J Dyn Diff Equat (2013) 25:505–533
DOI 10.1007/s10884-013-9293-6

Front-Like Entire Solutions for Monostable Reaction-Diffusion Systems

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Received: 21 December 2012 / Revised: 26 January 2013 / Published online: 26 February 2013

Abstract This paper is concerned with front-like entire solutions for monostable reaction-diffusion systems with cooperative and non-cooperative nonlinearities. In the cooperative case, the existence and asymptotic behavior of spatially independent solutions (SIS) are first proved. Further, combining a SIS and traveling fronts with different wave speeds and propagation directions, the existence and various qualitative properties of entire solutions are established by using the comparison principle. In the non-cooperative case, we introduce two auxiliary cooperative systems and establish a comparison theorem for the Cauchy problems of the three systems, and then prove the existence of entire solutions via using the comparison theorem, the traveling fronts and SIS of the auxiliary systems. Our results are applied to some biological and epidemiological models. To the best of our knowledge, it is the first work to study the entire solutions of non-cooperative reaction-diffusion systems.

Keywords Entire solution · Traveling wave solution · Cooperative system · Non-cooperative system · Monostable nonlinearity

Mathematics Subjec Classification (2000) 35K57 · 35B40 · 34K30 · 92D25 · 92D30

1 Introduction

This paper is concerned with entire solutions of the following m-dimensional reaction-diffusion system in $\mathbb{R}^N$:

$$u_t = D\Delta u + f(u), \quad x \in \mathbb{R}^N, t \in \mathbb{R}, \tag{1.1}$$

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where $m, N \in \mathbb{N}$,

$$u = (u_1, \cdots, u_m),\ \ f = (f_1, \cdots, f_m),\ \ D = \mathrm{diag}(d_1, \cdots, d_m),$$

and $(d_1, \cdots, d_m) \gg \mathbf{0} := (0, \cdots, 0) \in \mathbb{R}^m$. Here and in what follows, we always use the usual notations for the standard ordering in $\mathbb{R}^m$. As usual, system (1.1) is said to be cooperative on $I \subseteq \mathbb{R}^m$ if each $f_i(u)$ is non-decreasing in u_j on I for $1 \le j \ne i \le m$; otherwise, it is said to be non-cooperative on I.

One important topic for reaction-diffusion systems is the traveling wave solution that describes the phenomenon of wave propagation. In the past decades, many studies have led to almost complete description of traveling wave solutions of (1.1) with cooperative nonlinearity [22,31,34,42,43]. For example, Volpert et al. [34] gave a complete result about the monostable and bistable traveling fronts, and Tsai [31] investigated the global exponential stability of the bistable traveling fronts. In a series of papers, Weinberger et al. [22,42,43] studied spreading speeds and traveling fronts for general cooperative recursion systems. The second author, Wang [36], extended the results of spreading speeds and traveling waves for cooperative systems to a large class of non-cooperative systems. Related results on scalar non-monotone evolution equations, we refer to [8,18,21,25,35,45,49].

In addition to traveling wave solutions, another important topic in diffusion systems is the interactions of them, which is crucially related to the pattern formation problem. We refer to [5,6,19,26] for more details. Mathematically, this phenomenon can be described by the so-called front-like *entire solution* that is defined for all space and time and behaves like a combination of traveling fronts as $t \to -\infty$. On the other hand, from the dynamical points of view, the study of entire solutions is essential for a full understanding of the transient dynamics and the structures of the global attractor [27]. In the recent years, there were many works devoted to the interactions of traveling fronts and entire solutions for scalar reaction-diffusion (both spatially continuous and discrete) equations with and without delays, see e.g., [2–4,11,12,15,16,23,24,27,38–40,46,52].

More recently, Morita and Tachibana [28], Guo and Wu [14], Wang and Lv [37] and Wu [44] extended the existence of entire solutions for scalar equations to some specific *two component cooperative* reaction-diffusion model systems. The basic idea in these studies, similar to [2,12,24,38], is to use traveling fronts propagating from both directions of the x-axis to build sub- and supersolutions, and then prove the existence results by employing comparison principle. Unfortunately, it seems difficult, if not impossible, to construct such supersolutions for the *m-component* reaction-diffusion system (1.1). In fact, to the best of our knowledge, there have been no results on entire solutions for general cooperative reaction-diffusion systems and non-cooperative systems.

The purpose of this paper is to consider entire solutions of system (1.1) with cooperative or non-cooperative nonlinearity. In the cooperative case, the existence and asymptotic behavior of spatially independent solutions are first proved. Since it is difficult to use traveling fronts to construct supersolutions for the general *m-component* system, we extend the arguments developed in [15] for scalar KPP equations to system (1.1). More precisely, we construct appropriate upper estimates by virtue of the exact asymptotic behavior of the traveling fronts and spatially independent solution, and then prove the existence of entire solutions of (1.1) by using comparison principle (see Theorem 2.9). Various qualitative features of the entire solutions are also investigated (see Theorems 2.10 and 2.11). Although the argument is inspired by the work of Hamel and Nadirashvili [15], the technical details are different. In [15], the upper estimates were proved by the solution formulation of the linearization of the scalar KPP equation at the trivial equilibrium. Contrasting to [15], we use a general comparison principle to prove the upper estimates (see Lemma 2.13). Recently, the method was successfully applied in our previous work [48] to a multi-type SIS nonlocal epidemic model.

It is well known that the comparison principle is not applicable for the non-cooperative reaction-diffusion systems. To overcome this difficulty, we introduce two auxiliary cooperative systems with one reaction term lies above and another lies below the reaction term of (1.1), and establish a comparison theorem for the Cauchy problems of the three systems. The existence and qualitative features of entire solutions of (1.1) is then established by using the comparison theorem and considering a sequence of initial value problems of (1.1), where the combinations of traveling fronts and spatially independent solutions of the lower system (i.e. the auxiliary system with smaller reaction term) are taken as the initial values (see Theorem 3.6). We mention that these auxiliary systems have been used by the second author of this paper to establish the existence of traveling wave solutions of (1.1) recently, see Wang [36]. To the best of our knowledge, it is the first work to study the entire solutions of non-cooperative reaction-diffusion systems.

In biology and epidemiology, there are quite a few reaction-diffusion model systems of the form (1.1) with cooperative or non-cooperative nonlinearities. We shall illustrate our main results by discussing the following models in [1,32,33,36,41].

A. A Buffered System. In [32,33], Tsai and Sneyd presented a buffered system:

$$\begin{cases} \partial_t u_1 = d\Delta u_1 + g(u_1) + \sum_{i=1}^m [k_i^-(b_i^0 - v_i) - k_i^+ u_1 v_i], \\ \partial_t v_i = d_i \Delta v_i + k_i^-(b_i^0 - v_i) - k_i^+ u_1 v_i, \ i = 1, \cdots, n, \end{cases} \tag{1.2}$$

where $d, k_i^{\pm}, b_i^0 > 0$ and $d_i > 0$ are given parameters. They studied the existence, uniqueness and stability of traveling fronts of (1.2) by taking the typical bistable nonlinearity for the function g, i.e. $g(u_1) = u_1(u_1 - a)(1 - u_1)$ for some $a \in (0, 1)$. Note that (1.2) can be transformed to a cooperative system on $\mathbb{R}^+ \times \prod_{i=1}^n [0, b_i^0]$ under the change of variable $w_i = b_i^0 - v_i, i = 1, \cdots, n$. Other results related to the buffered system, we refer to [7,13,20] and the references therein.

B. An Epidemic Model. To study the fecally-orally transmitted diseases in the European Mediterranean regions, Capasso and Maddalena [1] introduced the epidemic model:

$$\begin{cases} \partial_t u_1 = d_1 \Delta u_1 - a_{11} u_1 + a_{12} u_2, \\ \partial_t u_2 = d_2 \Delta u_2 - a_{22} u_2 + g(u_1), \end{cases} \tag{1.3}$$

where $d_1, a_{11}, a_{12}, a_{22} > 0$ and $d_2 \geq 0$ are given parameters. The function $g(u_1)$ describes the infection rate of human under the assumption that total susceptible human population is constant. In general, $g(\cdot)$ is increasing on $[0, +\infty)$. But, if the "psychological" effect is considered (see, e.g., Xiao and Ruan [50]), then $g(\cdot)$ is a unimodal curve on $[0, +\infty)$, that is, $g(\cdot)$ achieves its maximum at some $u_{\max} > 0$, and is increasing on $[0, u_{\max}]$ and decreasing on $[u_{\max}, +\infty)$. When $d_2 = 0$ and g is monotone, Xu and Zhao [51] proved the existence, uniqueness and stability of bistable traveling fronts of (1.3) and Zhao and Wang [54] established the existence and non-existence of monostable traveling fronts. These results were then extended by Wu and Liu [45] to the non-monotone case by constructing two auxiliary monotone integral equations.

C. A Population Model. Weinberger et al. [41] discussed the reaction-diffusion model which describes the interaction between ungulates with linear density u_1 and grass with linear density u_2:

$$\begin{cases} \partial_t u_1 = d_1 \Delta u_1 + u_1[-\alpha - \delta u_1 + r_1 u_2], \\ \partial_t u_2 = d_2 \Delta u_2 + r_2 u_2[1 - u_2 + h(u_1)], \end{cases} \tag{1.4}$$

where $d_1, d_2, r_1, r_2, \alpha, \delta$ are all positive parameters. The function $h(u_1)$ models the increase in the specific growth rate of the grass due to the presence of ungulates. When the density u_1 is small the net effect of ungulates is increasingly beneficial, but as the density increases above a certain value, the benefits decrease with increasing. In Weinberger et al. [41] established the spreading speeds for (1.4) by employing comparison methods. Taking the non-monotone Ricker function $u_1 e^{-u_1}$ as $h(u_1)$, Wang [36] further characterized the spreading speed as the slowest speed of traveling wave solutions.

Throughout this paper, we always make the following assumptions:

(A_0) There exists $\mathbf{K} \gg \mathbf{0}$ such that $f(\mathbf{0}) = f(\mathbf{K}) = \mathbf{0}$, $f \in C^2([\mathbf{0}, \mathbf{K}], \mathbb{R}^m)$ and there is no other positive equilibrium of f between $\mathbf{0}$ and $\mathbf{K}$.

(A_1) One of the following holds:

(a) The matrix $f'(\mathbf{0})$ is cooperative and irreducible with $s(f'(\mathbf{0})) > 0$, where

$$s(f'(\mathbf{0})) := \max\{\Re\lambda : \det(\lambda I - f'(\mathbf{0})) = 0\};$$

(b) For each $\lambda \geq 0$, $A(\lambda) := D\lambda^2 + f'(\mathbf{0})$ is in block lower triangular form, the first diagonal block has a positive principal eigenvalue $M(\lambda)$, and $M(\lambda)$ is strictly larger than the principal eigenvalues of all other diagonal blocks. In addition, there is a positive eigenvector $v(\lambda) = (v_1(\lambda), \cdots, v_m(\lambda)) \gg 0$ of $A(\lambda)$ corresponding to $M(\lambda)$ and $v(\lambda)$ is continuous with respect to λ.

We mention that a square matrix is called to be cooperative if all off-diagonal entries are non-negative, and irreducible if it cannot be placed into block lower-triangular form by simultaneous row/column permutations (cf. Smith [29]).

If (A_1)(b) holds, by the argument of [36, Lemma 1.1], there exist two numbers $c_* > 0$ and $\lambda_* > 0$ such that

$$c_* = \frac{M(\lambda_*)}{\lambda_*} = \inf_{\lambda > 0} \frac{M(\lambda)}{\lambda}, \tag{1.5}$$

and for any $c > c_*$, there exists $\lambda_1 := \lambda_1(c) \in (0, \lambda_*)$ such that $M(\lambda_1) = c\lambda_1$ and $M(\lambda) < c\lambda$ for any $\lambda \in (\lambda_1, \lambda_*]$.

If (A_1)(a) holds, then the matrix $A(\lambda) = D\lambda^2 + f'(\mathbf{0})$ is also cooperative and irreducible. Hence

$$M(\lambda) = s(A(\lambda)) := \max\{\Re\lambda : \det(\lambda I - A(\lambda)) = 0\}$$

is a simple eigenvalue of $A(\lambda)$ with an eigenvector $v(\lambda) = (v_1(\lambda), \cdots, v_m(\lambda)) \gg 0$. In addition, $M(\lambda) = s(A(\lambda)) \geq s(f'(\mathbf{0})) > 0$ for any $\lambda \geq 0$ (see e.g., [29, Corollary 4.3.2]). From the argument of [7, Lemma 2.1], there also exist $c_* > 0$ and $\lambda_* > 0$ such that (1.5) holds, and for any $c > c_*$, there exists $\lambda_1 := \lambda_1(c) \in (0, \lambda_*)$ such that $M(\lambda_1) = c\lambda_1$ and $M(\lambda) < c\lambda$ for any $\lambda \in (\lambda_1, \lambda_*]$.

The rest of the paper is organized as follows. In Sect. 2, we consider the entire solutions of system (1.1) with monostable and cooperative nonlinearity. Section 3 is devoted to the entire solutions of (1.1) with monostable and non-cooperative nonlinearity. In Sect. 4, we apply our abstract results to the above models (1.2)–(1.4). Finally, conclusions and discussions are given in Sect. 5.

2 Entire Solutions for Cooperative Systems

In this section, we consider the entire solutions of (1.1) with monostable and cooperative nonlinearity. In addition to (A_0) and (A_1), we also need the following assumptions:

(A_2) System(1.1) is cooperative on $[\mathbf{0}, \mathbf{K}]$, that is, $\partial_j f_i(u) \geq 0$ for all $u \in [\mathbf{0}, \mathbf{K}]$ and $1 \leq j \neq i \leq m$.

(A_3) For any $k \in \mathbb{Z}^+$, $\rho_1, \cdots, \rho_k > 0$ and $\lambda_1, \cdots, \lambda_k \in [0, \lambda_*]$,

$$f\left(\min\{\mathbf{K}, \rho_1 v(\lambda_1) + \cdots + \rho_k v(\lambda_k)\}\right) \leq f'(\mathbf{0})\left[\rho_1 v(\lambda_1) + \cdots + \rho_k v(\lambda_k)\right].$$

Here, $v(\lambda) \gg 0$ is the eigenvector of $A(\lambda)$ corresponding to $M(\lambda)$.

Remark 2.1 It is easily seen that if $f(u) \leq f'(\mathbf{0})u$ for $u \in [\mathbf{0}, \mathbf{K}]$, then ($A_3$) holds spontaneously. We also note that if f is defined on $[0, +\infty)^m$, then (A_3) can be replaced by (A_3)*:

(A_3)* For any $k \in \mathbb{Z}^+$, $\rho_1, \cdots, \rho_k > 0$ and $\lambda_1, \cdots, \lambda_k \in [0, \lambda_*]$,

$$f\left(\rho_1 v(\lambda_1) + \cdots + \rho_k v(\lambda_k)\right) \leq f'(\mathbf{0})\left[\rho_1 v(\lambda_1) + \cdots + \rho_k v(\lambda_k)\right].$$

From the arguments of [7, Theorem 3.1] and [36, Theorem 2.1], we have the following result.

Proposition 2.2 *Let* (A_0)–(A_3) *hold. For every* $c \geq c_*$ *and* $\nu \in \mathbb{R}^N$ *with* $\|\nu\| = 1$, (1.1) *admits a traveling front*

$$\Phi_c(\xi) = (\phi_{1,c}(\xi), \cdots, \phi_{m,c}(\xi)), \ \xi = x \cdot \nu + ct,$$

which satisfies $\Phi_c(-\infty) = \mathbf{0}$, $\Phi_c(+\infty) = \mathbf{K}$ *and* $\Phi_c(\cdot) \gg \mathbf{0}$. *Furthermore, if* $c > c_*$, *then there holds*

$$\lim_{\xi \to -\infty} \Phi_c(\xi) e^{-\lambda_1(c)\xi} = v(\lambda_1(c)) \text{ and } \Phi_c(\xi) \leq v(\lambda_1(c)) e^{\lambda_1(c)\xi} \text{ for all } \xi \in \mathbb{R}.$$

In the remainder of this section, we first give some comparison theorems for sub and supersolutions of (1.1). We then state the main results for the cooperative system (Theorems 2.9–2.11) and establish the existence and asymptotic behavior of spatially independent solutions. Finally, we prove Theorems 2.9–2.11 by constructing appropriate subsolutions and upper estimates and using a general comparison principle.

2.1 Preliminaries

Consider the initial value problem of (1.1) with initial condition:

$$u(x, \tau) = \varphi(x), \quad x \in \mathbb{R}^N, \tag{2.1}$$

where $\tau \in \mathbb{R}$ is an any given constant.

Let $X = \mathrm{BUC}(\mathbb{R}^N, \mathbb{R}^m)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}^N$ into $\mathbb{R}^m$ with the supremum norm $\|\cdot\|_X$. For simplicity, we denote $W = [\mathbf{0}, \mathbf{K}]$ and $[\mathbf{0}, \mathbf{K}]_X = \left\{\phi \in X : \mathbf{0} \leq \phi(x) \leq \mathbf{K}, x \in \mathbb{R}^N\right\}$. Take $L = \max_{i=1,\cdots,m} \max\left\{|\partial_i f_i(u)| \big| u \in [\mathbf{0}, \mathbf{K}]\right\}$ and define

$$Q(u) = (Q_1(u), \cdots, Q_m(u)) = f(u) + Lu, \ u \in W.$$

Clearly, $Q(u)$ is non-decreasing in u for $u \in W$. We further define a family of linear operator

$$T(t) = \mathrm{diag}(T_1(t), \cdots, T_m(t)) : X \to X, \ t \geq 0, \tag{2.2}$$

by $T_i(0) = I$ and

$$(T_i(t)\phi)(x) = e^{-Lt} \int_{\mathbb{R}^N} \Psi_i(y, t)\phi(x - y)dy, \ \forall x \in \mathbb{R}^N, \ t > 0, \ \phi(x) \in \mathrm{BUC}(\mathbb{R}^N, \mathbb{R}),$$

where

$$\Psi_i(x,t) = \frac{1}{(4d_i\pi t)^{N/2}} \exp\left\{-\frac{\|x\|^2}{4d_i t}\right\},\ i = 1,\cdots,m.$$

The definitions of sub- and supersolutions of (1.1) are as follows.

Definition 2.3 A continuous function $u = (u_1,\cdots,u_m) : \mathbb{R}^N \times [\tau,+\infty) \to W$ is called a supersolution of (1.1) on $[\tau,+\infty)$ if

$$u(x,t) \geq T(t-\tau)u(x,\tau) + \int_{\tau}^{t} T(t-s)Q(u(x,s))ds, \quad \forall x \in \mathbb{R}^N, t > \tau, \tag{2.3}$$

A subsolution of (1.1) is defined by reversing the inequality.

Remark 2.4 Let $w = (w_1,\cdots,w_m) : \mathbb{R}^N \times [\tau,+\infty) \to W$ be a continuous function with the property that w_i is C^1 in t and C^2 in x. It is easy to see that if w satisfies

$$w_t \geq (or \leq) D\Delta w + f(w), \quad \forall x \in \mathbb{R}^N, t > \tau,$$

then w is a supersolution (or subsolution) of (1.1) on $[\tau,+\infty)$.

By Definition 2.3, we have the following results, see e.g., Fang and Zhao [7].

Lemma 2.5 *(i) For any $\varphi \in [\mathbf{0},\mathbf{K}]_X$, (1.1) admits a unique classical solution $u(x,t;\varphi)$ satisfying $u(x,\tau;\varphi) = \varphi(x)$ and $\mathbf{0} \leq u(x,t;\varphi) \leq \mathbf{K}$ for all $x \in \mathbb{R}^N$ and $t \geq \tau$.*
(ii) Let $w^+(x,t)$ and $w^-(x,t)$ be a supersolution and a subsolution of (1.1), respectively. If $w^+(\cdot,\tau) \geq w^-(\cdot,\tau)$, then $w^+(\cdot,t) \geq w^-(\cdot,t)$ for all $t \geq \tau$.

The following result follows from the standard parabolic estimates (Friedman [10]), see also Wang et al. [38, Proposition 4.3].

Lemma 2.6 *Suppose that $u(x,t;\varphi)$ is a solution of (1.1) with the initial value $\varphi \in [\mathbf{0},\mathbf{K}]_X$. Then there exists a positive constant M_1, independent of τ and φ, such that for any $x \in \mathbb{R}^N$ and $t > \tau + 1$,*

$$\left\|\frac{\partial u}{\partial t}(x,t;\varphi)\right\| \leq M_1,\ \left\|\frac{\partial^2 u}{\partial t x_i}(x,t;\varphi)\right\| \leq M_1,\ \left\|\frac{\partial^2 u}{\partial t^2}(x,t;\varphi)\right\| \leq M_1,$$
$$\left\|\frac{\partial u}{\partial x_i}(x,t;\varphi)\right\| \leq M_1,\ \left\|\frac{\partial^2 u}{\partial x_i t}(x,t;\varphi)\right\| \leq M_1,\ \left\|\frac{\partial^2 u}{\partial x_i x_j}(x,t;\varphi)\right\| \leq M_1$$
$$\left\|\frac{\partial^3 u}{\partial x_i^2 t}(x,t;\varphi)\right\| \leq M_1,\ \left\|\frac{\partial^3 u}{\partial x_i^2 x_j}(x,t;\varphi)\right\| \leq M_1,\ \forall i,j = 1,\cdots,N.$$

Similar to Lemma 2.5(ii), we have the following result.

Lemma 2.7 *Let $u^+ \in C\big(\mathbb{R}^N \times [\tau,+\infty), [0,+\infty)^m\big)$ and*

$$u^- \in C\big(\mathbb{R}^N \times [\tau,+\infty), (-\infty,K_1] \times \cdots \times (-\infty,K_m]\big)$$

be such that $u^+(\cdot,\tau) \geq u^-(\cdot,\tau)$ and

$$u_t^+ \geq D\Delta u^+ + f'(\mathbf{0})u^+, \quad \forall x \in \mathbb{R}^N, t > \tau,$$
$$u_t^- \leq D\Delta u^- + f'(\mathbf{0})u^-, \quad \forall x \in \mathbb{R}^N, t > \tau.$$

Then, $u^+(x,t) \geq u^-(x,t)$ for all $x \in \mathbb{R}^N$ and $t \geq \tau$.

2.2 Main results for cooperative systems

Before to state our main results, we give the following definition and notation.

Definition 2.8 Let $n \in \mathbb{N}$ and $p, p_0 \in \mathbb{R}^n$. We say that the functions $W_p(x,t) = \big(W_{1;p}(x,t), \cdots, W_{m;p}(x,t)\big)$ converge to $W_{p_0}(x,t) = \big(W_{1;p_0}(x,t), \cdots, W_{m;p_0}(x,t)\big)$ as $p \to p_0$ in the sense of the topology $\mathcal{T}$ if, for any compact set $S \subset \mathbb{R}^{N+1}$, the functions W_p, $\partial_t W_p$, $\partial_{x_i} W_p$, $\partial_{x_i^2} W_p$, $i = 1, \cdots, N$, converge uniformly in S to W_{p_0}, $\partial_t W_{p_0}$, $\partial_{x_i} W_{p_0}$, $\partial_{x_i^2} W_{p_0}$, $i = 1, \cdots, N$, as $p \to p_0$.

Notation For any $l \in \mathbb{Z}^+$, $\nu_i \in \mathbb{R}^N$, $i = 1, \cdots, l$, $A \in \mathbb{R}$ and $a \in \mathbb{R}$, denote by $T^i_{A,a}$ and $\tilde{T}^i_{A,a}$ $(i = 1, \cdots, l+1)$ the regions:

$$T^i_{A,a} := \big\{x \in \mathbb{R}^N \big| x \cdot \nu_i \ge A\big\} \times [a, +\infty),\ \ i = 1, \cdots, l,\ \ T^{l+1}_{A,a} := \mathbb{R}^N \times [a, +\infty),$$
$$\tilde{T}^i_{A,a} := \big\{x \in \mathbb{R}^N \big| x \cdot \nu_i \le A\big\} \times (-\infty, a],\ \ i = 1, \cdots, l,\ \ \tilde{T}^{l+1}_{A,a} := \mathbb{R}^N \times (-\infty, a].$$

Now, we state the main results of entire solutions for the cooperative system as follows.

Theorem 2.9 **(Existence)** *Let $(A_0) - (A_3)$ hold. Then, for any $l \in \mathbb{Z}^+$, $\nu_1, \cdots, \nu_l \in \mathbb{R}^N$ with $\|\nu_i\| = 1$, $h_1, \cdots, h_{l+1} \in \mathbb{R}$, $c_1, \cdots, c_l > c_*$, and $\chi_1, \cdots, \chi_{l+1} \in \{0, 1\}$ with $\chi_1 + \cdots + \chi_{l+1} \ge 2$, there exists an entire solution $U_p(x,t) := \big(U_{1;p}(x,t), \cdots, U_{m;p}(x,t)\big)$ of (1.1) such that*

$$\underline{u}(x,t) \le U_p(x,t) \le \min\big\{\mathbf{K}, \Pi(x,t)\big\}, \quad \forall (x,t) \in \mathbb{R}^{N+1}, \tag{2.4}$$

where $p := p_{\chi_1,\cdots,\chi_{l+1}} = \big(\chi_1 c_1, \chi_1 h_1, \chi_1 \nu_1, \cdots, \chi_l c_l, \chi_l h_l, \chi_l \nu_l, \chi_{l+1} h_{l+1}\big)$ and

$$\underline{u}(x,t) := \max\Big\{\max_{i=1,\cdots,l} \chi_i \Phi_{c_i}\big(x \cdot \nu_i + c_i t + h_i\big), \chi_{l+1}\Gamma(t + h_{l+1})\Big\},$$

$$\Pi(x,t) := \sum_{i=1}^{l} \chi_i v(\lambda_1(c_i)) e^{\lambda_1(c_i)(x \cdot \nu_i + c_i t + h_i)} + \chi_{l+1} v^* e^{\lambda^*(t + h_{l+1})}.$$

Here, $\Gamma(t)$ is the spatially independent solution of (1.1) obtained in Lemma 2.12, $\lambda^* = M(0)$ and $v^* = v(0)$.

By considering a combination of any finite number of traveling wave fronts coming from directions ν_i with speeds $c_i > c_*$ and a spatial variable independent solution, we obtain some new types of entire solutions of (1.1). It is clear that $U_p(x,t)$ depends on these parameters $p_{\chi_1,\cdots,\chi_{l+1}}$. For example, if $\chi_1 = \cdots = \chi_{l+1} = 1$, then $U_p(x,t)$ depends on $p_{1,\cdots,1} = (c_1, h_1, \nu_1, \cdots, c_l, h_l, \nu_l, h_{l+1})$. Here, we can not prove that $U_p(x,t)$ depends *continuously* on $p_{\chi_1,\cdots,\chi_{l+1}}$.

However, some other qualitative properties, such as the monotonicity of $U_p(x,t)$ with respect to t and x_i (see (i) and (ii) in Theorem 2.10 below), the asymptotic behavior of $U_p(x,t)$ when $t \to \pm\infty$ (see (iii) $-$ (vi) in Theorem 2.10 below), and the monotonicity and the limit of $U_p(x,t)$ with respect to h_i (see (vii) and (viii) in Theorem 2.10 below), can be obtained. In fact, we have the following results.

Theorem 2.10 **(Qualitative properties)** *Let all the assumptions of Theorem 2.9 hold and $U_p(x,t)$ be the entire solution of (1.1) obtained in Theorem 2.9, then the following properties hold.*

(i) $\mathbf{0} \ll U_p(x,t) \ll \mathbf{K}$ *and* $\partial_t U_p(x,t) \gg \mathbf{0}$ *for all* $(x,t) \in \mathbb{R}^{N+1}$.

(ii) *If* $l = 1$ *and* $\chi_1 = \chi_2 = 1$*, then*

$$\operatorname{sgn}\big(\partial_{x_i} U_p(x,t)\big) = \operatorname{sgn}(\nu_{1i}),\ i = 1, \cdots, N,\ \textit{for}\ (x,t) \in \mathbb{R}^{N+1},$$

where $\nu_1 = (\nu_{11}, \cdots, \nu_{1N}) \in \mathbb{R}^N$ *with* $\|\nu_1\| = 1$*. In particular, for* $N = 2$*, if we denote* $\nu_1 = (\cos\theta_1, \sin\theta_1)$*,* $\theta_1 \in [0, 2\pi)$*, then*

$$\partial_{x_1} U_p(x,t) = \begin{cases} \geq 0, & \theta_1 \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]; \\ \leq 0, & \theta_1 \in [\frac{\pi}{2}, \frac{3\pi}{2}], \end{cases} \quad \textit{and}\ \partial_{x_2} U_p(x,t) = \begin{cases} \geq 0, & \theta_1 \in [0, \pi]; \\ \leq 0, & \theta_1 \in [\pi, 2\pi]. \end{cases}$$

for $(x,t) \in \mathbb{R}^3$*. Similar results hold true for* $N = l = 2$ *and* $\chi_2 = 0$*.*

(iii) $\lim_{t\to-\infty} \sup_{\|x\|\leq A} \|U_p(x,t)\| = 0$ *for any* $A \in \mathbb{N}$*.*

(iv) *If* $\chi_{l+1} = 1$*, then for every* $x \in \mathbb{R}^N$*,*

$$U_p(x,t) \sim \Gamma(t + h_{l+1}) \sim v^* e^{\lambda^*(t+h_{l+1})}\ \textit{as}\ t \to -\infty.$$

(v) *If* $\chi_{l+1} = 0$*, then for every* $x \in \mathbb{R}^N$*,*

$$U_p(x,t) = O\big(e^{\vartheta(c_1,\cdots,c_l)t}\big)\ \textit{as}\ t \to -\infty,$$

where $\vartheta(c_1, \cdots, c_l) = \min_{i=1,\cdots,l}\{c_i\lambda_1(c_i)\}$*.*

(vi) *If* $\chi_{l+1} = 0$*, then* $\lim_{t\to+\infty} \sup_{\|x\|\leq A} \|U_p(x,t) - \mathbf{K}\| = 0$ *for any* $A \in \mathbb{R}_+$*, and if* $\chi_{l+1} = 1$*, then* $\lim_{t\to+\infty} \sup_{x\in\mathbb{R}^N} \|U_p(x,t) - \mathbf{K}\| = 0$*.*

(vii) *For any* $(x,t) \in \mathbb{R}^{N+1}$*,* $U_p(x,t)$ *is increasing with respect to* h_i*,* $i = 1, \cdots, l+1$*.*

(viii) $U_p(x,t)$ *converges to* $\mathbf{K}$ *as* $h_i \to +\infty$ *in* $\mathcal{T}$ *and uniformly on* $(x,t) \in T^i_{A,a}$ *for any* $A, a \in \mathbb{R}$*,* $i = 1, \cdots, l+1$*.*

By the assumption (A_1), it is easy to verify that $c\lambda_1(c) > \lambda^*$ for any $c > c_*$ (see the proof of Theorem 2.10 (iv)). Thus, by the statements (iv) and (v) of Theorem 2.10, we see that the entire solutions $U_p(x,t)$ with $\chi_{l+1} = 1$ and $U_p(x,t)$ with $\chi_{l+1} = 0$ have different decay rates when $t \to -\infty$, and hence they are completely different.

For the sake of simplicity, we denote $\boldsymbol{\chi} = (\chi_1, \cdots, \chi_{l+1})$. According to the assumption $\chi_1, \cdots, \chi_{l+1} \in \{0, 1\}$ with $\chi_1 + \cdots + \chi_{l+1} \geq 2$ in Theorem 2.9, we further denote the entire solution $U_p(x,t)$ of (1.1) by

$$U_p(x,t) := \begin{cases} U_{p_0}(x,t), & \text{if } \boldsymbol{\chi} = \big(1, \cdots, 1\big); \\ U_{p_i}(x,t), & \text{if } \boldsymbol{\chi} = \big(1, \cdots, 1, 0_i, 1, \cdots, 1\big),\ i = 1, \cdots, l+1, \\ U_{p_{i,j}}(x,t), & \text{if } \chi_i = \chi_j = 0,\ 1 \leq i \neq j \leq l+1, \\ & \quad \text{and } \chi_k = 1, \forall k \in \{1, \cdots, l+1\} \setminus \{i, j\}, \end{cases} \tag{2.5}$$

where $p_0 = p_{1,\cdots,1}$, $p_i = p_{1,\cdots,1,0_i,1,\cdots,1}$, $i = 1, \cdots, l+1$ and $p_{i,j} = p_{\chi_1,\cdots,\chi_{l+1}}$ with $\chi_i = \chi_j = 0$, $1 \leq i \neq j \leq l+1$ and $\chi_k = 1$, $\forall k \in \{1, \cdots, l+1\} \setminus \{i, j\}$.

Moreover, we have the following convergence results.

Theorem 2.11 (Convergence properties) *Assume* $(A_0)-(A_3)$*. Assume further that* $f'(u) \leq f'(\mathbf{0})$ *for* $u \in [\mathbf{0}, \mathbf{K}]$*. Then, from (2.5), the following properties hold.*

(i) *For any* $A, a \in \mathbb{R}$*,* $U_{p_0}(x,t)$ *converges to* $U_{p_i}(x,t)$ *as* $h_i \to -\infty$ *in* $\mathcal{T}$*, and uniformly on* $(x,t) \in \tilde{T}^i_{A,a}$*,* $i = 1, \cdots, l+1$*.*

(ii) *For any* $A, a \in \mathbb{R}$*,* $U_{p_i}(x,t) (i = 1, \cdots, l+1)$ *converges to* $U_{p_{i,j}}(x,t)$ *as* $h_j \to -\infty$ *in* $\mathcal{T}$*, and uniformly on* $(x,t) \in \tilde{T}^j_{A,a}$*,* $1 \leq i \neq j \leq l+1$*.*

(iii) *For any* $h_1, \cdots, h_l, h_1^*, \cdots, h_l^* \in \mathbb{R}$, *there exists* $(x_0, t_0) \in \mathbb{R}^{N+1}$, *depending on* $c_1, \cdots, c_l, h_1, \cdots, h_l, h_1^*, \cdots, h_l^*$ *such that*

$$U_{p_{l+1}}(x,t) = U_{p_{l+1}^*}(x + x_0, t + t_0) \text{ for all } (x,t) \in \mathbb{R}^{N+1}.$$

Here, $p_{l+1}^* = \big(c_1, h_1^*, \nu_1, \cdots, c_l, h_l^*, \nu_l, 0\big)$.

2.3 Existence of Spatially Independent Solutions

In this subsection, we consider the spatially independent solutions of (1.1) connecting $\mathbf{0}$ and $\mathbf{K}$, that is, solutions of the following ordinary differential problem:

$$\frac{d\Gamma(t)}{dt} = f\big(\Gamma(t)\big), \ \ t \in \mathbb{R}, \tag{2.6}$$

$$\Gamma(-\infty) = \mathbf{0}, \ \ \Gamma(+\infty) = \mathbf{K}, \tag{2.7}$$

where $\Gamma = (\Gamma_1, \cdots, \Gamma_m)$ and $f = (f_1, \cdots, f_m)$. Recall that $W = [\mathbf{0}, \mathbf{K}]$.

Note that (2.6) is a cooperative and irreducible system. The existence of such a heteroclinic orbit $\Gamma(t)$ can be established by using the theory of monotone dynamical systems (see Smith [29] and Zhao [53]). However, these results do not give the exponential decay rate of the solution at minus infinity. To overcome the shortcoming, we shall use the standard technique of monotone iteration scheme to prove the existence and asymptotic behavior of the solutions of (2.6) and (2.7).

Lemma 2.12 *Let* $(A_0) - (A_3)$ *hold. There exists a solution* $\Gamma(t) : \mathbb{R} \to W$ *of* (2.6) *and* (2.7) *such that*

$$\Gamma'(t) \gg \mathbf{0}, \ \lim_{t\to-\infty} \Gamma(t)e^{-\lambda^* t} = v^*, \ \text{and } \Gamma(t) \le e^{\lambda^* t} v^* \text{ for all } t \in \mathbb{R},$$

where $\lambda^* = M(0)$ *and* $v^* = v(0)$.

Proof Since the method is standard, we only sketch the outline. Let $C(\mathbb{R}, \mathbb{R}^m)$ be the spaces of continuous vector-valued functions on $\mathbb{R}$. Define the operator $F = (F_1, \cdots, F_m) : C(\mathbb{R}, W) \to C(\mathbb{R}, \mathbb{R}^m)$ by

$$F_i(u)(t) = \int_{-\infty}^{t} e^{-L(t-s)} Q_i(u)(s)ds, \ \ i = 1, \cdots, m.$$

Recall that

$$L = \max_{i=1,\cdots,m} \max\left\{|\partial_i f_i(u)| \big| u \in W\right\} \text{ and } Q_i(u)(t) = f_i(u(t)) + Lu_i(t), \ \ i = 1, \cdots, m.$$

It is easy to verify that each $Q_i(\cdot)$ is a nondecreasing map from $C(\mathbb{R}, W)$ to $C(\mathbb{R}, \mathbb{R})$ with respect to the point-wise ordering. The remainder of the proof is divided into the following three steps.

Step 1 The following observation is straightforward.

(i) $F : C(\mathbb{R}, W) \to C(\mathbb{R}, W)$;
(ii) $F(\phi)(t) \ge F(\psi)(t)$ for $\phi, \psi \in C(\mathbb{R}, W)$ with $\phi(t) \ge \psi(t)$;
(iii) $F(\phi)(t)$ is increasing in $\mathbb{R}$ if $\phi \in C(\mathbb{R}, W)$ is increasing in $\mathbb{R}$.

Step 2 For any fixed $\varepsilon \in \big(1, 2\big)$ and sufficiently large $q > 1$, define two functions as follows:

$$\overline{\phi}(t) = \big(\overline{\phi}_1(t), \cdots, \overline{\phi}_m(t)\big) \text{ and } \underline{\phi}(t) = \big(\underline{\phi}_1(t), \cdots, \underline{\phi}_m(t)\big),$$

where

$$\overline{\phi}_i(t) = \min\left\{K_i, v_i^* e^{\lambda^* t}\right\} \text{ and } \underline{\phi}_i(t) = \max\left\{0, v_i^* e^{\lambda^* t} - q v_i^* e^{\varepsilon\lambda^* t}\right\}, \ t \in \mathbb{R}.$$

Then, by direct computations, we obtain

$$\mathbf{0} \le \underline{\phi}(t) \le \overline{\phi}(t) \le \mathbf{K}, F(\overline{\phi})(t) \le \overline{\phi}(t) \text{ and } F(\underline{\phi})(t) \ge \underline{\phi}(t) \text{ for all } t \in \mathbb{R}.$$

Step 3 Using the monotone iteration technique, we can show that equation (2.6) admits a solution $\Gamma(t)$ which satisfies

$$\Gamma'(t) \ge \mathbf{0} \text{ and } \underline{\phi}(t) \le \Gamma(t) \le \overline{\phi}(t) \text{ for all } t \in \mathbb{R}.$$

Thus, $\Gamma(-\infty) = \mathbf{0}$, $\Gamma(+\infty) \in (\mathbf{0}, \mathbf{K}]$ and $\lim_{t\to-\infty} \Gamma(t)e^{-\lambda^* t} = v^*$, $\mathbf{0} \ll \Gamma(t) \le e^{\lambda^* t} v^*$ for all $t \in \mathbb{R}$. Moreover, one can easily verify that $\Gamma(+\infty) = \mathbf{K}$ for all $t \in \mathbb{R}$.

Next, we show that $\Gamma'(t) \gg \mathbf{0}$ for all $t \in \mathbb{R}$. Since $\partial_j f_i(u) \ge 0$ for all $u \in [\mathbf{0}, \mathbf{K}]$ and $1 \le j \ne i \le m$, by (2.6), we have

$$\Gamma_i''(t) = \partial_1 f_i\big(\Gamma(t)\big)\Gamma_1'(t) + \cdots + \partial_m f_i\big(\Gamma(t)\big)\Gamma_m'(t) \ge \partial_i f_i\big(\Gamma(t)\big)\Gamma_i'(t) \ge m_0\Gamma_i'(t), \ \forall t \in \mathbb{R},$$

where $m_0 = \min_{i=1,\cdots,m} \min\left\{\partial_i f_i(u)\big| u \in W\right\}$. Thus, for any $\tau \in \mathbb{R}$, we obtain

$$\Gamma_i'(t) \ge \Gamma_i'(\tau)e^{m_0(t-\tau)}, \ \forall t > \tau, \ i = 1, \cdots, m. \tag{2.8}$$

Suppose for the contrary that there exist $i_0 \in \{1, \cdots, m\}$ and $t_0 \in \mathbb{R}$ such that $\Gamma_{i_0}'(t_0) = 0$. It then follows from (2.8) that $\Gamma_{i_0}'(\tau) = 0$ for all $\tau < t_0$. Thus, $\Gamma_{i_0}(\tau) = \Gamma_{i_0}(t_0)$ for all $\tau \le t_0$ and hence $0 < \Gamma_{i_0}(t_0) = \Gamma_{i_0}(-\infty) = 0$. This contradiction shows that $\Gamma'(t) \gg \mathbf{0}$ for all $t \in \mathbb{R}$. The proof is complete. □

2.4 Proofs of Theorems 2.9–2.11

In this subsection, we will use the results of previous subsections to obtain an appropriate upper estimate for solutions of (1.1) and then prove Theorems 2.9–2.11.

For any $l, n \in \mathbb{Z}^+$, $\nu_1, \cdots, \nu_l \in \mathbb{R}^N$ with $\|\nu_i\| = 1$, $h_1, \cdots, h_{l+1} \in \mathbb{R}$, $c_1, \cdots, c_l > c_*$, and $\chi_1, \cdots, \chi_{l+1} \in \{0, 1\}$ with $\chi_1 + \cdots + \chi_{l+1} \ge 2$, we denote

$$\varphi^n(x) := \max\left\{\max_{i=1,\cdots,l} \chi_i \Phi_{c_i}\big(x \cdot \nu_i - c_i n + h_i\big), \chi_{l+1}\Gamma(-n + h_{l+1})\right\},$$

$$\underline{u}(x, t) := \max\left\{\max_{i=1,\cdots,l} \chi_i \Phi_{c_i}\big(x \cdot \nu_i + c_i t + h_i\big), \chi_{l+1}\Gamma(t + h_{l+1})\right\}, \ t \ge -n.$$

Let $U^n(x, t) = \big(U_1^n(x, t), \cdots, U_m^n(x, t)\big)$ be the unique solution of the following initial value problem of (1.1)

$$\begin{cases} u_t = D\Delta u + f(u), \ x \in \mathbb{R}^N, t > -n, \\ u(x, -n) = \varphi^n(x), \ x \in \mathbb{R}^N. \end{cases}$$

Then, by Lemma 2.5, we have

$$\underline{u}(x, t) \le U^n(x, t) \le \mathbf{K} \text{ for all } x \in \mathbb{R}^N \text{ and } t \ge -n.$$

The following lemma provides the appropriate upper estimate of $U^n(x, t)$.

Lemma 2.13 *Assume* $(A_0)-(A_3)$. *The function* $U^n(x,t)$ *satisfies*

$$U^n(x,t) \le \min\left\{\mathbf{K}, \Pi(x,t)\right\} \text{ for all } x \in \mathbb{R}^N \text{ and } t \ge -n,$$

where $\Pi(x,t)$ *is defined in Theorem 2.9.*

Proof Let $v^+(x,t) = \min\left\{\mathbf{K}, \Pi(x,t)\right\}$. From Proposition 2.2 and Lemma 2.12, we have

$$\begin{aligned} v^+(x,-n) &= \min\left\{\mathbf{K}, \Pi(x,-n)\right\} \\ &= \min\left\{\mathbf{K}, \sum_{i=1}^{l} \chi_i v(\lambda_1(c_i)) e^{\lambda_1(c_i)(x\cdot\nu_i - c_i n + h_i)} + \chi_{l+1} v^* e^{\lambda^*(-n+h_{l+1})}\right\} \\ &\ge \varphi^n(x) = U^n(x,-n), \ \forall x \in \mathbb{R}^N. \end{aligned}$$

By Lemma 2.5(ii), it is sufficient to show that $v^+(x,t)$ is a supersolution of (1.1) on $[-n,+\infty)$, that is,

$$v^+(x,t) \ge T(t+n)v^+(x,-n) + \int_{-n}^{t} T(t-s)Q(v^+(x,s))ds, \ \forall x \in \mathbb{R}^N, t > -n. \quad (2.9)$$

Note that $Q(u) = f(u) + Lu$ is non-decreasing in u for $\mathbf{0} \le u \le \mathbf{K}$. For $x \in \mathbb{R}^N, t > -n$, we have

$$\begin{aligned} & T_i(t+n)v_i^+(x,-n) + \int_{-n}^{t} T_i(t-s)Q_i(v^+(x,s))ds \\ & \le T_i(t+n)K_i + \int_{-n}^{t} T_i(t-s)Q_i(\mathbf{K})ds \\ & \le e^{-L(t+n)}K_i + \int_{-n}^{t} e^{-L(t-s)}LK_i ds = K_i. \end{aligned}$$

Consequently,

$$T(t+n)v^+(x,-n) + \int_{-n}^{t} T(t-s)Q(v^+(x,s))ds \le \mathbf{K}, \ \forall x \in \mathbb{R}^N, t > -n. \quad (2.10)$$

Note also that $A(0)v^* = \lambda^* v^*$ and

$$A(\lambda_1(c_i))v(\lambda_1(c_i)) = M(\lambda_1(c_i))v(\lambda_1(c_i)) = c_i\lambda_1(c_i)v(\lambda_1(c_i)), \ i = 1, \cdots, l.$$

It is easy to see that the function $\Pi(x,t)$ satisfies the linear equation:

$$\Pi_t = D\Delta\Pi + f'(\mathbf{0})\Pi(x,t).$$

Then, for any $x \in \mathbb{R}^N, t > -n$, $\Pi(x,t)$ satisfies the integral equation:

$$\Pi(x,t) = T(t+n)\Pi(x,-n) + \int_{-n}^{t} T(t-s)\left[f'(\mathbf{0})\Pi(x,s) + L\Pi(x,s)\right]ds.$$

By the assumption (A_3), we obtain

$$Q(v^+(x,t)) = f(v^+(x,t)) + Lv^+(x,t) \\ \le f'(\mathbf{0})\Pi(x,t) + Lv^+(x,t) \le f'(\mathbf{0})\Pi(x,t) + L\Pi(x,t),$$

and hence

$$T(t+n)v^+(x,-n) + \int_{-n}^{t} T(t-s)Q(v^+(x,s))ds \\ \le T(t+n)\Pi(x,-n) + \int_{-n}^{t} T(t-s)\big[f'(\mathbf{0})\Pi(x,s) + L\Pi(x,s)\big]ds \\ = \Pi(x,t). \tag{2.11}$$

Combining (2.10) and (2.11), (2.9) holds and the assertion follows from Lemma 2.5. This completes the proof. □

Remark 2.14 We note that if $f(u) \le f'(\mathbf{0})u$ for $u \in [\mathbf{0}, \mathbf{K}]$, then Lemma 2.13 is a direct consequence of Lemma 2.7. In fact, by $f(u) \le f'(\mathbf{0})u$ for $u \in [\mathbf{0}, \mathbf{K}]$, we have

$$U_t^n \le D\Delta U^n + f'(\mathbf{0})U^n, \ \forall x \in \mathbb{R}^N, t > -n.$$

Noting that $U^n(x,-n) = \varphi^n(x) \le \Pi(x,-n)$ for all $x \in \mathbb{R}^N$ and

$$\Pi_t = D\Delta\Pi + f'(\mathbf{0})\Pi(x,t), \ \forall x \in \mathbb{R}^N, t > -n.$$

It follows from Lemma 2.7 that $U^n(x,t) \le \Pi(x,t)$ and hence $U^n(x,t) \le \min\{\mathbf{K}, \Pi(x,t)\}$ for all $x \in \mathbb{R}^N$ and $t \ge -n$.

Now we give the proofs of Theorems 2.9–2.11.

Proof of Theorem 2.9 By Lemmas 2.5 and 2.13, we have

$$\underline{u}(x,t) \le U^n(x,t) \le U^{n+1}(x,t) \le \min\{\mathbf{K}, \Pi(x,t)\}$$

for all $x \in \mathbb{R}^N$ and $t \ge -n$. By the priori estimate of Lemma 2.6 and the diagonal extraction process, there exists a subsequence $\{U^{n_k}(x,t)\}_{k\in\mathbb{N}}$ of $\{U^n(x,t)\}_{n\in\mathbb{N}}$ such that $U^{n_k}(x,t)$ converges to a function $U_p(x,t) = \big(U_{1;p}(x,t), \cdots, U_{m;p}(x,t)\big)$ in the sense of topology $\mathcal{T}$. Since $U^n(x,t) \le U^{n+1}(x,t)$ for any $t > -n$, we have

$$\lim_{n\to+\infty} U^n(x,t) = U_p(x,t) \text{for any} (x,t) \in \mathbb{R}^{N+1}.$$

The limit function is unique, whence all of the functions $U^n(x,t)$ converge to the function $U_p(x,t)$ in the sense of topology $\mathcal{T}$ as $n \to +\infty$. Clearly, $U_p(x,t)$ is an entire solution of (1.1) satisfying (2.4). This completes the proof of Theorem 2.9. □

Proof of Theorem 2.10 The assertions for parts (ii)–(iii) and (vi)–(viii) are direct consequences of (2.4). Therefore, we only prove the results of parts (i), (iv) and (v).

(i) Clearly, $U_p(x,t) \gg 0$ for all $(x,t) \in \mathbb{R}^{N+1}$. Since

$$U^n(x,t) \ge \underline{u}(x,t) \ge \underline{u}(x,-n) = \varphi^n(x) = U^n(x,-n)$$

for all $(x,t) \in \mathbb{R}^N \times [-n,+\infty)$, by Lemma 2.5, we have $\frac{\partial}{\partial t}U^n(x,t) \geq 0$ for $(x,t) \in \mathbb{R}^N \times (-n,+\infty)$. This yields $\frac{\partial}{\partial t}U_p(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^{N+1}$. Noting that

$$\begin{aligned}\frac{\partial^2 U_{i;p}}{\partial t^2} &= d_i\Delta(U_{i;p})_t + \partial_1 f_i\big(U_p\big)(U_{1;p})_t + \cdots + \partial_m f_i\big(U_p\big)(U_{m;p})_t \\ &\geq d_i\Delta(U_{i;p})_t + \partial_i f_i\big(U_p\big)(U_{i;p})_t \\ &\geq d_i\Delta(U_{i;p})_t + m_0(U_{i;p})_t,\ \ i = 1,\cdots,m,\end{aligned}$$

where $m_0 = \min_{i=1,\cdots,m,u\in W} \partial_i f_i(u)$, we obtain for any $\tau \in \mathbb{R}$,

$$(U_{i;p})_t(x,t) \geq e^{m_0(t-\tau)} \int\limits_{\mathbb{R}^N} \Psi_i(x-y,t-\tau)(U_{i;p})_t(y,\tau)dy \geq 0,\ \ \forall x \in \mathbb{R}^N, t > \tau. \quad (2.12)$$

Assume, by contradiction, that there exist $i_0 \in \{1,\cdots,m\}$ and $(x_0,t_0) \in \mathbb{R}^{N+1}$ such that $(U_{i_0;p})_t(x_0,t_0) = 0$, it then follows from (2.12) that $(U_{i_0;p})_t(x_0,\tau) = 0$ for all $\tau \leq t_0$. Hence $U_{i_0;p}(x_0,t) = U_{i_0;p}(x_0,t_0)$ for all $t \leq t_0$, which implies that $\lim_{t\to-\infty} U_{i_0;p}(x_0,t) = U_{i_0;p}(x_0,t_0)$. But following from (2.4),

$$U_{i_0;p}(x_0,t_0) > 0 \text{ and } \lim_{t\to-\infty} U_{i_0;p}(x_0,t) = 0.$$

This contradiction yields that $\frac{\partial}{\partial t}U_p(x,t) \gg 0$ for all $(x,t) \in \mathbb{R}^{N+1}$. Following from $\frac{\partial}{\partial t}U_p(x,t) \gg 0$ for $(x,t) \in \mathbb{R}^{N+1}$, we have $U_p(x,t) \ll \mathbf{K}$ for $(x,t) \in \mathbb{R}^{N+1}$.

(iv) When $\chi_{l+1} = 1$, by (2.4), we have

$$\begin{aligned}&\max\left\{\max_{i=1,\cdots,l} \chi_i\Phi_{c_i}\big(x\cdot\nu_i + c_i t + h_i\big), \Gamma(t+h_{l+1})\right\} \\ &\leq U_p(x,t) \leq \sum_{i=1}^{l} \chi_i v(\lambda_1(c_i))e^{\lambda_1(c_i)(x\cdot\nu_i+c_i t+h_i)} + v^* e^{\lambda^*(t+h_{l+1})}.\end{aligned}$$

We claim that $c\lambda_1(c) > \lambda^*$ for any $c > c_*$. In fact, if $(A_1)(a)$ holds, then $M(\lambda) > M(0) = \lambda^*$ (see, e.g., [29, Corollary 4.3.2]), since $A(\lambda) > A(0)$ for any $\lambda > 0$. In view of $M(\lambda_1(c)) = c\lambda_1(c)$ and $\lambda_1(c) > 0$ for any $c > c_*$, we obtain $c\lambda_1(c) > \lambda^*$ for any $c > c_*$. If $(A_1)(b)$ holds, it is easily seen that $c\lambda_1(c) > \lambda^*$ for any $c > c_*$([17, Theorem 8.1.18]). Therefore, the assertion follows from the fact

$$\lim_{t\to-\infty} \Gamma(t)e^{-\lambda^* t} = v^* \text{ and } \lim_{\xi\to-\infty} \Phi_{c_i}(\xi)e^{-\lambda_1(c_i)\xi} = v(\lambda_1(c_i)),\ \ i = 1,\cdots,l.$$

The proof of part (v) is similar to that of part (iv) and omitted. This completes the proof of Theorem 2.10. □

Proof of Theorem 2.11 (i) We only prove the case that $U_{p_0}(t)$ converges to $U_{p_1}(t)$ in the sense of topology $\mathcal{T}$ as $h_1 \to -\infty$, and uniformly on $(x,t) \in \widetilde{T}^1_{A,a}$. The proofs for the other cases are similar.

For $(\chi_1,\cdots,\chi_{l+1}) = (1,\cdots,1)$, we denote $\varphi^n(x)$ by $\varphi^n_{p_0}(x)$ and $U^n(x,t)$ by $U^n_{p_0}(x,t)$, respectively. Similarly, when $(\chi_1,\cdots,\chi_{l+1}) = (0,1,\cdots,1)$, we denote $\varphi^n(x)$ by $\varphi^n_{p_1}(x)$ and $U^n(x,t)$ by $U^n_{p_1}(x,t)$, respectively. Let

$$W^n(x,t) = U^n_{p_0}(x,t) - U^n_{p_1}(x,t),\ \ (x,t) \in \mathbb{R}^N \times (-n,+\infty),$$

then $\mathbf{0} \leq W^n(x,t) \leq \mathbf{K}$ for all $(x,t) \in \mathbb{R}^N \times (-n, +\infty)$. Since $f'(u) \leq f'(\mathbf{0})$ for $u \in [\mathbf{0}, \mathbf{K}]$, we get

$$\begin{aligned} \frac{\partial W^n}{\partial t} &= D\Delta W^n + f(U^n_{p_0}(x,t)) - f(U^n_{p_1}(x,t)) \\ &= D\Delta W^n + f'\big(U^n_{p_0}(x,t) + (1-\theta_3)W^n(x,t)\big)W^n(x,t) \\ &\leq D\Delta W^n + f'(\mathbf{0})W^n(x,t), \ \forall x \in \mathbb{R}^N, t > -n, \end{aligned}$$

where $\theta_3 \in (0,1)$. Define the function

$$\widehat{W}(x,t) = v(\lambda_1(c_1))e^{\lambda_1(c_1)(x\cdot\nu_1+c_1t+h_1)}, \ (x,t) \in \mathbb{R}^{N+1}.$$

Since $A(\lambda_1(c_1))v(\lambda_1(c_1)) = M(\lambda_1(c_1))v(\lambda_1(c_1)) = c_1\lambda_1(c_1)v(\lambda_1(c_1))$, direct computations show that

$$\frac{\partial \widehat{W}}{\partial t} = D\Delta\widehat{W} + f'(\mathbf{0})\widehat{W}(x,t), \ \forall x \in \mathbb{R}^N, t \in \mathbb{R}.$$

Moreover, by Proposition 2.2, we have

$$\begin{aligned} W^n(x,-n) &= U^n_{p_0}(x,-n) - U^n_{p_1}(x,-n) \\ &\leq \Phi_{c_1}\big(x\cdot\nu_1 - c_1n + h_1\big) \\ &\leq v(\lambda_1(c_1))e^{\lambda_1(c_1)(x\cdot\nu_1-c_1n+h_1)} = \widehat{W}(x,-n). \end{aligned}$$

It then follows from Lemma 2.7 that

$$0 \leq W^n(x,t) = U^n_{p_0}(x,t) - U^n_{p_1}(x,t) \leq \widehat{W}(x,t) = v(\lambda_1(c_1))e^{\lambda_1(c_1)(x\cdot\nu_1+c_1t+h_1)}$$

for all $(x,t) \in \mathbb{R}^N \times [-n, +\infty)$. Since $\lim_{n\to+\infty} U^n_{p_i}(x,t) = U_{p_i}(x,t)$, $i = 0, 1$, we get

$$\mathbf{0} \leq U_{p_0}(x,t) - U_{p_1}(x,t) \leq v(\lambda_1(c_1))e^{\lambda_1(c_1)(x\cdot\nu_1+c_1t+h_1)} \text{ for all } (x,t) \in \mathbb{R}^{N+1},$$

which implies that $U_{p_0}(x,t)$ converges to $U_{p_1}(x,t)$ as $h_1 \to -\infty$ uniformly on $(x,t) \in \widetilde{T}^1_{A,a}$ for any $A, a \in \mathbb{R}$. For any sequence h^ℓ_1 with $h^\ell_1 \to -\infty$ as $\ell \to +\infty$, the functions $U_{p^\ell_0}(x,t)$, $p^\ell_0 := (c_1, h^\ell_1, \nu_1, \cdots, c_l, h_l, \nu_l, h_{l+1})$, converge to a solution of (1.1) (up to extraction of some subsequence) in the sense of topology $\mathcal{T}$, which turns out to be $U_{p_1}(x,t)$. The limit does not depend on the sequence h^ℓ_1, whence all of the functions $U_{p_0}(x,t)$ converge to $U_{p_1}(x,t)$ in the sense of topology $\mathcal{T}$ as $h_1 \to -\infty$, and the assertion of this part follows.

The proof of part (ii) is similar to that of part (i), and omitted. Moreover, the proof of part (iii) is straightforward. This completes the proof of Theorem 2.11. □

3 Entire Solutions for Non-Cooperative Systems

In this section, we consider the entire solutions of (1.1) with monostable and non-cooperative nonlinearity. First, we introduce two auxiliary cooperative reaction-diffusion systems and establish a comparison theorem for the Cauchy problems of the three systems. Then, we prove the existence and qualitative properties of entire solutions using the comparison theorem.

Throughout this section, in addition to (A_0) and (A_1), we also make the following assumptions:

$(A_2)'$ There exist $\mathbf{K}^{\pm} = (K_1^{\pm}, \cdots, K_m^{\pm}) \gg 0$ with $0 \ll \mathbf{K}^- \leq \mathbf{K} \leq \mathbf{K}^+$ and two continuous and twice piecewise continuous differentiable functions $f^+, f^- : [\mathbf{0}, \mathbf{K}^+] \to \mathbb{R}^m$ such that $f \in C^2\big([\mathbf{0}, \mathbf{K}^+], \mathbb{R}^m\big)$, $f^{\pm}(\mathbf{0}) = f^+(\mathbf{K}^+) = f^-(\mathbf{K}^-) = \mathbf{0}$, and

$$f^-(u) \leq f(u) \leq f^+(u) \text{ for all } u \in [\mathbf{0}, \mathbf{K}^+].$$

$(A_3)'$ There is no other positive equilibrium of $f^{\pm}$ between $\mathbf{0}$ and $\mathbf{K}^{\pm}$, and $f(u)$ and $f^{\pm}(u)$ have the same Jacobian matrix $f'(\mathbf{0})$ at $u = \mathbf{0}$.
$(A_4)'$ $\partial_j f_i^{\pm}(u) \geq 0$ for all $u \in [\mathbf{0}, \mathbf{K}^+]$, $1 \leq j \neq i \leq m$.
$(A_5)'$ For any $k \in \mathbb{Z}^+$, $\rho_1, \cdots, \rho_k > 0$ and $\lambda_1, \cdots, \lambda_k \in [0, \lambda_*]$,

$$f^+\Big(\min\big\{\mathbf{K}^+, \rho_1 v(\lambda_1) + \cdots + \rho_k v(\lambda_k)\big\}\Big) \leq f'(\mathbf{0})\big[\rho_1 v(\lambda_1) + \cdots + \rho_k v(\lambda_k)\big].$$

Remark 3.1 Clearly, if $f^+(u) \leq f'(\mathbf{0})u$ for $u \in [\mathbf{0}, \mathbf{K}^+]$, then $(A_5)'$ holds. We remark that when (1.1) is cooperative, then $f^{\pm} = f$ and $\mathbf{K}^{\pm} = \mathbf{K}$. We also note that if f is defined on $[0, +\infty)^m$, then $(A_5)'$ can be replaced by $(A_5)^*$:

$(A_5)^*$ For any $k \in \mathbb{Z}^+$, $\rho_1, \cdots, \rho_k > 0$ and $\lambda_1, \cdots, \lambda_k \in [0, \lambda_*]$,

$$f^+\big(\rho_1 v(\lambda_1) + \cdots + \rho_k v(\lambda_k)\big) \leq f'(\mathbf{0})\big[\rho_1 v(\lambda_1) + \cdots + \rho_k v(\lambda_k)\big].$$

Denote $W^+ = [\mathbf{0}, \mathbf{K}^+]$. It is easy to verify that for any $\varphi \in [\mathbf{0}, \mathbf{K}^+]_X$, system (1.1) admits a unique solution $u(x, t; \varphi)$ satisfying $u(\cdot, \tau; \varphi) = \varphi(\cdot)$ and $\mathbf{0} \leq u(x, t; \varphi) \leq \mathbf{K}^+$ for all $x \in \mathbb{R}^N$ and $t \geq \tau$.

Now, we consider the following two auxiliary cooperative reaction-diffusion systems

$$u_t = D\Delta u + f^+(u), \quad x \in \mathbb{R}^N, t \in \mathbb{R}, \tag{3.1}$$
$$u_t = D\Delta u + f^-(u), \quad x \in \mathbb{R}^N, t \in \mathbb{R}. \tag{3.2}$$

Take $\widetilde{L} = \max_{u \in W^+, i=1,\cdots,m} \max\big\{|\partial_i f_i^+(u)|, |\partial_i f_i^-(u)|\big\}$ and define

$$\widetilde{Q}(u) = (\widetilde{Q}_1(u), \cdots, \widetilde{Q}_m(u)) = f(u) + \widetilde{L}u, \ u \in W^+$$
$$\widetilde{Q}^{\pm}(u) = (\widetilde{Q}_1^{\pm}(u), \cdots, \widetilde{Q}_m^{\pm}(u)) = f^{\pm}(u) + \widetilde{L}u, \ u \in W^+.$$

Clearly, $\widetilde{Q}^{\pm}(u)$ is non-decreasing in u for $u \in W^+$ and

$$\widetilde{Q}^-(u) \leq \widetilde{Q}(u) \leq \widetilde{Q}^+(u) \text{ for any } u \in W^+.$$

We further define the operator $\widetilde{T}(t) = \operatorname{diag}\big(\widetilde{T}_1(t), \cdots, \widetilde{T}_m(t)\big)$ as (2.2) by replacing L with $\widetilde{L}$.

The following comparison theorem plays an important role in the proof of our main result for the non-cooperative system.

Lemma 3.2 *Let $u, u^{\pm} \in C(\mathbb{R}^N \times [\tau, +\infty), W^+)$ be such that*

$$u^-(x,t) \leq \widetilde{T}(t-\tau)u^-(x,\tau) + \int_\tau^t \widetilde{T}(t-s)\widetilde{Q}^-(u^-(x,s))ds, \quad \forall x \in \mathbb{R}^N, t > \tau, \tag{3.3}$$

$$u(x,t) = \widetilde{T}(t-\tau)u(x,\tau) + \int_\tau^t \widetilde{T}(t-s)\widetilde{Q}(u(x,s))ds, \quad \forall x \in \mathbb{R}^N, t > \tau, \tag{3.4}$$

$$u^+(x,t) \geq \widetilde{T}(t-\tau)u^+(x,\tau) + \int_\tau^t \widetilde{T}(t-s)\widetilde{Q}^+(u^+(x,s))ds, \quad \forall x \in \mathbb{R}^N, t > \tau, \tag{3.5}$$

and $u^-(x,\tau) \le u(x,\tau) \le u^+(x,\tau)$. *Then, there holds*

$$u^-(x,t) \le u(x,t) \le u^+(x,t) \text{ for all } x \in \mathbb{R}^N \text{ and } t > \tau.$$

Proof We first prove $u(x,t) \le u^+(x,t)$ for all $x \in \mathbb{R}^N$ and $t > \tau$. Let $w(x,t) = u(x,t) - u^+(x,t)$ and define

$$L_i = \max_{u \in W^+, j=1,\cdots,m} \frac{\partial \widetilde{Q}_i^+(u)}{\partial u_j}, \ i = 1, \cdots, m, \text{ and } [r]_+ = \max\{r, 0\} \text{ for any } r \in \mathbb{R}.$$

Since $w(\cdot,\tau) \le 0$ and $\widetilde{Q}^+(u)$ is non-decreasing in u for $u \in W^+$, by (3.4) and (3.5), we obtain

$$\begin{aligned}
w_i(x,t) &\le \widetilde{T}_i(t-\tau)w_i(x,\tau) + \int_\tau^t \widetilde{T}_i(t-s)\big[\widetilde{Q}_i(u(x,s)) - \widetilde{Q}_i^+(u^+(x,s))\big]ds \\
&\le \int_\tau^t \widetilde{T}_i(t-s)\big[\widetilde{Q}_i^+(u(x,s)) - \widetilde{Q}_i^+(u^+(x,s))\big]ds \\
&= \int_\tau^t \widetilde{T}_i(t-s)\left(\sum_{j=1}^m w_j(x,s)\int_0^1 \frac{\partial}{\partial u_j}\widetilde{Q}_i^+(u^+(x,s)+\theta w(x,s))d\theta\right)ds \\
&\le \int_\tau^t \widetilde{T}_i(t-s)\left(L_i\sum_{j=1}^m [w_j(x,s)]_+\right)ds, \ \forall x \in \mathbb{R}^N, t > \tau.
\end{aligned}$$

Consequently,

$$[w_i(x,t)]_+ \le \int_\tau^t \widetilde{T}_i(t-s)\left(L_i\sum_{j=1}^m [w_j(x,s)]_+\right)ds, \ \forall x \in \mathbb{R}^N, t > \tau. \tag{3.6}$$

Let $\varpi(x,t) = \sum_{i=1}^m [w_i(x,t)]_+$. It follows from (3.6) that

$$\begin{aligned}
\varpi(x,t) &\le \sum_{i=1}^m \int_\tau^t \widetilde{T}_i(t-s)L_i\varpi(x,s)ds \\
&\le \int_\tau^t \sum_{i=1}^m \int_{\mathbb{R}^N} L_i\Psi_i(x-y,t-s)\varpi(y,s)ds \\
&= \int_\tau^t \int_{\mathbb{R}^N} P(x-y,t-s)\varpi(y,s)ds,
\end{aligned}$$

where $P(y,s) = \sum_{i=1}^m L_i\Psi_i(y,s)$. Using the same argument as in [30, Lemma 3.2], we obtain $\varpi(x,t) = 0$, and hence $u(x,t) \le u^+(x,t)$ for all $x \in \mathbb{R}^N$ and $t > \tau$. Similarly, we can prove that $u^-(x,t) \le u(x,t)$ for all $x \in \mathbb{R}^N$ and $t > \tau$. This completes the proof. □

The following result is a direct consequence of Lemma 3.2, see also Fife [9].

Corollary 3.3 *Let $u, u^{\pm} \in C(\mathbb{R}^N \times [\tau, +\infty), W^+)$ be such that $u_i, u_i^{\pm}$ is C^1 in t and C^2 in x. If*

$$u_t^- \le D\Delta u^- + f^-(u^-), \quad \forall x \in \mathbb{R}^N, t > \tau,$$
$$u_t = D\Delta u + f(u), \quad \forall x \in \mathbb{R}^N, t > \tau,$$
$$u_t^+ \ge D\Delta u^+ + f^+(u^+), \quad \forall x \in \mathbb{R}^N, t > \tau,$$

and $u^-(x,\tau) \le u(x,\tau) \le u^+(x,\tau)$, then $u^-(x,t) \le u(x,t) \le u^+(x,t)$ for all $x \in \mathbb{R}^N, t > \tau$.

From the argument of Wang [36, Theorem 2.1], we have the following result.

Proposition 3.4 *Let $(A_0)-(A_1)$ and $(A_2)'-(A_5)'$ hold. For any $c > c_*$ and $\nu \in \mathbb{R}^N$ with $\|\nu\| = 1$, (3.2) has a non-decreasing traveling wave solution*

$$\Phi_c^-(x\cdot\nu + ct) = \big(\phi_{1,c}^-(x\cdot\nu + ct), \cdots, \phi_{m,c}^-(x\cdot\nu + ct)\big),$$

which satisfies $\Phi_c^-(\cdot) \gg \mathbf{0}$, $\Phi_c^-(-\infty) = \mathbf{0}$, $\Phi_c^-(+\infty) = \mathbf{K}^-$ and

$$\lim_{\xi\to-\infty} \Phi_c^-(\xi)e^{-\lambda_1(c)\xi} = v(\lambda_1(c)),\ \ \Phi_c^-(\xi) \le v(\lambda_1(c))e^{\lambda_1(c)\xi} \text{ for all } \xi \in \mathbb{R}. \tag{3.7}$$

Here, c_*, $\lambda_1(c)$ and $v(\lambda_1(c))$ are given as in Sect. 1.

We also consider the following ordinary differential system

$$u'(t) = f^-(u), \quad t \in \mathbb{R}. \tag{3.8}$$

By Lemma 2.12, the following result holds.

Lemma 3.5 *Let $(A_0)-(A_1)$ and $(A_2)'-(A_5)'$ hold. There exists a solution $\Gamma^-(t) : \mathbb{R} \to W^+$ of (3.8) which satisfies $\Gamma^-(-\infty) = \mathbf{0}$ and $\Gamma^-(+\infty) = \mathbf{K}^-$. Furthermore,*

$$(\Gamma^-)'(t) \gg \mathbf{0},\ \ \lim_{t\to-\infty} \Gamma^-(t)e^{-\lambda^* t} = v^* \text{ and } \Gamma^-(t) \le e^{\lambda^* t}v^* \text{ for all } t \in \mathbb{R},$$

where $\lambda^* = M(0)$ and $v^* = v(0)$.

The following theorem contains the main results of this section.

Theorem 3.6 *Let $(A_0)-(A_1)$ and $(A_2)'-(A_5)'$ hold. For any $l \in \mathbb{Z}^+$, $\nu_1, \cdots, \nu_l \in \mathbb{R}^N$ with $\|\nu_i\| = 1$, $h_1, \cdots, h_{l+1} \in \mathbb{R}$, $c_1, \cdots, c_l > c_*$, and $\chi_1, \cdots, \chi_{l+1} \in \{0, 1\}$ with $\chi_1 + \cdots + \chi_{l+1} \ge 1$, there exists an entire solution $U(x,t) := \big(U_1(x,t), \cdots, U_m(x,t)\big)$ of (1.1) such that*

$$u^-(x,t) \le U(x,t) \le \min\big\{\mathbf{K}^+, \Pi(x,t)\big\} \tag{3.9}$$

for all $(x,t) \in \mathbb{R}^{N+1}$, where

$$u^-(x,t) = \max\Big\{\max_{i=1,\cdots,l} \chi_i\Phi_{c_i}^-\big(x\cdot\nu_i + c_i t + h_i\big), \chi_{l+1}\Gamma^-(t + h_{l+1})\Big\},$$

$$\Pi(x,t) = \sum_{i=1}^{l} \chi_i v(\lambda_1(c_i))e^{\lambda_1(c_i)(x\cdot\nu_i + c_i t + h_i)} + \chi_{l+1}v^* e^{\lambda^*(t+h_{l+1})}.$$

Furthermore, the following statements hold:

(i) $U(x,t) \gg 0$ *for* $(x,t) \in \mathbb{R}^{N+1}$ *and* $\lim_{t\to-\infty}\sup_{\|x\|\le A}\|U(x,t)\| = 0$ *for any* $A \in \mathbb{R}_+$.

(ii) *If* $\chi_{l+1} = 1$*, then* $\liminf_{t\to+\infty} \inf_{x\in\mathbb{R}} U(x,t) \geq K^-$ *and for every* $x \in \mathbb{R}^N$,

$$U(x,t) \sim v^* e^{\lambda^*(t+h_{l+1})} \text{ as } t \to -\infty.$$

(iii) *If* $\chi_{l+1} = 0$*, then* $\liminf_{t\to+\infty} \inf_{\|x\|\leq A} U(x,t) \geq K^-$ *for any* $A \in \mathbb{R}_+$ *and for every* $x \in \mathbb{R}^N$,

$$U(x,t) = O\big(e^{\vartheta(c_1,\cdots,c_l)t}\big) \text{ as } t \to -\infty,$$

where $\vartheta(c_1,\cdots,c_l) = \min_{i=1,\cdots,l}\big\{c_i\lambda_1(c_i)\big\}$.

Proof Let $W^n(x,t) = \big(W_1^n(x,t),\cdots,W_m^n(x,t)\big)$ be the unique solution of the following initial value problem

$$\begin{cases} u_t = D\Delta u + f(u), \ \ x \in \mathbb{R}^N, t > -n, \\ u(x,-n) = \widetilde{\varphi}^n(x), \ \ x \in \mathbb{R}^N, \end{cases}$$

where

$$\widetilde{\varphi}^n(x) := \max\left\{\max_{i=1,\cdots,l} \chi_i\Phi_{c_i}^-\big(x\cdot\nu_i - c_i n + h_i\big), \chi_{l+1}\Gamma^-(-n+h_{l+1})\right\}.$$

We first show the following claim.

Claim The function $W^n(x,t)$ satisfies

$$u^-(x,t) \leq W^n(x,t) \leq u^+(x,t) := \min\big\{\mathbf{K}^+, \Pi(x,t)\big\} \text{ for all } x \in \mathbb{R}^N, t > -n. \quad (3.10)$$

In fact, from Proposition 3.4 and Lemma 3.5, we see that

$$u^-(x,-n) = \widetilde{\varphi}^n(x) = W^n(x,-n) \leq \min\big\{\mathbf{K}^+, \Pi(x,-n)\big\} = u^+(x,-n), \ \forall x \in \mathbb{R}.$$

By Lemma 3.2, it suffices to show that for any $x \in \mathbb{R}^N, t > -n$,

$$u^-(x,t) \leq \widetilde{T}(t+n)u^-(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)\widetilde{Q}^-(u^-(x,s))ds, \quad (3.11)$$

$$u^+(x,t) \geq \widetilde{T}(t+n)u^+(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)\widetilde{Q}^+(u^+(x,s))ds. \quad (3.12)$$

Now we prove (3.11). Note that the function $\widetilde{u}(x,t) := \chi_j\Phi_{c_j}^-\big(x\cdot\nu_j + c_j t + h_j\big)$ $(j = 1,\cdots,l)$, satisfies the equation

$$\widetilde{u}_t = D\Delta\widetilde{u} + f^-(\widetilde{u}),$$

or the integral equation

$$\widetilde{u}(x,t) = \widetilde{T}(t+n)\widetilde{u}(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)\widetilde{Q}^-(\widetilde{u}(x,s))ds.$$

Since $u^-(x,t) \geq \widetilde{u}(x,t)$ for $x \in \mathbb{R}^N$, $t \geq -n$, and $\widetilde{Q}^-(u) = f^-(u) + \widetilde{L}u$ is non-decreasing in u for $u \in W^+$, we have

$$\begin{aligned}&\widetilde{T}(t+n)u^-(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)\widetilde{Q}^-(u^-(x,s))ds\\ &\geq \widetilde{T}(t+n)\widetilde{u}(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)\widetilde{Q}^-(\widetilde{u}(x,s))ds\\ &= \widetilde{u}(x,t),\ \forall x \in \mathbb{R}^N, t > -n,\end{aligned}$$

that is,

$$\widetilde{T}(t+n)u^-(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)\widetilde{Q}^-(u^-(x,s))ds \geq \chi_j \Phi^-_{c_j}\big(x \cdot \nu_i + c_j t + h_j\big). \quad (3.13)$$

Similarly, we can show that for $x \in \mathbb{R}^N$, $t > -n$,

$$\widetilde{T}(t+n)u^-(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)\widetilde{Q}^-(u^-(x,s))ds \geq \chi_{l+1}\Gamma^-(t + h_{l+1}). \quad (3.14)$$

Hence, (3.11) follows from (3.13) and (3.14).

Next, we prove (3.12). Since $\widetilde{Q}^+(u) = f^+(u) + \widetilde{L}u$ is non-decreasing in u for $u \in W^+$, we get for $x \in \mathbb{R}^N$, $t > -n$,

$$\begin{aligned}&\widetilde{T}_i(t+n)u_i^+(x,-n) + \int_{-n}^{t} \widetilde{T}_i(t-s)\widetilde{Q}_i^+(u^+(x,s))ds\\ &\leq e^{-\widetilde{L}(t+n)}K_i^+ + \int_{-n}^{t} e^{-\widetilde{L}(t-s)}K_i^+\widetilde{L}ds = K_i^+,\ i = 1, \cdots, m.\end{aligned}$$

Consequently,

$$\widetilde{T}(t+n)u^+(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)\widetilde{Q}^+(u^+(x,s))ds \leq \mathbf{K}^+,\ \forall x \in \mathbb{R}^N, t > -n. \quad (3.15)$$

Note that $\Pi(x,t)$ satisfies the integral equation:

$$\Pi(x,t) = \widetilde{T}(t+n)\Pi(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)\big[f'(\mathbf{0})\Pi(x,s) + \widetilde{L}\Pi(x,s)\big]ds. \quad (3.16)$$

By the assumption $(A_5)'$, we obtain

$$\widetilde{Q}^+(u^+(x,t)) = f^+(u^+(x,t)) + \widetilde{L}u^+(x,t) \leq f'(\mathbf{0})\Pi(x,t) + \widetilde{L}\Pi(x,t).$$

It follows from (3.16) that

$$\begin{aligned}
&\widetilde{T}(t+n)u^+(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)\widetilde{Q}^+(u^+(x,s))ds \\
&\le \widetilde{T}(t+n)\Pi(x,-n) + \int_{-n}^{t} \widetilde{T}(t-s)[f'(\mathbf{0})\Pi(x,s) + \widetilde{L}\Pi(x,s)]ds \\
&= \Pi(x,t).
\end{aligned} \tag{3.17}$$

Combining (3.15) and (3.17), (3.12) holds. Therefore, the claim follows from Lemma 3.2.

Moreover, $W^n(x,t)$ satisfies the regular estimates as in Lemma 2.5, that is, there exists a positive constant M, independent of n, such that for any $x \in \mathbb{R}^N$ and $t > -n+1$,

$$\left\| \frac{\partial W^n}{\partial t}(x,t) \right\|, \ \left\| \frac{\partial^2 W^n}{\partial t x_i}(x,t) \right\|, \ \left\| \frac{\partial^2 W^n}{\partial t^2}(x,t) \right\|, \ \left\| \frac{\partial W^n}{\partial x_i}(x,t) \right\|, \ \left\| \frac{\partial^2 W^n}{\partial x_i t}(x,t) \right\| \le M,$$

and

$$\left\| \frac{\partial^2 W^n}{\partial x_i x_j}(x,t) \right\|, \ \left\| \frac{\partial^3 W^n}{\partial x_i^2 t}(x,t) \right\|, \ \left\| \frac{\partial^3 W^n}{\partial x_i^2 x_j}(x,t) \right\| \le M, \ \forall i,j = 1,\cdots,N.$$

By using the diagonal extraction process, there exists a subsequence $\{W^{n_k}(x,t)\}_{k\in\mathbb{N}}$ of $\{W^n(x,t)\}_{n\in\mathbb{N}}$ such that $W^{n_k}(x,t)$ converges to a function $U(x,t) = \big(U_1(x,t),\cdots,U_m(x,t)\big)$ in the sense of topology $\mathcal{T}$. Clearly, $U(x,t)$ is an entire solution of (1.1). By virtue of (3.10), we have

$$u^-(x,t) \le U(x,t) \le \min\left\{\mathbf{K}^+, \Pi(x,t)\right\} \text{ for all } (x,t) \in \mathbb{R}^{N+1}.$$

From (3.9), it is easy to see that the assertion of part (i) holds. Note that $c\lambda_1(c) \ge \lambda^*$ for any $c > c_*$, and

$$\lim_{t\to-\infty} \Gamma^-(t)e^{-\lambda^* t} = v^*, \quad \lim_{\xi\to-\infty} \Phi^-_{c_i}(\xi)e^{-\lambda_1(c_i)\xi} = v(\lambda_1(c_i)), \ i = 1,\cdots,l.$$

The assertions for parts (ii) and (iii) are direct consequences of (3.9). The proof is complete. □

4 Applications

In this section, we apply our main results developed in Sect. 2–3 to the models (1.2)–(1.4). In particular, we shall carefully discuss the sufficient conditions to ensure $(A_2)' - (A_5)'$ when (A_2) is not satisfied.

4.1 A Buffered System

Consider the buffered system (1.2). For simplicity, we consider the case $n = 1$, i.e.

$$\begin{cases} \partial_t u_1 = d_1\Delta u_1 + g(u_1) + k_1(b - v_1) - k_2 u_1 v_1, \\ \partial_t v_1 = d_2\Delta v_1 + k_1(b - v_1) - k_2 u_1 v_1, \end{cases} \tag{4.1}$$

where d_1, d_2, k_1, k_2, b are positive constants. Our choice of the function g is the typical monostable nonlinearity, i.e. $g(u_1) = u_1(1-u_1)$. Let $w_1 = u_1$ and $w_2 = b - v_1$, then (4.1) can be transformed to

$$\begin{cases} \partial_t w_1 = d_1 \Delta w_1 + w_1(1 - w_1) + k_1 w_2 - k_2 w_1(b - w_2), \\ \partial_t w_2 = d_2 \Delta w_2 - k_1 w_2 + k_2 w_1(b - w_2). \end{cases} \tag{4.2}$$

System (4.2) has only two equilibria $\mathbf{0} = (0, 0)$ and $\mathbf{K} = \big(1, k_2 b/(k_2+k_1)\big)$ and is cooperative on $[\mathbf{0}, \mathbf{K}]$. Let $D = \mathrm{diag}(d_1, d_2)$, and

$$f(w_1, w_2) = \big(w_1(1 - w_1) + k_1 w_2 - k_2 w_1(b - w_2),\ -k_1 w_2 + k_2 w_1(b - w_2)\big).$$

Assume $d_1 \geq d_2,\ 1 > k_2 b$ and $k_1 \geq k_2$. We claim that the conditions (A_0), $(A_1)(a)$, (A_2) and $(A_3)^*$ hold for (4.2). In fact, it is easy to see that

$$f'(\mathbf{0}) = \begin{pmatrix} 1 - k_2 b & k_1 \\ k_2 b & -k_1 \end{pmatrix}.$$

Obviously, $f'(\mathbf{0})$ is cooperative and irreducible, and

$$s(f'(\mathbf{0})) = \frac{1 - k_2 b - k_1 + \sqrt{(1 - k_2 b + k_1)^2 + 4k_1}}{2} > 0.$$

Hence, (A_0), $(A_1)(a)$ and (A_2) hold for (4.2). Moreover, for any $\lambda \geq 0$,

$$A(\lambda) := D\lambda^2 + f'(\mathbf{0}) = \begin{pmatrix} d_1\lambda^2 + 1 - k_2 b & k_1 \\ k_2 b & d_2\lambda^2 - k_1 \end{pmatrix}.$$

Direct computation shows that

$$\begin{aligned} M(\lambda) &= s(A(\lambda)) \\ &= \frac{d_1\lambda^2 + d_2\lambda^2 + 1 - k_2 b - k_1 + \sqrt{[(d_1 - d_2)\lambda^2 + 1 - k_2 b + k_1]^2 + 4k_2 k_1 b}}{2} > 0, \end{aligned}$$

and the eigenvector $v(\lambda)$ corresponding to $M(\lambda)$ is

$$v(\lambda) := (v_1(\lambda), v_2(\lambda)) = \big(M(\lambda) - d_2\lambda^2 + k_1, k_2 b\big) \gg (0, 0).$$

Take $c_* = \frac{M(\lambda_*)}{\lambda_*} = \inf_{\lambda>0} \frac{M(\lambda)}{\lambda}$. Next, we check the condition $(A_3)^*$ (see Remark 2.1). Since $d_1 \geq d_2$ and $1 > k_2 b$, for any $\lambda \geq 0$,

$$\begin{aligned} \frac{v_1(\lambda)}{v_2(\lambda)} &= \frac{M(\lambda) - d_2\lambda^2 + k_1}{k_2 b} \\ &= \frac{1}{2k_2 b}\left[(d_1 - d_2)\lambda^2 + 1 - k_2 b + k_1 + \sqrt{[(d_1 - d_2)\lambda^2 + 1 - k_2 b + k_1]^2 + 4k_2 k_1 b}\right] \\ &> \frac{1}{2}\left[1 - k_2 b + k_1 + \sqrt{[1 - k_2 b + k_1]^2 + 4k_2 k_1 b}\right] \geq k_1. \end{aligned}$$

For any $k \in \mathbb{Z}^+$, $\rho_1, \cdots, \rho_k > 0$ and $\lambda_1, \cdots, \lambda_k \in [0, \lambda_*]$, denote

$$(z_1, z_2) := \big(\rho_1 v_1(\lambda_1) + \cdots + \rho_k v_1(\lambda_k), \rho_1 v_2(\lambda_1) + \cdots + \rho_k v_2(\lambda_k)\big) \gg (0, 0).$$

Consequently, $(A_3)^*$ is equivalent to the following two inequalities

$$\begin{aligned} &z_1(1 - z_1) + k_1 z_2 - k_2 z_1(b - z_2) \leq (1 - k_2 b) z_1 + k_1 z_2, \\ &-k_1 z_2 + k_2 z_1(b - z_2) \leq k_2 b z_1 - k_1 z_2 \end{aligned}$$

or

$$z_1 \geq k_2 z_2 \text{ and } -k_2 z_1 z_2 \leq 0. \tag{4.3}$$

Since $\frac{v_1(\lambda)}{v_2(\lambda)} \geq k_1$ for any $\lambda \geq 0$, we have $z_1/z_2 \geq k_1$. Hence, (4.3) holds if $k_1 \geq k_2$. Therefore, $(A_3)^*$ holds.

From Proposition 2.2 and Lemma 2.12, we see that (4.2) has a traveling wave front $\Phi_c(\xi)$ for any $c \geq c_*$ and a spatially independent solution $\Gamma(t)$. Furthermore, by Theorems 2.9 and 2.10, (4.2) has other types of entire solutions which are different from the traveling wave fronts and spatially independent solution. In fact, we have the following result.

Theorem 4.1 *If $d_1 \geq d_2$, $1 > k_2 b$ and $k_1 \geq k_2$, then the conclusions of Theorems 2.9 and 2.10 are valid for (4.2), i.e., for any $l \in \mathbb{Z}^+$, $\nu_1, \cdots, \nu_l \in \mathbb{R}^N$ with $\|\nu_i\| = 1$, $h_1, \cdots, h_{l+1} \in \mathbb{R}$, $c_1, \cdots, c_l > c_*$, and $\chi_1, \cdots, \chi_{l+1} \in \{0, 1\}$ with $\chi_1 + \cdots + \chi_{l+1} \geq 2$, (4.2) admits an entire solution $W_p(x,t) = \big(W_{1;p}(x,t), W_{2;p}(x,t)\big)$, where $p = p_{\chi_1,\cdots,\chi_{l+1}}$.*

Moreover, the monotonicity with respect to t and x_i, the asymptotic behavior when $t \to \pm\infty$, and the monotonicity and the limit with respect to h_i in Theorem 2.10 hold true for $W_p(x,t)$ as for $U_p(x,t)$.

4.2 An Epidemic Model

Consider the epidemic model (1.3). Scaling time and absorbing the appropriate constants into u_2, system (1.3) can be rewritten as

$$\begin{cases} \partial_t u_1(x,t) = \tilde{d}_1 \Delta u_1(x,t) - u_1(x,t) + \gamma u_2(x,t), \\ \partial_t u_2(x,t) = \tilde{d}_2 \Delta u_2(x,t) - \beta u_2(x,t) + g(u_1(x,t)), \end{cases} \tag{4.4}$$

where $\tilde{d}_1 = d_1/a_{11} > 0$, $\tilde{d}_2 = d_2/a_{11}^2 > 0$, $\gamma = a_{12}/a_{11}^2 > 0$ and $\beta = a_{22}/a_{11} > 0$. For convenience, we denote $\tilde{d}_i$ by d_i, $i = 1, 2$.

We assume

(H_1) $g \in C^2([0,+\infty),[0,+\infty))$, $g(0) = g(k) - \frac{\beta}{\gamma}k = 0$, $g(u) > \frac{\beta}{\gamma}u$ for $u \in (0,k)$, and $g(u) \leq g'(0)u$ for $u \in [0,k]$, where $k > 0$ is a constant.

(H_2) One of the following holds:

- (a) $g(u)$ is increasing for $u > 0$;
- (b) There exists a number $u_{\max} > 0$ such that $g(u)$ is increasing for $0 < u \leq u_{\max}$ and decreasing for $u > u_{\max}$.

Let $\mathbf{K} = (k, g(k)/\beta)$, $D = \mathrm{diag}(d_1, d_2)$ and $f(u_1,u_2) = \big(-u_1 + \gamma u_2, -\beta u_2 + g(u_1)\big)$. Clearly, $f(\mathbf{0}) = f(\mathbf{K}) = \mathbf{0}$ and

$$f'(\mathbf{0}) = \begin{pmatrix} -1 & \gamma \\ g'(0) & -\beta \end{pmatrix}.$$

From (H_1), we see $g'(0) > \frac{\beta}{\gamma} > 0$. It is easy to see that $f(u) \leq f'(\mathbf{0})u$ for $u \in [\mathbf{0},\mathbf{K}]$, $f'(\mathbf{0})$ is cooperative and irreducible, and

$$s(f'(\mathbf{0})) = \frac{-(\beta+1) + \sqrt{(\beta+1)^2 + 4(\gamma g'(0) - \beta)}}{2} > 0.$$

Thus, the conditions (A_0) and $(A_1)(a)$ hold for (4.4). Furthermore, for any $\lambda \geq 0$,

$$A(\lambda) := D\lambda^2 + f'(\mathbf{0}) = \begin{pmatrix} d_1\lambda^2 - 1 & \gamma \\ g'(0) & d_2\lambda^2 - \beta \end{pmatrix}$$

and

$$M(\lambda)=s(A(\lambda))=\frac{d_1\lambda^2+d_2\lambda^2-\beta-1+\sqrt{[(d_1\lambda^2-1)-(d_2\lambda^2-\beta)]^2+4\gamma g'(0)}}{2}>0.$$

Clearly, $\inf_{\lambda>0}\frac{M(\lambda)}{\lambda}$ exists and denote by c_*.

Theorem 4.2 *Assume* (H_1). *The following statements hold:*

(i) If $(H_2)(a)$ *or* $(H_2)(b)$ *holds and* $k\le u_{\max}$, *then the conclusions of Theorems 2.9 and 2.10 are valid for* (4.4). *If, in addition,* $g'(u)\le g'(0)$ *for* $u\in[0,k]$, *then the conclusions of Theorem 2.11 hold true for (4.4).*

(ii) If $(H_2)(b)$ *holds and* $k>u_{\max}$, *then the conclusions of Theorem 3.6 hold for (4.4).*

If (H_1) and $(\mathrm{H}_2)(a)$ or $(\mathrm{H}_2)(b)$ hold and $k\le u_{\max}$, then system (4.4) is cooperative on $[\mathbf{0},\mathbf{K}]$. It is easy to verify that $(\mathrm{A}_2)-(\mathrm{A}_3)$ hold. If, in addition, $g'(u)\le g'(0)$ for $u\in[0,k]$, then $f'(u)\le f'(\mathbf{0})$ for $u\in[\mathbf{0},\mathbf{K}]$. Therefore, the statement (i) of Theorem 4.2 holds true.

When (H_1), $(\mathrm{H}_2)(b)$ hold and $k>u_{\max}$, system (4.4) is non-cooperative on $[\mathbf{0},\mathbf{K}]$. Take

$$u_{\min}=\inf\left\{u\in(0,u_{\max}]\,\middle|\,g(u)=g\left(\frac{\gamma}{\beta}g(u_{\max})\right)\right\}.$$

Clearly, $u_{\min}>0$. We define two functions $f^{\pm}(u)$ as follows:

$$f^{\pm}(u)=\big(-u_1+\gamma u_2,-\beta u_2+g^{\pm}(u_1)\big),$$

where

$$g^{+}(u_1)=\begin{cases}g(u_1), & u_1\in\big[0,u_{\max}\big],\\ g(u_{\max}), & u_1\in\big[u_{\max},\frac{\gamma}{\beta}g(u_{\max})\big]\end{cases}$$

and

$$g^{-}(u_1)=\begin{cases}g(u_1), & u_1\in\big[0,u_{\min}\big],\\ g\big(u_{\min}\big), & u_1\in\big[u_{\min},\frac{\gamma}{\beta}g(u_{\max})\big].\end{cases}$$

Clearly, $g^{+}(u_1)\le g'(0)u_1$ for $u_1\in\big[0,\frac{\gamma}{\beta}g(u_{\max})\big]$. Hence, $f^{+}(u)\le f'(\mathbf{0})u$ for $u\in[\mathbf{0},\mathbf{K}^{+}]$ which yields that $(\mathrm{A}_5)'$ holds. One can further check the conditions $(\mathrm{A}_2)'-(\mathrm{A}_4)'$ with $\mathbf{K}=\big(k,g(k)/\beta\big)$,

$$\mathbf{K}^{+}=\left(\frac{\gamma}{\beta}g(u_{\max}),g(u_{\max})\right)\text{ and }\mathbf{K}^{-}=\left(\frac{\gamma}{\beta}g\big(u_{\min}\big),g\big(u_{\min}\big)\right).$$

Therefore, the statement (ii) of Theorem 4.2 holds true.

We remark that two specific functions

$$g_1(u)=\frac{\omega u}{1+\nu u}\text{ and }g_2(u)=\frac{\omega u}{1+\nu u^2},$$

which have been widely used in the mathematical biology literature, satisfies the above conditions for a wide range of parameters ω and ν. In fact, we have the following statements:

(a) if $\omega\gamma>\beta$, then the function

$$f(u_1,u_2)=\big(-u_1+\gamma u_2,-\beta u_2+g_1(u_1)\big)$$

satisfies the conditions (H_1) and $(\mathrm{H}_2)(a)$ with $k=\frac{\omega\gamma-\beta}{\beta\nu}$;

(b) if $\omega\gamma > \beta$, then the function

$$f(u_1, u_2) = \big(- u_1 + \gamma u_2, -\beta u_2 + g_2(u_1)\big)$$

satisfies the conditions (H_1) and $(\mathrm{H}_2)(b)$ with

$$k = \sqrt{\frac{\omega\gamma - \beta}{\beta\nu}} \text{ and } u_{\max} = \sqrt{\frac{1}{\nu}}.$$

Furthermore, it is easy to see that if $\omega\gamma \le 2\beta$, then $k \le u_{\max}$, and if $\omega\gamma > 2\beta$, then $k > u_{\max}$.

4.3 A Population Model

Consider the model (1.4) by taking the non-monotone Ricker function $u_1 e^{-u_1}$ as $h(u_1)$. Let $w_1 = u_1$ and $w_2 = u_2 - 1$, then (1.4) reduces to

$$\begin{cases} \partial_t w_1 = d_1 \Delta w_1 + w_1(r_1 - \alpha - \delta w_1 + r_1 w_2), \\ \partial_t w_2 = d_2 \Delta w_2 + r_2(1 + w_2)[-w_2 + h(w_1)], \end{cases} \tag{4.5}$$

where $h(w_1) = w_1 e^{-w_1}$ and $d_1, d_2, r_1, r_2, \alpha, \delta$ are all positive parameters. Similar to [36], we assume

$$r_1 > \alpha,\ d_1 \ge d_2 \text{ and } \delta \ge \frac{r_1 r_2}{r_1 + r_2 - \alpha}. \tag{4.6}$$

In the nonnegative quadrant, (4.5) has only two equilibria $\mathbf{0} = (0, 0)$ and $\mathbf{K} = (K_1, K_2)$ which satisfy

$$r_1 K_1 e^{-K_1} = \delta K_1 + \alpha - r_1 \text{ and } K_2 = K_1 e^{-K_1}. \tag{4.7}$$

Let $D = \mathrm{diag}(d_1, d_2)$ and

$$f(w) = \big(w_1(r_1 - \alpha - \delta w_1 + r_1 w_2),\ r_2(1 + w_2)[-w_2 + w_1 e^{-w_1}]\big).$$

For any $\lambda \ge 0$,

$$A(\lambda) := D\lambda^2 + f'(\mathbf{0}) = \begin{pmatrix} d_1\lambda^2 + r_1 - \alpha & 0 \\ r_2 & d_2\lambda^2 - r_2 \end{pmatrix}.$$

Direct computation shows that $M(\lambda) = d_1\lambda^2 + r_1 - \alpha > 0$ and the eigenvector $v(\lambda)$ corresponding to $M(\lambda)$ is

$$v(\lambda) := (v_1(\lambda), v_2(\lambda)) = \big((d_1 - d_2)\lambda^2 + r_1 + r_2 - \alpha,\ r_2\big) \gg (0, 0).$$

Hence, the conditions (A_0) and $(\mathrm{A}_1)(b)$ hold for (4.5). Take $c_* = \frac{M(\lambda_*)}{\lambda_*} = \inf_{\lambda>0} \frac{M(\lambda)}{\lambda}$. Note that $h(w_1) = w_1 e^{-w_1}$ achieves its maximum at $h_m = 1$, and is increasing on $[0, h_m]$ and decreasing on $[h_m, +\infty)$.

Theorem 4.3 *Assume (4.6). The following statements hold:*

(i) If $K_1 \le 1$, then the conclusions of Theorems 2.9 and 2.10 are valid for (4.5).

(ii) If $K_1 > 1$, then the conclusions of Theorem 3.6 hold true for (4.5).

When $K_1 \le 1$, system (4.5) is a cooperative system on $[\mathbf{0}, \mathbf{K}]$, i.e., (A_2) holds. We need to check the condition $(A_3)^*$ (see Remark 2.1). For any $k \in \mathbb{Z}^+$, $\rho_1, \cdots, \rho_k > 0$ and $\lambda_1, \cdots, \lambda_k \in [0, \lambda_*]$, denote

$$(z_1, z_2) := \big(\rho_1 v_1(\lambda_1) + \cdots + \rho_k v_1(\lambda_k), \rho_1 v_2(\lambda_1) + \cdots + \rho_k v_2(\lambda_k)\big) \gg (0, 0).$$

Consequently, $(A_3)^*$ is equivalent to the following two inequalities

$$z_1[r_1 - \alpha - \delta z_1 + r_1 z_2] \le (r_1 - \alpha) z_1, \tag{4.8}$$
$$r_2(1 + z_2)\big(- z_2 + z_1 e^{-z_1}\big) \le r_2(z_1 - z_2) \tag{4.9}$$

or

$$\delta z_1 \ge r_1 z_2, \tag{4.10}$$
$$e^{z_1}(z_1 + z_2^2) \ge z_1(1 + z_2). \tag{4.11}$$

Since for any $\lambda \ge 0$,

$$\frac{v_1(\lambda)}{v_2(\lambda)} = \frac{(d_1 - d_2)\lambda^2 + r_1 + r_2 - \alpha}{r_2} \ge \frac{r_1 + r_2 - \alpha}{r_2},$$

we have

$$\frac{z_1}{z_2} \ge \frac{r_1 + r_2 - \alpha}{r_2}.$$

Note also that $z_1 > 0$ and $e^{z_1} > 1 + z_1$. Thus, the following two equalities suffice to verify (4.10) and (4.11):

$$\delta \frac{r_1 + r_2 - \alpha}{r_2} \ge r_1 \text{ and } z_1 z_2^2 + \big(z_1 - \frac{1}{2} z_2\big)^2 + \frac{3}{4} z_2^2 \ge 0,$$

which are true provided that (4.6) holds.

If $K_1 > 1$, system (4.5) is non-cooperative on $[\mathbf{0}, \mathbf{K}]$. Similar to [36,41], we define two functions $f^{\pm}(u)$ as follows:

$$f^{\pm}(w) = \big(w_1(r_1 - \alpha - \delta w_1 + r_1 w_2), r_2(1 + w_2)[-w_2 + h^{\pm}(w_1)]\big),$$

where

$$h^+(w_1) = \begin{cases} w_1 e^{-w_1}, & w_1 \in [0, 1], \\ e^{-1}, & w_1 > 1, \end{cases}$$

and

$$h^-(w_1) = \begin{cases} w_1 e^{-w_1}, & w_1 \in [0, h_0], \\ K_1^+ e^{-K_1^+}, & w_1 > h_0. \end{cases}$$

Here $K_1^+ > K_1$ and $h_0 \in (0, 1]$ are the unique roots of the equations

$$\delta K_1^+ + \alpha - r_1 - r_1 h^+(K_1^+) = 0 \text{ and } h_0 e^{-h_0} - K_1^+ e^{-K_1^+} = 0,$$

respectively. It is easy to verify that $(A_2)' - (A_4)'$ hold with $\mathbf{K} = \big(K_1, K_1 e^{-K_1}\big)$ and $\mathbf{K}^{\pm} = \big(K_1^{\pm}, K_1^{\pm} e^{-K_1^{\pm}}\big)$, where $K_1^- \in (0, K_1)$ is the unique root of the equation

$$\delta K_1^- + \alpha - r_1 - r_1 h^-(K_1^-) = 0.$$

Next, we check the condition $(A_5)^*$ (see Remark 3.1). Let

$$(z_1, z_2) := \big(\rho_1 v_1(\lambda_1) + \cdots + \rho_k v_1(\lambda_k), \rho_1 v_2(\lambda_1) + \cdots + \rho_k v_2(\lambda_k)\big) \gg (0, 0).$$

Consequently, $(A_5)^*$ is equivalent to the following two inequalities

$$z_1[r_1 - \alpha - \delta z_1 + r_1 z_2] \le (r_1 - \alpha) z_1, \tag{4.12}$$

$$r_2(1 + z_2)\big(-z_2 + h^+(z_1)\big) \le r_2(z_1 - z_2). \tag{4.13}$$

Note that (4.8) and (4.9) hold and $h^+(z_1) = z_1 e^{-z_1}$ for $z_1 \in (0, 1]$. To verify the above two inequalities, we only need to show (4.13) holds for $z_1 > 1$, i.e.,

$$(1 + z_2)(-z_2 + e^{-1}) \le z_1 - z_2,$$

that is,

$$e(z_1 + z_2^2) \ge 1 + z_2 \text{ for } z_1 > 1.$$

It suffices to show that

$$2(1 + z_2^2) \ge 1 + z_2,$$

which holds obviously.

5 Conclusion and Discussion

In this paper, we establish the existence and qualitative properties of front-like entire solutions for m-dimensional monostable cooperative reaction-diffusion systems in $\mathbb{R}^N$. The same issues for some class of monostable non-cooperative systems are also considered. The main results are applied to some biological and epidemiological models. Though the case that the condition (A_2) does not hold is studied, our main results are invalid for some classical non-cooperative systems, such as L-V competition system and L-V predator-prey system. Besides, uniqueness and stability of entire solutions of diffusion systems and the continuous dependence of such entire solutions on parameters, such as wave speeds and wave directions, seem to be very interesting and challenging problems.

We mention that the assumption $(d_1, \cdots, d_m) \gg \mathbf{0} := (0, \cdots, 0) \in \mathbb{R}^m$ (i.e. (1.1) is non-degenerate) is crucial for our main results. When some but not all diffusion coefficients are zero (i.e. (1.1) is partially degenerate), system (1.1) has weak regularity and compactness. For example, if $d_i = 0$ for some $i \in \{1, \cdots, m\}$, then u_i is not smooth enough with respect to x due to zero diffusion coefficient and hence the prior estimate for u_i is not valid (see Lemma 2.6). Recently, in [44], we considered the entire solution of the reaction-diffusion system modeling man-environment-man epidemics with bistable nonlinearity:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + \alpha v(x,t), \\ \frac{\partial v(x,t)}{\partial t} = -\beta v(x,t) + g(u(x,t)). \end{cases} \tag{5.1}$$

To obtain the entire solution, we established the following prior estimate of solutions of (5.1), see [44, Theorem 3.3].

Proposition 5.1 *Suppose that $w(x,t) = (u(x,t), v(x,t))$ is a solution of (5.1) with initial value $\varphi \in [\mathbf{0}, \mathbf{K}]_X$, then there exists a positive constant $M > 0$ such that for any $\varphi \in [\mathbf{0}, \mathbf{K}]_X$,*

$x \in \mathbb{R}$ *and* $t > 1$,

$$|u_t(x,t)| \le M,\ |u_{tt}(x,t)| \le M,\ |u_{tx}(x,t)| \le M,\ |u_x(x,t)| \le M,$$
$$|u_{xt}(x,t)| \le M,\ |u_{xx}(x,t)| \le M,\ |u_{xxx}(x,t)| \le M, |u_{xxt}(x,t)| \le M,$$
$$|v_t(x,t)| \le M,\ |v_x(x,t)| \le M,\ |v_{tx}(x,t)| \le M,\ |v_{tt}(x,t)| \le M.$$

As mention above, $v(x,t)$ in general is not C^1 in x when $v(0,\cdot) \in C(\mathbb{R}; [0; K_2])$. Hence, the estimates for v_x, v_{tx} and u_{xxx} are not valid. Here, we correct this mistake. We shall prove that v, v_t and u_{xx} possess a property which is similar to a global Lipschitz condition with respect to x. In fact, we have the following result.

Proposition 5.2 *Suppose that $w(x,t) = (u(x,t), v(x,t))$ is a solution of (5.1) with initial value $\varphi = (\varphi_1, \varphi_2) \in C\big(\mathbb{R}, [\mathbf{0}, \mathbf{K}]\big)$, then there exists a positive constant $M > 0$, independent of φ, such that for any $x \in \mathbb{R}$ and $t > 1$,*

$$\big|u_t(x,t)\big|,\ \big|u_{tt}(x,t)\big|,\ \big|u_{tx}(x,t)\big|,\ \big|u_x(x,t)\big| \le M,$$
$$\big|u_{xt}(x,t)\big|,\ \big|u_{xx}(x,t)\big|,\ \big|u_{xxt}(x,t)\big| \le M,$$
$$\big|v_t(x,t)\big|,\ \big|v_{tt}(x,t)\big| \le M.$$

If, in addition, there exists a constant $L' > 0$ such that for any $\eta > 0$, $\sup_{x\in\mathbb{R}} |\varphi_2(x+\eta) - \varphi_2(x)| \le L'\eta$, then for any $\eta > 0$,

$$\sup_{x\in\mathbb{R}, t\ge 1} \big|v(x+\eta,t) - v(x,t)\big| \le M'\eta,\quad \sup_{x\in\mathbb{R}, t\ge 1} |v_t(x+\eta,t) - v_t(x,t)| \le M'\eta,$$

and

$$\sup_{x\in\mathbb{R}, t\ge 1} |u_{xx}(x+\eta,t) - u_{xx}(x,t)| \le M'\eta,$$

where $M' > 0$ is a constant which is independent of φ and η.

It turns out that the results in [44] hold for the bistable partially degenerate system (5.1). More recently, we have extended the results to a class of two component monostable cooperative partially degenerate reaction-diffusion systems [47]. However, it seems difficult to establish such results for general partially degenerate reaction-diffusion systems. Thus, an interesting problem is to adress the entire solutions of general partially degenerate reaction-diffusion systems.

Acknowledgment The authors thank the anonymous referee for their valuable comments and suggestions that help the improvement of the manuscript. Shi-Liang Wu was supported by the Scientific Research Program Funded by Shaanxi Provincial Education Department (No. 12JK0860) and the Fundamental Research Funds for the Central Universities (N0. K50511700002).

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