1}$, then

$$f^i(\mathbf{u}(t)) \geq \eta \sum_{i=1}^{n} u_i(t), \quad \text{for} \ \ t \in [0, \omega].$$

Lemma 2.4 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \lambda \Gamma \eta \|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_1}.$$

We now determine Ω_{r_2}. $\mathbf{F}_\infty = 0$ implies that $f^i_\infty = 0$, $i = 1, ..., n$. It follows from Lemma 2.5 that $\hat{f}^i_\infty = 0$, $i = 1, ..., n$. Therefore there is an $r_2 > 2r_1$ such that

$$\hat{f}^i(r_2) \leq \varepsilon r_2, \ i = 1, ..., n,$$

where the constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.6. Thus, we have by Lemma 2.6 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \lambda \varepsilon \hat{C} \|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

By Lemma 2.1,

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0 \quad \textit{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1.$$

It follows from the additivity of the fixed point index that $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (1.2). □

4 Proof of Theorem 1.2

Proof Part (a). Fix a number $r_1 > 0$. Lemma 2.7 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\|, \ \text{for} \ \mathbf{u} \in \partial\Omega_{r_1}, \lambda > \lambda_0.$$

If $\mathbf{F}_0 = 0$, then $f^i_0 = 0$, $i = 1, ..., n$. It follows from Lemma 2.5 that

$$\hat{f}^i_0 = 0, \ i = 1, ..., n.$$

Therefore, we can choose $0 < r_2 < r_1$ so that

$$\hat{f}^i(r_2) \leq \varepsilon r_2, \ i = 1, ..., n,$$

where the constant $\varepsilon > 0$ satisfies

$$\lambda\varepsilon\hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.6. We have by Lemma 2.6 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \lambda\varepsilon\hat{C}\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

If $\mathbf{F}_\infty = 0$, then $f^i_\infty = 0$, $i = 1, ..., n$. It follows from Lemma 2.5 that $\hat{f}^i_\infty = 0$, $i = 1, ..., n$. Therefore there is an $r_3 > 2r_1$ such that

$$\hat{f}^i(r_3) \leq \varepsilon r_3, \ i = 1, ..., n,$$

where the constant $\varepsilon > 0$ satisfies

$$\lambda\varepsilon\hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.6. Thus, we have by Lemma 2.6 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \lambda\varepsilon\hat{C}\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0, \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1 \quad \textit{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 1.$$

Thus $i(\mathbf{T}_\lambda, \Omega_{r_1} \setminus \bar{\Omega}_{r_2}, K) = -1$ and $i(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = 1$. Hence, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ according to $\mathbf{F}_0 = 0$ or $\mathbf{F}_\infty = 0$, respectively. Consequently, (1.2) has a positive solution for $\lambda > \lambda_0$.

Part (b). Fix a number $r_1 > 0$. Lemma 2.8 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \text{ for } \mathbf{u} \in \partial\Omega_{r_1}, \ 0 < \lambda < \lambda_0.$$

If $\mathbf{F}_0 = \infty$, there exists a component f^i of $\mathbf{F}$ such that $f^i_0 = \infty$. Therefore, there is a positive number $r_2 < r_1$ such that

$$f^i(\mathbf{u}) \geq \eta\|\mathbf{u}\|$$

for $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{R}^n_+$ and $\|\mathbf{u}\| \leq r_2$, where $\eta > 0$ is chosen so that

$$\lambda\Gamma\eta > 1.$$

Then

$$f^i(\mathbf{u}(t)) \geq \eta \sum_{i=1}^{n} u_i(t),$$

for $\mathbf{u} = (u_1, ..., u_n) \in \partial\Omega_{r_2},\ \ t \in [0, \omega]$. Lemma 2.4 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \lambda\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

If $\mathbf{F}_\infty = \infty$, there exists a component f^i of $\mathbf{F}$ such that $f^i_\infty = \infty$. Therefore, there is an $\hat{H} > 0$ such that

$$f^i(\mathbf{u}) \geq \eta\|\mathbf{u}\|$$

for $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{R}^n_+$ and $\|\mathbf{u}\| \geq \hat{H}$, where $\eta > 0$ is chosen so that

$$\lambda\Gamma\eta > 1.$$

Let $r_3 = \max\{2r_1, \frac{\hat{H}}{\sigma}\}$, where $\sigma = \min_{i=1,...,n}\{\frac{\sigma_i^L(1-\sigma_i^l)}{1-\sigma_i^L}\}$. If $\mathbf{u} = (u_1, ..., u_n) \in \partial\Omega_{r_3}$, then

$$\min_{0\leq t\leq\omega} \sum_{i=1}^{n} u_i(t) \geq \sigma\|\mathbf{u}\| = \sigma r_3 \geq \hat{H},$$

which implies that

$$f^i(\mathbf{u}(t)) \geq \eta \sum_{i=1}^{n} u_i(t) \text{ for t} \in [0, \omega].$$

It follows from Lemma 2.4 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \lambda\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{\mathrm{r}_3}.$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1, \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0 \quad \text{and} \quad \mathrm{i}(\mathbf{T}_\lambda, \Omega_{\mathrm{r}_3}, \mathrm{K}) = 0,$$

and hence, $i(\mathbf{T}_\lambda, \Omega_{r_1} \setminus \bar{\Omega}_{r_2}, K) = 1$ and $i(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = -1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ according to $\mathbf{F}_0 = \infty$ or $\mathbf{F}_\infty = \infty$, respectively. Consequently, (1.2) has a positive solution for $0 < \lambda < \lambda_0$.

Part (c). Fix two numbers $0 < r_3 < r_4$. Lemma 2.7 implies that there exists a $\lambda_0 > 0$ such that we have, for $\lambda > \lambda_0$,

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\|, \ \text{for} \ \mathbf{u} \in \partial\Omega_{\mathrm{r_i}}, \ \ (\mathrm{i} = 3, 4).$$

Since $\mathbf{F}_0 = 0$ and $\mathbf{F}_\infty = 0$, it follows from the proof of Theorem 1.2 (a) that we can choose $0 < r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \text{ for } \mathbf{u} \in \partial\Omega_{r_i}, \ (i = 1, 2).$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1, \ i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1,$$

and

$$i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 0, \ i(\mathbf{T}_\lambda, \Omega_{r_4}, K) = 0$$

and hence, $i(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = -1$ and $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_4}, K) = 1$. Thus, $\mathbf{T}_\lambda$ has two fixed points $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ such that $\mathbf{u}_1(t) \in \Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ and $\mathbf{u}_2(t) \in \Omega_{r_2} \setminus \bar{\Omega}_{r_4}$, which are the desired distinct positive periodic solutions of (1.2) for $\lambda > \lambda_0$ satisfying

$$r_1 < \|\mathbf{u}_1\| < r_3 < r_4 < \|\mathbf{u}_2\| < r_2.$$

Part (d). Fix two numbers $0 < r_3 < r_4$. Lemma 2.8 implies that there exists a $\lambda_0 > 0$ such that we have, for $0 < \lambda < \lambda_0$,

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \text{ for } \mathbf{u} \in \partial\Omega_{\mathrm{r_i}}, \ (\mathrm{i} = 3, 4).$$

Since $\mathbf{F}_0 = \infty$ and $\mathbf{F}_\infty = \infty$, it follows from the proof of Theorem 1.2 (b) that we can choose $0 < r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\|, \text{ for } \mathbf{u} \in \partial\Omega_{r_i}, \ (i = 1, 2).$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0, \ i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0,$$

and

$$i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 1, \ i(\mathbf{T}_\lambda, \Omega_{r_4}, K) = 1$$

and hence, $i(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = 1$ and $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_4}, K) = -1$. Thus, $\mathbf{T}_\lambda$ has two fixed points $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ such that $\mathbf{u}_1(t) \in \Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ and $\mathbf{u}_2(t) \in \Omega_{r_2} \setminus \bar{\Omega}_{r_4}$, which are the desired distinct positive periodic solutions of (1.2) for $0 < \lambda < \lambda_0$ satisfying

$$r_1 < \|\mathbf{u}_1\| < r_3 < r_4 < \|\mathbf{u}_2\| < r_2.$$

Part (e). Since $\mathbf{F}_0 < \infty$ and $\mathbf{F}_\infty < \infty$, then $f_0^i < \infty$ and $f_\infty^i < \infty$, $i = 1, ..., n$. It is easy to show (see [13]) that there exists an $\varepsilon > 0$ such that

$$f^i(\mathbf{u}) \le \varepsilon\|\mathbf{u}\| \text{ for } \mathbf{u} \in \mathbb{R}_+^{\mathrm{n}}, \ \ \mathrm{i} = 1, ..., \mathrm{n}.$$

Assume $\mathbf{v}(t)$ is a positive solution of (1.2). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0$, where

$$\lambda_0 = \frac{1}{\sum_{i=1}^n \frac{1}{1-\sigma_i^l} \int_0^\omega b_i(s)ds\varepsilon}.$$

In fact, for $0 < \lambda < \lambda_0$, since $\mathbf{T}_\lambda \mathbf{v}(t) = \mathbf{v}(t)$ for $t \in [0, \omega]$, we find

$$\begin{aligned}
\|\mathbf{v}\| &= \|\mathbf{T}_\lambda \mathbf{v}\| \\
&= \sum_{i=1}^n \max_{0 \le t \le \omega} T_\lambda^i \mathbf{v}(t) \\
&\le \sum_{i=1}^n \frac{1}{1-\sigma_i^l} \lambda \int_0^\omega b_i(s) f^i(\mathbf{v}(s-\tau(s)))ds \\
&\le \sum_{i=1}^n \frac{1}{1-\sigma_i^l} \int_0^\omega b_i(s)ds\lambda\varepsilon\|\mathbf{v}\| \\
&< \|\mathbf{v}\|,
\end{aligned}$$

which is a contradiction.

Part (f). Since $\mathbf{F}_0 > 0$ and $\mathbf{F}_\infty > 0$, there exist two components f^i and f^j of $\mathbf{F}$ such that $f_0^i > 0$ and $f_\infty^j > 0$. It is easy to show (see [13]) that there exist positive numbers η, r_1 such that

$$f^i(\mathbf{u}) \ge \eta\|\mathbf{u}\| \ \text{ for } \ \mathbf{u} \in \mathbb{R}_+^n, \ \|\mathbf{u}\| \le r_1 \tag{4.2}$$

and

$$f^j(\mathbf{u}) \ge \eta\|\mathbf{u}\| \ \text{ for } \ \mathbf{u} \in \mathbb{R}_+^n, \ \|\mathbf{u}\| \ge \sigma r_1, \tag{4.3}$$

here $\sigma = \min_{i=1,\dots,n}\{\frac{\sigma_i^L(1-\sigma_i^l)}{1-\sigma_i^L}\}$. Assume $\mathbf{v}(t) = (v_1, ..., v_n)$ is a positive solution of (1.2). We will show that this leads to a contradiction for $\lambda > \lambda_0 = \frac{1}{\Gamma\eta}$. In fact, if $\|\mathbf{v}\| \le r_1$, (4.2) implies that

$$f^i(\mathbf{v}(t)) \ge \eta \sum_{i=1}^n v_i(t), \ \text{ for } \ t \in [0, \omega].$$

On the other hand, if $\|\mathbf{v}\| > r_1$, then

$$\min_{0\le t\le\omega}\sum_{i=1}^{n} v_i(t) \ge \sigma\|\mathbf{v}\| > \sigma r_1,$$

which, together with (4.3), implies that

$$f^j(\mathbf{v}(t)) \ge \eta\sum_{i=1}^{n} v_i(t), \quad \text{for} \ \ t \in [0,\omega].$$

Since $\mathbf{T}_\lambda\mathbf{v}(t) = \mathbf{v}(t)$ for $t \in [0,\omega]$, it follows from Lemma 2.4 that, for $\lambda > \lambda_0$,

$$\begin{aligned}\|\mathbf{v}\| &= \|\mathbf{T}_\lambda\mathbf{v}\| \\ &\ge \lambda\Gamma\eta\|\mathbf{v}\| \\ &> \|\mathbf{v}\|,\end{aligned}$$

which is a contradiction. $\square$

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Eingegangen am 4. Juni 2005