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TRAVELING WAVES OF DIFFUSIVE PREDATOR-PREY SYSTEMS: DISEASE OUTBREAK PROPAGATION

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Abstract. We study the traveling waves of reaction-diffusion equations for a diffusive SIR model. The existence of traveling waves is determined by the basic reproduction number of the corresponding ordinary differential equations and the minimal wave speed. Our proof is based on Schauder fixed point theorem and Laplace transform.

1. **Introduction.** Kermack and McKendrik [32] proposed a simple deterministic susceptible-infected-removed (SIR) model for an infectious disease outbreak in a closed population consisting of susceptible individuals ($S(t)$), infected individuals ($I(t)$) and removed individuals ($R(t)$). The Kermack-McKendrick SIR model when individuals move randomly is given by the following reaction-diffusion system

$$\partial_t S = d_1 \partial_{xx} S - \beta SI, \quad (1)$$

$$\partial_t I = d_2 \partial_{xx} I + \beta SI - \gamma I, \quad (2)$$

$$\partial_t R = d_3 \partial_{xx} R + \gamma I, \quad (3)$$

where β is the transmission coefficient, γ is the recovery/remove rate, and d_1, d_2 and d_3 are the diffusion rates of the susceptible, infective and removed individuals, respectively. The model captures the essential transmission dynamics using the mass action and predicts infection propagation from the initial source of an outbreak.

As the model focuses on the outbreak situation and ignores the natural and death process, the model system (1-3) has infinitely many disease-free equilibria $(S_{-\infty}, 0, 0)$ with arbitrary $S_{-\infty} > 0$. A traveling wave solution is a special type of solutions with the form $(S(x+ct), I(x+ct), R(x+ct))$, and represents the transition from the initial disease-free equilibrium $(S_{-\infty}, 0, 0)$ to another disease free state $(S_{+\infty}, 0, 0)$ with $S_{+\infty}$ being determined by the transmission rate and the disease

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specific recovery rate, as well as possibly the mobility of individuals. For applications to the disease control and prevention, it is important to determine whether traveling waves exist and what the propagation speed c is.

Note that R does not appear in the first two equations (1) and (2), it suffices to consider the two dimensional system for (S, I). Such a system with a kinetic planar vector field provides a simple example of the general diffusive predator-prey system. When the diffusion rate d_1 is zero, Källén [30] showed that a nontrivial traveling wave solution exists if and only if the basic reproduction number $R_0 := \beta S_{-\infty}/\gamma$ is no less than one and the traveling speed c exceeds a minimal value $c^* := 2\sqrt{d_2(\beta S_{-\infty} - \gamma)}$. Hosono and Ilyas [26] proved the existence of a traveling wave with any positive speed if the diffusion rate of the infected class $d_2 = 0$. These two results were later extended to the non-degenerate case $d_1 \neq 0$ and $d_2 \neq 0$ by Hosono and Ilyas [27] with the aid of the shooting technique and invariant manifold theory developed by Dunbar [18, 19]. In particular, it was proved that if the basic reproduction number $\beta S_{-\infty}/\gamma > 1$, then for each $c \geq c^* = 2\sqrt{d_2(\beta S_{-\infty} - \gamma)}$ system (1-2) has a traveling wave solution $(S(x+ct), I(x+ct))$ satisfying $S(\pm\infty) = S_{\pm\infty}, I(\pm\infty) = 0, S_{-\infty} > S_{\infty}$. On the other hand, there is no traveling solution for (1-2) if $\beta S_{-\infty}/\gamma \leq 1$.

Here, we consider the SIR disease outbreak model with the standard incidence, while the recovereds are removed from the population and thus not involved in the contact and disease transmission. We refer to [8, 61] for the detailed epidemiological consideration of the corresponding ODE model, and here we focus on the diffusive system

$$\partial_t S = d_1 \partial_{xx} S - \beta SI/(S+I), \tag{4}$$
$$\partial_t I = d_2 \partial_{xx} I + \beta SI/(S+I) - \gamma I, \tag{5}$$
$$\partial_t R = d_3 \partial_{xx} R + \gamma I. \tag{6}$$

Again R does not appear in the first two equations and we only need to consider the two dimensional system for S and I. In this paper, we shall show the analogous existence and non-existence results for (4-5) hold. Specifically, we show that if $R_0 := \beta/\gamma > 1$, then for each $c > c^* = 2\sqrt{d_2(\beta - \gamma)}$ there exists a nonnegative constant $S_\infty < S_{-\infty}$ such that system (4-5) has a traveling wave solution $(S(x+ct), I(x+ct))$ satisfying $S(\pm\infty) = S_{\pm\infty}, I(\pm\infty) = 0, S_{-\infty} > S_\infty$ and no traveling wave solution for $0 < c < c^*$. In addition, we show that if $R_0 := \beta/\gamma \leq 1$, there is no traveling solution for (4-5).

Our method in this paper is mainly based on that in Wang and Wu [62] and several early studies. We will employ the Schauder fixed point theorem (to prove the existence theorem) for an equivalent non-monotone abstract operator, and the challenging and difficult task is to construct and verify a suitable invariant convex set for the non-monotone operator. Much of the existing theoretical development on traveling waves has been built on the applications of the powerful theory of monotone dynamical systems and some comparison arguments; see, for examples [13, 50, 58] and references therein. Recently, a number of results on the existence of traveling solutions for non-monotone spatial models have been obtained (see, e.g. [17, 37, 40, 57]), where suitable invariant sets are carefully constructed and the order-preserving properties of the involved operators are proved. The non-existence proof is based on an argument applying two-side Laplace transform in [12, 14]. On the other hand, the existence proof in [26, 27] was based on the shooting method developed in [18, 19] and similar methods was used in [4, 29].

The paper is organized as follows. In the following three sections, we will present and prove the existence and non-existence theorem respectively. In Section 3, we will introduce notations of differential and integral operators and state some properties of these operators. The traveling wave solution is a fixed point of a map defined in terms of the integral operators. Next, we construct an invariant convex cone and verify the continuity and compactness of our map. Finally, we apply Schauder fixed point theorem to prove the existence theorem. In Section 4, we prove the non-existence theorem for two different cases: one is $R_0 > 1$ and $c < c^*$; the other is $R_0 \leq 1$. Conclusions and discussions are given in Section 5.

2. Main results.

Because R does not appear in the system of equations for the susceptible individuals S and infected individuals I, we omit the R equation and study the following system with S and I only:

$$\partial_t S = d_1 \partial_{xx} S - \beta SI/(S+I), \tag{7}$$

$$\partial_t I = d_2 \partial_{xx} I + \beta SI/(S+I) - \gamma I. \tag{8}$$

We look for the non-trivial and non-negative traveling wave solutions $(S(x+ct), I(x+ct))$ which satisfy the following boundary conditions at infinity:

$$S(-\infty) = S_{-\infty},\ S(\infty) < S_{-\infty},\ I(\pm\infty) = 0. \tag{9}$$

The ordinary differential system describing the traveling waves (or wave profiles) is given below:

$$cS' = d_1 S'' - \beta SI/(S+I), \tag{10}$$

$$cI' = d_2 I'' + \beta SI/(S+I) - \gamma I. \tag{11}$$

Our main theorem is stated as follows:

Theorem 2.1. *If $R_0 := \beta/\gamma > 1$ and $c > c^* := 2\sqrt{d_2(\beta-\gamma)}$, then there exists a non-trivial and non-negative traveling wave solutions (S, I) such that the boundary conditions (9) are satisfied. Furthermore, S is monotonically decreasing, $0 \leq I(x) \leq S(-\infty) - S(\infty)$ for all $x \in \mathbb{R}$, and*

$$\int_{-\infty}^{\infty} \gamma I(x)dx = \int_{-\infty}^{\infty} \frac{\beta S(x)I(x)}{S(x)+I(x)}dx = c[S(-\infty) - S(\infty)]. \tag{12}$$

If $R_0 = \beta/\gamma \leq 1$ or $c < c^* := 2\sqrt{d_2(\beta-\gamma)}$, then there exist no non-trivial and non-negative traveling wave solution (S, I) satisfying the boundary conditions (9).

Remark 1. It is easy to see that (cf. [59]) $R_0 := \beta/\gamma$ is the basic reproduction number of the corresponding ordinary differential system. It is also obvious that the minimal wave speed $c^* := 2\sqrt{d_2(\beta-\gamma)}$ does not depend on the diffusion coefficient of S. This speed is calculated from the characteristic function of the linearized equation for I

$$f(\lambda) := -d_2\lambda^2 + c\lambda - (\beta - \gamma), \tag{13}$$

at the disease free equilibrium $(S_{-\infty}, 0)$. It is obvious that this function f has two different roots if and only if $c > c^*$.

3. **Existence theorem.**

We first prove the existence of non-trivial and non-negative traveling waves. Throughout this section, we always assume $R_0 := \beta/\gamma > 1$ and c is a fixed number greater than $c^* := 2\sqrt{d_2(\beta-\gamma)}$. We denote by

$$\lambda_0 := \frac{c - \sqrt{c^2 - 4d_2(\beta-\gamma)}}{2d_2} \tag{14}$$

the smaller positive root of the characteristic function f defined in (13).

3.1. **Differential operators and their inverses.**

In this subsection, we introduce two second-order linear differential operators Δ_1 and Δ_2 and their inverses Δ_1^{-1} and Δ_2^{-1}. Given $\alpha_i > 0$ with $i = 1$ or 2, it is easily seen that the equation

$$-d_i\lambda^2 + c\lambda + \alpha_i = 0 \tag{15}$$

has two roots

$$\lambda_i^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_i\alpha_i}}{2d_i}. \tag{16}$$

Note that λ_i^- is negative and λ_i^+ is positive, and note also that λ_i^+ is greater than $-\lambda_i^-$. We then choose α_i to be sufficiently large such that

$$-\lambda_i^- > \lambda_0$$

where λ_0 is given in (14). We also need

$$\alpha_1 > \beta, \quad \alpha_2 > \gamma$$

two conditions which will be used in the proof of Lemma 3.4. Denote

$$\rho_i := d_i(\lambda_i^+ - \lambda_i^-) = \sqrt{c^2 + 4d_i\alpha_i}. \tag{17}$$

We now define the second-order linear differential operator Δ_i as

$$\Delta_i h := -d_i h'' + ch' + \alpha_i h \tag{18}$$

for any $h \in C^2(\mathbb{R})$. The corresponding integral operator Δ_i^{-1} is defined by

$$(\Delta_i^{-1}h)(x) := \frac{1}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h(y)dy + \frac{1}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)}h(y)dy \tag{19}$$

for any $h \in C_{\mu^-,\mu^+}(\mathbb{R})$ with $\mu^- > \lambda_i^-$ and $\mu^+ < \lambda_i^+$, where

$$C_{\mu^-,\mu^+}(\mathbb{R}) := \{h \in C(\mathbb{R}) : \sup_{x\le 0}|h(x)e^{-\mu^- x}| + \sup_{x\ge 0}|h(x)e^{-\mu^+ x}| < \infty\}. \tag{20}$$

Note that the asymptotic conditions in the definition of $C_{\mu^-,\mu^+}(\mathbb{R})$ guarantee the integral $\Delta_i^{-1}h$ to be well defined. In the following lemma, we prove that Δ_i^{-1} is actually the inverse operator of Δ_i.

Lemma 3.1. *Let $i = 1$ or 2. We have*

$$\Delta_i^{-1}(\Delta_i h) = h \tag{21}$$

for any $h \in C^2(\mathbb{R})$ such that $h, h', h'' \in C_{\mu^-,\mu^+}(\mathbb{R})$ with $\mu^- > \lambda_i^-$ and $\mu^+ < \lambda_i^+$, and

$$\Delta_i(\Delta_i^{-1}h) = h \tag{22}$$

for any $h \in C_{\mu^-,\mu^+}(\mathbb{R})$ with $\mu^- > \lambda_i^-$ and $\mu^+ < \lambda_i^+$.

Proof. It follows from the definitions of Δ_i and Δ_i^{-1} in (18) and (19) that

$$\begin{aligned}[\Delta_i^{-1}(\Delta_i h)](x) &= \frac{1}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}[-d_i h''(y) + ch'(y) + \alpha_i h(y)]dy \\ &+\frac{1}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)}[-d_i h''(y) + ch'(y) + \alpha_i h(y)]dy.\end{aligned}$$

Making use of integration by parts, we obtain

$$\int_{-\infty}^{x} e^{\lambda_i^-(x-y)} h'(y)dy = h(x) + \lambda_i^- \int_{-\infty}^{x} e^{\lambda_i^-(x-y)} h(y)dy,$$

and

$$\int_{-\infty}^{x} e^{\lambda_i^-(x-y)} h''(y)dy = h'(x) + \lambda_i^- h(x) + (\lambda_i^-)^2 \int_{-\infty}^{x} e^{\lambda_i^-(x-y)} h(y)dy.$$

Therefore, we have

$$\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}[-d_i h''(y) + ch'(y) + \alpha_i h(y)]dy = -d_i h'(x) + (-d_i\lambda_i^- + c)h(x).$$

Here we have used the fact that λ_i^- is the root of the equation $-d_i\lambda^2 + c\lambda + \alpha_i = 0$; see (15) and (16). Similarly, it can be shown that

$$\int_{x}^{\infty} e^{\lambda_i^+(x-y)}[-d_i h''(y) + ch'(y) + \alpha_i h(y)]dy = d_i h'(x) + (d_i\lambda_i^+ - c)h(x).$$

Applying the above two equalities to the expression of $\Delta_i^{-1}(\Delta_i h)$ gives

$$[\Delta_i^{-1}(\Delta_i h)](x) = \frac{d_i(\lambda_i^+ - \lambda_i^-)}{\rho_i} h(x) = h(x),$$

where in the last equality we have used the definition of ρ_i in (17). This proves (21). For $h \in C_{\mu^-,\mu^+}(\mathbb{R})$ with $\mu^- > \lambda_i^-$ and $\mu^+ < \lambda_i^+$, we can differentiate (19) twice to obtain

$$\begin{aligned}(\Delta_i^{-1}h)'(x) &= \frac{\lambda_i^-}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)} h(y)dy + \frac{\lambda_i^+}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)} h(y)dy, \\ (\Delta_i^{-1}h)''(x) &= \frac{(\lambda_i^-)^2}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)} h(y)dy + \frac{(\lambda_i^+)^2}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)} h(y)dy \\ &+\frac{\lambda_i^-}{\rho_i}h(x) - \frac{\lambda_i^+}{\rho_i}h(x).\end{aligned}$$

Noting that $\lambda_i^\pm$ are two roots of the equation (15), it is readily seen from (17), (19) and the above two equalities that

$$-d_i(\Delta_i^{-1}h)''(x) + c(\Delta_i^{-1}h)'(x) + \alpha_i(\Delta_i^{-1}h)(x) = h(x).$$

This ends our proof of the lemma. □

In the subsequent subsections, we will construct a convex invariant cone whose lower bound is of the form

$$g(x) := \begin{cases} e^{\lambda x}(1 - Me^{\varepsilon x}), & x \le x^*, \\ 0, & x \ge x^*, \end{cases}$$

where $x^* := -\ln M/\varepsilon$ is the point at which g is not differentiable. If we introduce the notation $\vee$ to denote the maximum of two numbers, then we can write g in a simple formula

$$g(x) = e^{\lambda x}(1 - Me^{\varepsilon x}) \vee 0.$$

The following lemma will be used in the next subsection to prove cone invariance.

Lemma 3.2. *Let $i = 1$ or 2. Given any $M > 0$, $\varepsilon > 0$ and λ such that $\lambda_i^- < \lambda < \lambda + \varepsilon < \lambda_i^+$, we have*

$$\Delta_i^{-1}(\Delta_i g) \geq g \tag{23}$$

for $g(x) := e^{\lambda x}(1 - Me^{\varepsilon x}) \vee 0$. Note that g is not differentiable at $x^* := -\ln M/\varepsilon$ and the differential operator $\Delta_i g$ is defined in the sense of distribution. In view of (19) and $\lambda_i^- < \lambda < \lambda+\varepsilon < \lambda_i^+$, the integral $\Delta_i^{-1}(\Delta_i g)$ is well defined because $\Delta_i g$ is continuous everywhere except at the point $x^* := -\ln M/\varepsilon$ and $e^{-\lambda x}|\Delta_i g|$ is bounded as $x \to -\infty$ and $e^{-(\lambda+\varepsilon)x}|\Delta_i g|$ is bounded as $x \to \infty$.

Proof. Let $x^* := -\ln M/\varepsilon$ be the point where g is not differentiable. For convenience, we denote

$$f_i(k) := -d_i k^2 + ck + \alpha_i = d_i(k - \lambda_i^-)(\lambda_i^+ - k), \tag{24}$$

which has two roots $\lambda_i^{\pm}$ as defined in (16). It is easily seen from (18) that

$$(\Delta_i g)(x) = \begin{cases} f_i(\lambda)e^{\lambda x} - Mf_i(\lambda+\varepsilon)e^{(\lambda+\varepsilon)x}, & x < x^*, \\ 0, & x > x^*. \end{cases} \tag{25}$$

We will consider the two cases $x \leq x^*$ and $x \geq x^*$ respectively. When $x \leq x^*$, we have from (19) and (25) that

$$[\Delta_i^{-1}(\Delta_i g)](x) = f_i(\lambda)A(\lambda) - Mf_i(\lambda+\varepsilon)A(\lambda+\varepsilon), \tag{26}$$

where

$$\begin{aligned} A(k) &:= \frac{1}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)+ky}dy + \frac{1}{\rho_i}\int_{x}^{x^*} e^{\lambda_i^+(x-y)+ky}dy \\ &= \frac{e^{kx}(\lambda_i^+ - \lambda_i^-)}{\rho_i(k-\lambda_i^-)(\lambda_i^+ - k)} - \frac{e^{kx^*+\lambda_i^+(x-x^*)}}{\rho_i(\lambda_i^+ - k)} \end{aligned}$$

for $k = \lambda$ or $\lambda+\varepsilon$. In view of (17) and (24), it follows from the above equality that

$$f_i(k)A(k) = e^{kx} - \frac{k-\lambda_i^-}{\lambda_i^+ - \lambda_i^-}e^{kx^*+\lambda_i^+(x-x^*)}.$$

Applying this to (26) and on account of $Me^{\varepsilon x^*} = 1$, we obtain

$$\begin{aligned} [\Delta_i^{-1}(\Delta_i g)](x) &= [e^{\lambda x} - \frac{\lambda-\lambda_i^-}{\lambda_i^+ - \lambda_i^-}e^{\lambda x^*+\lambda_i^+(x-x^*)}] \\ &\quad -[Me^{(\lambda+\varepsilon)x} - \frac{\lambda+\varepsilon-\lambda_i^-}{\lambda_i^+ - \lambda_i^-}e^{\lambda x^*+\lambda_i^+(x-x^*)}] \\ &= [e^{\lambda x} - Me^{(\lambda+\varepsilon)x}] + \frac{\varepsilon}{\lambda_i^+ - \lambda_i^-}e^{\lambda x^*+\lambda_i^+(x-x^*)} \\ &\geq e^{\lambda x} - Me^{(\lambda+\varepsilon)x}. \end{aligned}$$

This proves (23) for $x \leq x^*$. When $x \geq x^*$, we have from (19) and (25) that

$$[\Delta_i^{-1}(\Delta_i g)](x) = f_i(\lambda)B(\lambda) - Mf_i(\lambda+\varepsilon)B(\lambda+\varepsilon), \tag{27}$$

where

$$B(k) := \frac{1}{\rho_i}\int_{-\infty}^{x^*} e^{\lambda_i^-(x-y)+ky}dy = \frac{e^{kx^*+\lambda_i^-(x-x^*)}}{\rho_i(k-\lambda_i^-)}$$

for $k = \lambda$ or $\lambda + \varepsilon$. In view of (17) and (24), it follows from the above equality that

$$f_i(k)B(k) = \frac{\lambda_i^+ - k}{\lambda_i^+ - \lambda_i^-}e^{kx^*+\lambda_i^-(x-x^*)}.$$

Applying this to (27) and on account of $Me^{\varepsilon x^*} = 1$, we obtain

$$\begin{aligned}
[\Delta_i^{-1}(\Delta_i g)](x) &= \frac{\lambda_i^+ - \lambda}{\lambda_i^+ - \lambda_i^-}e^{\lambda x^*+\lambda_i^-(x-x^*)} - \frac{\lambda_i^+ - \lambda - \varepsilon}{\lambda_i^+ - \lambda_i^-}e^{\lambda x^*+\lambda_i^-(x-x^*)} \\
&= \frac{\varepsilon}{\lambda_i^+ - \lambda_i^-}e^{\lambda x^*+\lambda_i^-(x-x^*)} \\
&\geq 0.
\end{aligned}$$

This gives (23) in the case $x \geq x^*$. □

3.2. **Cone invariance.** Given μ such that

$$\lambda_0 < \mu < -\lambda_i^- < \lambda_i^+, i = 1, 2.$$

(see the definitions of λ_0 and λ_i^- in (14) and (16)), we have

$$\lambda_i^- < -\mu < \mu < \lambda_i^+, i = 1, 2.$$

Now we introduce the functional space

$$B_\mu(\mathbb{R}, \mathbb{R}^2) := \{\phi = (\phi_1, \phi_2) :\ \phi_i \in C(\mathbb{R}) \ \&\ \sup_{x\in\mathbb{R}} e^{-\mu|x|}|\phi_i(x)| < \infty, \forall i = 1, 2\} \quad (28)$$

equipped with the norm

$$|\phi|_\mu := \big(\sup_{x\in\mathbb{R}} e^{-\mu|x|}|\phi_1(x)|\big) \bigvee \big(\sup_{x\in\mathbb{R}} e^{-\mu|x|}|\phi_2(x)|\big). \quad (29)$$

Recall that the symbol $\vee$ denotes the maximum of two numbers. It is easily seen from (20) that

$$B_\mu(\mathbb{R}, \mathbb{R}^2) = C_{-\mu,\mu}(\mathbb{R}) \times C_{-\mu,\mu}(\mathbb{R}).$$

We then define a map $F = (F_1, F_2)$ on the space $B_\mu(\mathbb{R}, \mathbb{R}^2)$: given $\phi = (\phi_1, \phi_2) \in B_\mu(\mathbb{R}, \mathbb{R}^2)$, let

$$F_1(\phi_1, \phi_2) := \Delta_1^{-1}[\alpha_1\phi_1 - \beta\phi_1\phi_2/(\phi_1 + \phi_2)]; \quad (30)$$

$$F_2(\phi_1, \phi_2) := \Delta_2^{-1}[\alpha_2\phi_2 + \beta\phi_1\phi_2/(\phi_1 + \phi_2) - \gamma\phi_2]. \quad (31)$$

Since $\lambda_i^- < -\mu < \mu < \lambda_i^+$ for $i = 1$ and 2, the nonlinear map F is well defined for any nonnegative ϕ. To construct a convex invariant cone, we have to specify its boundary, which are the super-solutions and sub-solutions given as below. Recall the definition of λ_0 in (14). Let $S_{-\infty} > 0$ be as in the boundary conditions (9), we define for $x \in \mathbb{R}$ the following

$$S_+(x) := S_{-\infty}; \quad (32)$$

$$S_-(x) := S_{-\infty}(1 - M_1e^{\varepsilon_1 x}) \vee 0; \quad (33)$$

$$I_+(x) := e^{\lambda_0 x}; \quad (34)$$

$$I_-(x) := e^{\lambda_0 x}(1 - M_2e^{\varepsilon_2 x}) \vee 0, \quad (35)$$

where $M_1, M_2, \varepsilon_1, \varepsilon_2$ are four positive constants to be determined in the following lemma.

Lemma 3.3. *Given sufficiently large $M_1 > 0$, $M_2 > 0$ and sufficiently small $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, we have*

$$-\beta I_+ \geq -d_1 S_-'' + cS_-' \tag{36}$$

for $x \leq x_1 := -\varepsilon_1^{-1} \ln M_1$, and

$$\beta S_- I_-/(S_- + I_-) - \gamma I_- \geq -d_2 I_-'' + cI_-' \tag{37}$$

for $x \leq x_2 := -\varepsilon_2^{-1} \ln M_2$.

Proof. In view of (33) and (34), the first inequality (36) is the same as

$$-\beta e^{\lambda_0 x} \geq S_{-\infty} e^{\varepsilon_1 x}(d_1 M_1 \varepsilon_1^2 - cM_1\varepsilon_1)$$

for all $x \leq x_1 := -\varepsilon_1^{-1} \ln M_1$. The above inequality can be written as

$$S_{-\infty} M_1 \varepsilon_1 (c - d_1\varepsilon_1) \geq \beta e^{(\lambda_0 - \varepsilon_1)x}.$$

Note that $x \leq x_1 := -\varepsilon_1^{-1} \ln M_1$. It suffices to prove

$$S_{-\infty} M_1 \varepsilon_1 (c - d_1\varepsilon_1) \geq \beta e^{-\varepsilon_1^{-1}(\lambda_0 - \varepsilon_1)\ln M_1},$$

which is obviously true if we choose $M_1 = 1/\varepsilon_1$ and let $\varepsilon_1 > 0$ be sufficiently small. Now we intend to prove the second inequality (37) for $x < x_2 := -\varepsilon_2^{-1} \ln M_2$, which, by subtracting both side by $(\beta - \gamma)I$, is the same as

$$-\beta I_-^2/(S_- + I_-) \geq -d_2 I_-'' + cI_-' - (\beta - \gamma)I_- = -M_2 f(\lambda_0 + \varepsilon_2) e^{(\lambda_0 + \varepsilon_2)x},$$

where f is defined in (13) with λ_0 as its smaller root. We first choose a sufficiently small $\varepsilon_2 \in (0, \lambda_0)$ such that $f(\lambda_0 + \varepsilon_2) > 0$. Then, we assume M_2 is sufficiently large that $x_2 < x_1$ holds. In view of (33) and (35), the above inequality can be written as

$$-\frac{\beta e^{2\lambda_0 x}(1 - M_2 e^{\varepsilon_2 x})^2}{S_{-\infty}(1 - M_1 e^{\varepsilon_1 x}) + e^{\lambda_0 x}(1 - M_2 e^{\varepsilon_2 x})} \geq -M_2 f(\lambda_0 + \varepsilon_2) e^{(\lambda_0 + \varepsilon_2)x},$$

which is equivalent to

$$M_2 f(\lambda_0 + \varepsilon_2)[S_{-\infty}(1 - M_1 e^{\varepsilon_1 x}) + e^{\lambda_0 x}(1 - M_2 e^{\varepsilon_2 x})] \geq \beta e^{(\lambda_0 - \varepsilon_2)x}(1 - M_2 e^{\varepsilon_2 x})^2.$$

Note that $x \leq x_2 := -\varepsilon_2^{-1} \ln M_2$, we only need to prove

$$M_2 f(\lambda_0 + \varepsilon_2) S_{-\infty}(1 - M_1 M_2^{-\varepsilon_1/\varepsilon_2}) \geq \beta M_2^{-(\lambda_0 - \varepsilon_2)/\varepsilon_2},$$

which is true for large M_2 because as $M_2 \to \infty$, the left-hand side tends to infinity and the right-hand side vanishes (recall that $0 < \varepsilon_2 < \lambda_0$). This ends the proof of our lemma. □

With the aid of the super-solutions and sub-solutions, we are now ready to define a convex cone Γ as

$$\Gamma := \{(S, I) \in B_\mu(\mathbb{R}, \mathbb{R}^2) : \; S_- \leq S \leq S_+ \;\&\; I_- \leq I \leq I_+\}. \tag{38}$$

Since $\mu > \lambda_0 > 0$, it is easily seen that Γ is uniformly bounded with respect to the norm $|\cdot|_\mu$ defined in (29). The following lemma shows that this cone is invariant under the map $F = (F_1, F_2)$ defined in (30) and (31).

Lemma 3.4. *The operator $F = (F_1, F_2)$ maps Γ into Γ, namely, for any $(S, I) \in B_\mu(\mathbb{R}, \mathbb{R}^2)$ such that $S_- \leq S \leq S_+$ and $I_- \leq I \leq I_+$, we have*

$$S_- \leq F_1(S, I) \leq S_+$$

and

$$I_- \leq F_2(S, I) \leq I_+.$$

Proof. Since $\alpha_1 S - \beta SI/(S+I) \le \alpha_1 S_+ = \Delta_1 S_+$; see the definition of Δ_1 in (18), we obtain from (21) in Lemma 3.1 and (30) that

$$F_1(S,I) \le \Delta_1^{-1}(\Delta_1 S_+) = S_+.$$

By (36) in Lemma 3.3, we have for $x \le x_1$,

$$\alpha_1 S - \beta SI/(S+I) \ge \alpha_1 S_- - \beta I_+ \ge \alpha_1 S_- - d_1 S_-'' + cS_-' = \Delta_1 S_-.$$

When $x \ge x_1$, it follows from $\alpha_1 > \beta$ (recalling the choice of α_1 in the paragraph after (16)) and $S_-(x) = 0$ that

$$\alpha_1 S - \beta SI/(S+I) \ge \alpha_1 S - \beta S \ge 0 = \Delta_1 S_-.$$

Coupling the above two inequalities and making use of (23) in Lemma 3.2 yield

$$F_1(S,I) \ge \Delta_1^{-1}(\Delta_1 S_-) = S_-.$$

Since $\alpha_2 > \gamma$ (by the choice of α_2) and λ_0 is the root of f defined in (13), we have

$$\alpha_2 I + \beta SI/(S+I) - \gamma I \le \alpha_2 I_+ + \beta I_+ - \gamma I_+ = \alpha_2 I_+ - d_2 I_+'' + cI_+' = \Delta_2 I_+.$$

In view of (21) in Lemma 3.1, we obtain from the above inequality that

$$F_2(S,I) \le \Delta_2^{-1}(\Delta_2 I_+) = I_+.$$

By (37) in Lemma 3.3 and monotonicity of $\beta SI/(S+I)$ with respect to both variables S and I, we obtain

$$\alpha_2 I + \frac{\beta SI}{S+I} - \gamma I \ge \alpha_2 I_- + \frac{\beta S_- I_-}{S_- + I_-} - \gamma I_- \ge \alpha_2 I_- - d_2 I_-'' + cI_-' = \Delta_2 I_-$$

for $x \le x_2$. When $x \ge x_2$, it is readily seen from $\alpha_2 > \gamma$ and $I_-(x) = 0$ that

$$\alpha_2 I + \beta SI/(S+I) - \gamma I \ge \alpha_2 I - \gamma I \ge 0 = \Delta_2 I_-.$$

A combination of the above two inequalities and (23) in Lemma 3.2 yields

$$F_2(S,I) \ge \Delta_2^{-1}(\Delta_2 I_-) \ge I_-.$$

This ends our proof of the lemma. □

3.3. **Continuity and compactness.** Before applying Schauder fixed point theorem, we should verify F is continuous and compact on Γ.

Lemma 3.5. *The map $F = (F_1, F_2) : \Gamma \to \Gamma$ defined in (30) and (31) is continuous and compact with respect to the norm $|\cdot|_\mu$ defined in (29).*

Proof. For any $(S_1, I_1) \in \Gamma$ and $(S_2, I_2) \in \Gamma$, since

$$\left|\frac{\beta S_1 I_1}{S_1 + I_1} - \frac{\beta S_2 I_2}{S_2 + I_2}\right| = \beta\left|\frac{(S_1 - S_2)I_1 I_2 + S_1 S_2 (I_1 - I_2)}{(S_1 + I_1)(S_2 + I_2)}\right| \le \beta(|S_1 - S_2| + |I_1 - I_2|),$$

we have

$$\left|\left(\alpha_1 S_1 - \frac{\beta S_1 I_1}{S_1 + I_1}\right) - \left(\alpha_1 S_2 - \frac{\beta S_2 I_2}{S_2 + I_2}\right)\right| \le (\alpha_1 + \beta)(|S_1 - S_2| + |I_1 - I_2|).$$

Consequently, it follows from the definition (30) that

$$|F_1(S_1, I_1)(x) - F_1(S_2, I_2)(x)| e^{-\mu|x|} \le \frac{\alpha_1 + \beta}{\rho_1}(|S_1 - S_2|_\mu + |I_1 - I_2|_\mu) C(x),$$

where

$$C(x) := e^{-\mu|x|}\left[\int_{-\infty}^{x} e^{\lambda_1^-(x-y) + \mu|y|} dy + \int_x^\infty e^{\lambda_1^+(x-y) + \mu|y|} dy\right].$$

To prove the continuity of F_1, it suffices to prove $C(x)$ is uniformly bounded for $x \in \mathbb{R}$. Since $\lambda_1^- < -\mu < \mu < \lambda_1^+$, it can be obtained from L'Hôpital's rule that

$$C(-\infty) = \frac{1}{\mu + \lambda_1^+} - \frac{1}{\mu + \lambda_1^-}$$

and

$$C(\infty) = \frac{1}{\lambda_1^+ - \mu} + \frac{1}{\mu - \lambda_1^-}.$$

Hence, we conclude that C is uniformly bounded on $\mathbb{R}$ and consequently, F_1 is continuous with respect to the norm $|\cdot|_\mu$. Similarly, we can show that F_2 is also continuous with respect to the norm $|\cdot|_\mu$. To prove the compactness of F, we shall make use of Arzela-Ascoli theorem and a standard diagonal process. Let $I_k := [-k, k]$ with $k \in \mathbb{N}$ be a compact interval on $\mathbb{R}$ and temporarily we regard Γ as a bounded subset of $C(I_k, \mathbb{R}^2)$ equipped with the maximum norm. Since F maps Γ into Γ, it is obvious that F is uniformly bounded. We will use the following two inequalities to show that F is equi-continuous. Namely, from the definition of F_i in (30-31) and the definition of Δ_i^{-1} in (19) we have for any $(S, I) \in \Gamma$,

$$\begin{aligned} |[F_1(S,I)]'(x)| &\leq \frac{-\lambda_1^- \alpha_1 S_{-\infty}}{\rho_1} \int_{-\infty}^{x} e^{\lambda_1^-(x-y)} dy + \frac{\lambda_1^+ \alpha_1 S_{-\infty}}{\rho_1} \int_{x}^{\infty} e^{\lambda_1^+(x-y)} dy \\ &= \frac{2\alpha_1 S_{-\infty}}{\rho_1}, \end{aligned}$$

and

$$\begin{aligned} |[F_2(S,I)]'(x)| &\leq \frac{-\lambda_2^-(\alpha_2 + \beta - \gamma)}{\rho_2} \int_{-\infty}^{x} e^{\lambda_2^-(x-y) + \lambda_0 y} dy \\ &\quad + \frac{\lambda_2^+(\alpha_2 + \beta - \gamma)}{\rho_2} \int_{x}^{\infty} e^{\lambda_2^+(x-y) + \lambda_0 y} dy \\ &= \frac{(\alpha_2 + \beta - \gamma) e^{\lambda_0 x}}{\rho_2} \left(\frac{-\lambda_2^-}{\lambda_0 - \lambda_2^-} + \frac{\lambda_2^+}{\lambda_2^+ - \lambda_0} \right) \\ &= \frac{c\lambda_0 + 2\alpha_2}{\rho_2} e^{\lambda_0 x}. \end{aligned}$$

Here in the last step we have made use of the facts that λ_0 defined in (14) is a root of f in (13) and $\lambda_2^\pm$ defined in (16) are the roots of f_2 in (24). Let $\{u_n\}$ be a sequence of Γ, which can be also viewed as a bounded subset of $C(I_k)$ with $I_k := [-k, k]$. Since F is uniformly bounded and equi-continuous, by the Arzela-Ascoli theorem and the standard diagonal process, we can extract a subsequence $\{u_{n_k}\}$ such that $v_{n_k} := F u_{n_k}$ converges in $C(I_k)$ for any $k \in \mathbb{N}$. Let v be the limit of v_{n_k}. It is readily seen that $v \in C(\mathbb{R}, \mathbb{R}^2)$. Furthermore, since $F(\Gamma) \subset \Gamma$ by Lemma 3.4 and Γ is closed, it follows that $v \in \Gamma$. Now we come back to the norm $|\cdot|_\mu$ defined in (29). Note that $\mu > \lambda_0 > 0$, it follows that $\|I_+\|_\mu$, the norm of I_+ defined in (34), is bounded. Indeed, Γ is uniformly bounded with respect to the norm $|\cdot|_\mu$. Hence, the norm $\|v_{n_k} - v\|_\mu$ is uniformly bounded for all $k \in \mathbb{N}$. Given any $\varepsilon > 0$, we can find an integer $M > 0$ independent of v_{n_k} such that

$$e^{-\mu|x|} |v_{n_k}(x) - v(x)| < \varepsilon$$

for any $|x| > M$ and $k \in \mathbb{N}$. Since v_{n_k} converges to v on the compact interval $[-M, M]$ with respect to the maximum norm, there exists $K \in \mathbb{N}$ such that

$$e^{-\mu|x|} |v_{n_k}(x) - v(x)| < \varepsilon$$

for any $|x| \le M$ and $k > K$. The above two inequalities imply that v_{n_k} converges to v with respect to the norm $|\cdot|_\mu$. This proves the compactness of the map F. □

3.4. **Existence proof.** In this subsection, we give the proof of the first part in the statement of Theorem 2.1. Firstly, since F is continuous and compact on Γ by Lemma 3.4 and Lemma 3.5, it follows from the Schauder fixed point theorem that F has a fixed point $(S, I) \in \Gamma$ such that

$$S = F_1(S,I) = \Delta_1^{-1}[\alpha_1 S - \beta SI/(S+I)]; \tag{39}$$

$$I = F_2(S,I) = \Delta_2^{-1}[\alpha_2 I + \beta SI/(S+I) - \gamma I]. \tag{40}$$

Since $S, I \in C_{-\mu,\mu}(\mathbb{R})$ and $\lambda_i^- < -\mu < \mu < \lambda_i^+$ for any $i = 1, 2$, it is readily seen from (22) in Lemma 3.1 that

$$\begin{aligned} \Delta_1 S &= \alpha_1 S - \beta SI/(S+I), \\ \Delta_2 I &= \alpha_2 I + \beta SI/(S+I) - \gamma I. \end{aligned}$$

Recalling the definition of Δ_i (with $i = 1, 2$) in (18), we conclude that (S, I) satisfy the equations (10) and (11). For the sake of convenience, we repeat and relabel them as

$$cS' = d_1 S'' - \beta SI/(S+I); \tag{41}$$

$$cI' = d_2 I'' + \beta SI/(S+I) - \gamma I. \tag{42}$$

Next, we will verify the boundary conditions (9). Since $S_- \le S \le S_+$ and $I_- \le I \le I_+$, we obtain from the definitions of $S_\pm$ and $I_\pm$ in (32-35) and the squeeze theorem that $S(x) \to S_{-\infty}$ and $I(x) \sim e^{\lambda_0 x}$ as $x \to -\infty$. Furthermore, in view of the integral representation of first derivative

$$(\Delta_i^{-1} h)'(x) = \frac{\lambda_i^-}{\rho_i} \int_{-\infty}^{x} e^{\lambda_i^-(x-y)} h(y)dy + \frac{\lambda_i^+}{\rho_i} \int_{x}^{\infty} e^{\lambda_i^+(x-y)} h(y)dy$$

for any $h \in C_{-\mu,\mu}(\mathbb{R})$, we obtain from (39), (40) and L'Hôpital's rule that $S'(x) \to 0$ and $I'(x) \to 0$ as $x \to -\infty$. Finally, from (41) and (42), it follows that the second derivatives S'' and I'' also vanish at $-\infty$. We list the asymptotic behavior of S and I below. As $x \to -\infty$,

$$S(x) \to S_{-\infty},\ I(x) \sim e^{\lambda_0 x},\ S'(x) \to 0,\ I'(x) \to 0,\ S''(x) \to 0,\ I''(x) \to 0. \tag{43}$$

Now we investigate asymptotic behavior of S and I as $x \to \infty$. An integration of (41) from $-\infty$ to x gives

$$d_1 S'(x) = c[S(x) - S_{-\infty}] + \int_{-\infty}^{x} \frac{\beta S(y) I(y)}{S(y) + I(y)} dy.$$

Since $S(x)$ is uniformly bounded, the integral on the right-hand side should be uniformly bounded; otherwise $S'(x) \to \infty$ as $x \to \infty$, which implies $S(x) \to \infty$ as $x \to \infty$, a contradiction. Thus, we obtain that $\beta SI/(S+I)$ is integrable on $\mathbb{R}$, which together with the above equality yields S' is uniformly bounded on $\mathbb{R}$. Note that (41) implies

$$(e^{-cx/d_1} S')' = e^{-cx/d_1}(S'' - cS'/d_1) = e^{-cx/d_1} \frac{\beta SI}{d_1(S+I)}.$$

Integrating the above equality from x to infinity gives

$$e^{-cx/d_1} S'(x) = -\int_{x}^{\infty} e^{-cy/d_1} \frac{\beta S(y) I(y)}{d_1[S(y) + I(y)]} dy.$$

Hence, S is non-increasing. Furthermore, since S and I are non-trivial; see (43), the integral on the right-hand side of the above equality can not be identically zero, which implies S' is non-trivial and $S(\infty) < S(-\infty)$. We are now ready to study asymptotic behavior of $I(x)$ as $x \to \infty$. From (42) we have

$$I(x) = \frac{1}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)}dy + \frac{1}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)}dy, \tag{44}$$

where

$$\lambda^{\pm} := \frac{c \pm \sqrt{c^2+4d_2\gamma}}{2d_2}$$

and

$$\rho := d_2(\lambda^+ - \lambda^-) = \sqrt{c^2+4d_2\gamma}.$$

Remark that $\lambda^{\pm}$ are the two roots of the characteristic equation

$$-d_2\lambda^2 + c\lambda + \gamma = 0.$$

It is readily seen that $\lambda^- < 0 < \lambda^+$. We would also like to mention that the integral in (44) is well defined because of the estimate $\beta SI/(S+I) \le \beta S_{-\infty}$ and Lebesgue's dominated convergence theorem. Since $\beta SI/(S+I)$ is integrable on $\mathbb{R}$, it follows from the integral equation (44) and Fubini's theorem that I is also integral on $\mathbb{R}$, and

$$\int_{-\infty}^{\infty} I(x)dx = \frac{1}{\gamma}\int_{-\infty}^{\infty}\frac{\beta S(x)I(x)}{S(x)+I(x)}dx. \tag{45}$$

Furthermore, since

$$I'(x) = \frac{\lambda^-}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)}dy + \frac{\lambda^+}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)}dy,$$

we have from $\lambda^- < 0 < \lambda^+$, $\beta SI/(S+I) \le \beta I$ and $\rho = d_2(\lambda^+ - \lambda_-)$ that

$$|I'(x)| \le \frac{\beta}{d_2}\int_{-\infty}^{\infty} I(x)dx.$$

Since I' is uniformly bounded and $I \ge 0$ is integrable on $\mathbb{R}$, it is easily seen that $I(x) \to 0$ as $x \to \infty$; otherwise, we can find a number $\varepsilon > 0$, a sequence $x_n \to \infty$ and a number $\delta > 0$ (since I' is uniformly bounded) such that $I(x) > \varepsilon$ for all $|x - x_n| < \delta$, which contradicts the integrability of I on $\mathbb{R}$. Integrating (42) on the real line, it then follows from (43) and (45) that $I'(x) \to 0$ as $x \to \infty$ (noting that this can be also obtained from the integral representation of I' in the last two equation above by L'Hôpital's rule). Again, from (42) we obtain $I''(x) \to 0$ as $x \to \infty$. Since $\beta SI/(S+I)$ is integrable on the real line, it follows from (41) and (43) that S' is uniformly bounded, which in turn implies S'' is also uniformly bounded. Since $S' \le 0$ is integrable on $\mathbb{R}$, it can be proved that $S'(x) \to 0$ as $x \to \infty$. This, together with (41) gives $S''(x) \to 0$ as $x \to \infty$. We list the asymptotic behavior of S and I below. As $x \to \infty$,

$$S(x) \to S_\infty < S_{-\infty},\ I(x) \to 0,\ S'(x) \to 0,\ I'(x) \to 0,\ S''(x) \to 0,\ I''(x) \to 0. \tag{46}$$

Moreover, an integration of (41) on the real line yields

$$\int_{-\infty}^{\infty}\frac{\beta S(x)I(x)}{S(x)+I(x)}dx = c[S(-\infty) - S(\infty)]. \tag{47}$$

Finally, we intend to prove the inequality $I(x) \le S(-\infty) - S(\infty)$ for all $x \in \mathbb{R}$. Since $I(x) \sim e^{\lambda_0 x}$ as $x \to -\infty$ and $I(x) \to 0$ as $x \to \infty$, we can define

$$J(x) := I(x) + \frac{\gamma}{c}\int_{-\infty}^{x} I(y)dy + \frac{\gamma}{c}\int_{x}^{\infty} e^{c/d_2(x-y)} I(y)dy. \tag{48}$$

It follows from (43), (45), (46), (47) and L'Hôpital's rule that

$$\lim_{x\to-\infty} J(x) = 0, \ \lim_{x\to\infty} J(x) = \frac{\gamma}{c}\int_{-\infty}^{\infty} I(x)dx = S(-\infty) - S(\infty).$$

Similarly, we obtain by differentiating (48) once, the asymptotic formulas (43) and (46), and L'Hôpital's rule that

$$J'(x) = I'(x) + \frac{\gamma}{d_2}\int_{x}^{\infty} e^{c/d_2(x-y)} I(y)dy$$

and

$$\lim_{x\to-\infty} J'(x) = 0, \ \lim_{x\to\infty} J'(x) = 0.$$

Furthermore, By differentiating (48) twice, it is readily seen from the differential equation for I in (42) that

$$-d_2J'' + cJ' = -d_2I'' + cI' + \gamma I = \beta SI/(S+I).$$

An integration of the above equation from x to ∞ gives

$$J'(x) = \frac{1}{d_2}\int_{x}^{\infty} e^{c/d_2(x-y)} \frac{\beta S(y)I(y)}{S(y)+I(y)}dy.$$

Here we have used the fact that $J'(\infty) = 0$. Since $J(\infty) = S(-\infty) - S(\infty)$ by the asymptotic formula obtained from the equation (48), we obtain from the above equality that $J(x) \le S(-\infty) - S(\infty)$ for all $x \in \mathbb{R}$. Recall $I(x) \le J(x)$ by definition (48), it follows that $I(x) \le S(-\infty) - S(\infty)$ for all $x \in \mathbb{R}$. This ends the proof of all statements in the first half of Theorem 2.1.

4. **Non-existence.** We repeat and relabel the differential equations (10) and (11) and the boundary conditions (9) here.

$$cS' = d_1S'' - \beta SI/(S+I); \tag{49}$$

$$cI' = d_2I'' + \beta SI/(S+I) - \gamma I. \tag{50}$$

The boundary conditions subjected to the system are given by

$$S(-\infty) = S_{-\infty}, \ S(\infty) < S_{-\infty}, \ I(\pm\infty) = 0. \tag{51}$$

It is easily seen from (50) and (51) that I satisfies the integral equation (44). For the sake of convenience, we rewrite it as

$$I(x) = \frac{1}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)} \frac{\beta S(y)I(y)}{S(y)+I(y)}dy + \frac{1}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)} \frac{\beta S(y)I(y)}{S(y)+I(y)}dy, \tag{52}$$

where

$$\lambda^{\pm} := \frac{c \pm \sqrt{c^2 + 4d_2\gamma}}{2d_2}$$

and

$$\rho := d_2(\lambda^+ - \lambda^-) = \sqrt{c^2 + 4d_2\gamma}.$$

Remark that $\lambda^{\pm}$ are the two roots of the characteristic equation

$$-d_2\lambda^2 + c\lambda + \gamma = 0.$$

It is readily seen that $\lambda^- < 0 < \lambda^+$. Note that the integral in (52) is well defined because $\beta SI/(S+I)$ vanishes as infinity; see (51). By (52), the derivative of I has the following integral representation:

$$I'(x) = \frac{\lambda^-}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)}dy + \frac{\lambda^+}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)}dy.$$

An application of L'Hôpital's rule and (51) to the above equation yields $I'(\pm\infty) = 0$. Applying this and (51) to (50) gives $I''(\pm\infty) = 0$. We list the asymptotic behavior of I as below.

$$I(\pm\infty) = 0,\ I'(\pm\infty) = 0,\ I''(\pm\infty) = 0. \tag{53}$$

4.1. **The case $R_0 > 1$ and $c < c^*$.** We prove by contradiction that if $R_0 > 1$ and $c < c^*$, then there does not exist a non-trivial and non-negative traveling wave solution pair satisfying the boundary conditions (9). Let (S, I) be the solution to (10) and (11). Since $\beta S/(S+I) \to \beta$ as $x \to -\infty$, there exists $\bar{x}$ such that

$$\beta S/(S+I) - \gamma > \delta := (\beta - \gamma)/2 > 0$$

for all $x < \bar{x}$. Applying this to (50) yields

$$cI' - d_2 I'' > \delta I \geq 0 \tag{54}$$

for all $x < \bar{x}$. Since $cI' - d_2 I''$ is integrable at $-\infty$ by (53), Lebesgue's dominated convergence theorem and the above inequality implies that I is also integrable at $-\infty$. Define

$$K(x) := \int_{-\infty}^{x} I(y)dy.$$

An integration of (54) yields

$$\delta K(x) \leq cI(x) - d_2 I'$$

for all $x < \bar{x}$. A further integration together with non-negativeness of I gives

$$\int_{-\infty}^{x} K(y)dy \leq c/\delta K(x)$$

for all $x < \bar{x}$. Since K is non-decreasing, we have

$$\eta K(x-\eta) \leq \int_{\eta}^{x} K(y)dy \leq c/\delta K(x)$$

for all $\eta > 0$ and all $x < \bar{x}$. Hence, there exists a large $\eta > 0$ such that

$$K(x-\eta) < K(x)/2$$

for all $x < \bar{x}$. Denote $\mu_0 := (\ln 2)/\eta > 0$ and let

$$L(x) := K(x)e^{-\mu_0 x}.$$

It follows that

$$L(x-\eta) < L(x)$$

for all $x < \bar{x}$, which implies $L(x) = K(x)e^{-\mu_0 x}$ is bounded as $x \to -\infty$. Applying (53) to (54) yields

$$cI' > d_2 I'',\ cI > d_2 I',\ cK > d_2 I.$$

Hence, we conclude that $I(x)e^{-\mu_0 x}$, $I'(x)e^{-\mu_0 x}$ and $I''(x)e^{-\mu_0 x}$ are all bounded as $x \to -\infty$. In view of (53), they are actually uniformly bounded on the whole real line. Moreover, since $I/(S+I) \leq 1$ and $S(x) + I(x) \to S_{-\infty}$ as $x \to -\infty$,

$e^{-\mu_0 x}I(x)/[S(x)+I(x)]$ is also uniformly bounded on the real line. Now, we can introduce two-side Laplace transform on the equation (50):

$$f(\mu)\int_{-\infty}^{\infty} e^{-\mu x}I(x)dx = -\int_{-\infty}^{\infty} e^{-\mu x}\frac{\beta I(x)^2}{S(x)+I(x)}dx,$$

where f is the characteristic function defined in (13) and we have made use of the fact $\beta SI/(S+I) = \beta I - \beta I^2/(S+I)$. The integrals on both side of the above equality are well defined for any $\mu \in (0, \mu_0)$. Since $e^{-\mu_0 x}I(x)/[S(x)+I(x)]$ is uniformly bounded on the real line and $f(\mu)$ is always negative for all $\mu \in \mathbb{R}$ (noting that $c < c^* = 2\sqrt{4d_2(\beta-\gamma)}$), the two Laplace integrals can be analytically continued to the whole right half plane; otherwise the integral on the left has a singularity at $\mu = \mu^* \in \mathbb{R}$ and it is analytic for all $\mu < \mu^*$ (cf. [12, 62]). However, since $e^{-\mu_0 x}I(x)/[S(x)+I(x)]$ is uniformly bounded, the integral on the right is actually analytic for all $\mu < \mu^* + \mu_0$, a contradiction. Thus, the above equality holds for all $\mu > 0$ and can be rewritten as

$$\int_{-\infty}^{\infty} e^{-\mu x}I(x)[f(\mu) + \frac{\beta I(x)}{S(x)+I(x)}]dx = 0.$$

This again leads to a contradiction because $f(\mu) + \frac{\beta I(x)}{S(x)+I(x)} \to -\infty$ as $\mu \to \infty$, but $e^{-\mu x}I(x)$ is always non-negative for all $\mu \in \mathbb{R}$; see [12, 62] for early ideas in different settings. Finally, we conclude that if $R_0 > 1$ and $c < c^*$ then there does not exist a non-trivial and non-negative traveling wave solutions satisfying the boundary conditions (9).

4.2. **The case $R_0 \le 1$.** When $R_0 = \beta/\gamma \le 1$, then $\beta S(x)I(x)/[S(x)+I(x)] \le \gamma I(x)$ for all $x \in \mathbb{R}$. From (50) we have

$$(e^{-c/d_2 x}I')' = -\frac{1}{d_2}e^{-c/d_2 x}[\frac{\beta S(x)I(x)}{S(x)+I(x)} - \gamma I(x)] \ge 0,$$

which implies that the function $e^{-c/d_2 x}I'(x)$ is non-decreasing. Since $I'(\infty) = 0$ by (53) and $e^{-c/d_2 x} \to 0$ as $x \to \infty$, it follows that $I'(x) \le 0$ for all $x \in \mathbb{R}$. Again from $I(\pm\infty) = 0$ in (53) we obtain $I(x) = 0$ for all $x \in \mathbb{R}$, a contradiction. Therefore, we have proved that if $R_0 \le 1$, then there does not exist a non-trivial and non-negative traveling wave solution satisfying the boundary conditions (9).

5. Conclusion and discussion.

In this paper, we used a special type of disease outbreak model to show how to obtain the existence and non-existence of non-trivial traveling wave solutions for general predator-prey systems with spatial random movements characterized by the usual diffusion operator.

For the model under consideration, the traveling wave describes the disease propagation into the susceptible population from an initial disease-free equilibrium to the final, also disease-free, equilibrium. We proved that whether there is such a traveling wave is totally determined by the kinetic dynamics, and more specifically by the basic reproduction number $R_0 := \beta/\gamma$ calculated from the corresponding ordinary differential system at the initial disease-free equilibrium. We further established that the minimal wave speed $c^* = 2\sqrt{d_2(\beta-\gamma)}$ is determined by the mobility of the infected individuals and is linearly determined. Such a result for linear determinacy has been established for reaction-diffusion equations with kinetics given by a cooperative vector field for more general reaction-diffusion systems admitting certain comparison principles. Extensions to reaction-diffusion systems with certain competitive kinetics have been obtained by using the standard ordering relation with respect to competitive systems and the recent work [21] shows that these extensions may remain valid for much more general competitive reaction-diffusion systems with interspecific retarded competition (for which the kinetic system, a delay differential system, does not generate monotone dynamical systems in infinite dimensional phase spaces). Our work here indicated similar extensions can also be obtained for certain predator-prey systems. As such, we have a relatively complete picture about the existence of traveling waves for three building blocks (cooperation, competition and predation) of more complex biological dynamics.

5.1. **Open problems.** Note that we have established that c^* is indeed the minimal wave speed, but it remains to show that c^* coincides with the asymptotic speed of propagation. The pioneering work of Fisher [22] obtained such a result on the asymptotic speed of propagation for a logistic-based reaction-diffusion model for the spread of an advantageous gene in a spatially extended population, and this work has been extended to the most general format involving order-preserving and monotone dynamical systems. It remains to be shown that such a result is valid for many predator-prey systems, and in particular for epidemic models. Should we verify that c^* is indeed the asymptotic speed of propagation for the spatial epidemic model, we can conclude that for a given initial condition corresponding to a spatially localized disease, the epidemic model solution will evolve into a wave of infective individuals moving into the susceptible region with the constant velocity given by c^*.

It would be interesting to know the value of $S(\infty) = S_\infty$ for the traveling wave solutions which gives the number of susceptible individuals that remain after the infective wave has passed. For the system (1-2) with mass action incidence, it can be shown that

$$S(\infty) - S(-\infty) = \frac{\gamma}{\beta} \ln \frac{S(\infty)}{S(-\infty)}$$

providing the diffusion d_1 for susceptible individuals is zero; see [30] or [35]. The above formula coincides with the final size relation for the corresponding ordinary differential system without diffusion (see, for example, [8]). For the general system (1-2) with nonzero diffusion terms, however, it seems impossible to obtain a simple formula connecting $S(-\infty)$ and $S(\infty)$; see a discussion of this problem in [35]. The situation could become easier for the considered system (7-8) with standard incidence function where the recovered individuals are removed from the disease infection process. In this case, S should be considered as the size of the susceptible populations who will be eventually exposed to the disease and thus become infected. As such, for the kinetic ODE model, we do have $S(\infty) = 0$; see [61].

The existence of traveling waves in the limit case $c = c^*$ may be established using a standard limiting argument, such as the one provided in [21], when $R_0 := \beta/\gamma > 1$. Here we sketch a somehow different version of the argument. First, choose μ which lies in the interval of $\lambda_0(c^*)$ and $-(\lambda_1^-(c^*) \vee \lambda_2^-(c^*))$, recalling the definition of λ_0 and $\lambda_i^\pm$ in (14) and (16). For each $c \in (c^*, c^* + \delta)$ with δ given and small enough, we can find a traveling wave solution pair $(S_c, I_c) \in \Gamma(c) \subset B_\mu(\mathbb{R}, \mathbb{R}^2)$ such that $F(S_c, I_c) = (S_c, I_c)$; see the definition of the functional space $B_\mu(\mathbb{R}, \mathbb{R}^2)$ in (28), the map F in (30)-(31), and the convex cone Γ in (38). We define a new cone $\Gamma^* \subset B_\mu(\mathbb{R}, \mathbb{R}^2)$ consists of the function pairs bounded by the following four functions:

$$\begin{aligned} S_+^*(x) &:= S_{-\infty}, \\ S_-^*(x) &:= S_{-\infty}(1 - M_1^* e^{\varepsilon_1^* x}) \vee 0, \\ I_+^*(x) &:= \begin{cases} e^{\lambda_0(c^*)x}, & x \geq 0, \\ e^{\lambda_0(c^*+\delta)x}, & x \leq 0, \end{cases} \\ I_-^*(x) &:= 0, \end{aligned}$$

where M_1^* and ε_1^* are constructed such that (36) is satisfied for $c = c^*$; see the proof in Lemma 3.3. Since $\lambda_0(c)$ is a decreasing function of c and $(S_-^*)'$ is non-positive, it is readily seen that (36) is satisfied for all $c \in [c^*, c^* + \delta]$. Moreover, Γ^* constructed above contains the union $\cup_{c^*<c\leq c+\delta}\Gamma(c)$. It is also noted that Γ^* is uniformly bounded in $B_\mu(\mathbb{R}, \mathbb{R}^2)$ with respect to the norm $|\cdot|_\mu$ defined in (29). Thus, we could find a subsequence, still denoted by (S_c, I_c), which converges to (S, I) in $B_\mu(\mathbb{R}, \mathbb{R}^2)$. Since $-\lambda_i^-(c^*) < -\mu < \mu < \lambda_i^+(c)$ from our choice of μ, we can take limit $c \to c^*$ in the equations $F(S_c, I_c) = (S_c, I_c)$ and apply Lebesgue's dominant theorem to obtain $F^*(S, I) = (S, I)$, where the star denotes the map F defined in (30)-(31) with $c = c^*$, namely, the second-order differential operators $\Delta_i^* h = -d_i h'' + c^* h' + \alpha_i h$. Therefore, we have proved that (S, I) is a fixed point of the map F^*, and consequently, it is a traveling solution pair. A similar argument as in the proof of existence theorem shows that this fixed point actually satisfies the equality (12) and $I(\pm\infty) = 0$. It seems difficult to show the limit is non-trivial because we have infinity many equilibria and the standard argument as in [21] fails. However, should we find a nontrivial uniform lower bound for the peak value of I_c for any $c \in (c^*, c^* + \delta]$, we can choose $I_c(0)$ to be uniformly bounded by translation and conclude that the limit is nontrivial (i.e., $I(0) > 0$). It is worth noting that the infected number at peak time for the corresponding ordinary differential system is

$$I_{peak} = S_{-\infty} R_0^{-R_0/(R_0-1)}(R_0 - 1),$$

where $S_{-\infty}$ could also be viewed as the final size and $R_0 := \beta/\gamma$ is the basic reproduction number. It would be interesting to determine peak value of the traveling wave solutions for the reaction-diffusion system.

5.2. **A brief survey of the literature.** We conclude this paper with a short survey on the traveling waves of epidemiological models, so we can indicate how our results here can be further generalized. We refer to [48, 50, 51] and references therein for more comprehensive reviews on spatial dynamics in epidemiological models. When the incidence function in the Kermack–McKendrick model is replaced by a general function $f(S)I$, Kennedy and Aris [35] conducted some linear analysis for the special case with zero diffusion rate of susceptible individuals, as well as some numerical simulations for the general case. Using phase plane analysis, the singular perturbation method and the center manifold theory, Smith and Zhao [55] proved the existence of traveling wave solutions when the diffusive rate for the susceptible individuals is small. For an arbitrary diffusion coefficient, Huang [29] established an existence result by using shooting method and a Lyapunov function.

In a discussion attached to the paper of Bartlett [6], Kendall [33] proposed a generalized Kermack–McKendrick model with the incidence function replaced by a non-local integral. In [34], Kendall approximated the integral by a diffusion term and provided a threshold condition for the existence of traveling wave solutions.

Mollison [42] then studied the special case when the weight in the Kendall's integral is an exponential function and the remove rate γ is zero. The general case was subsequently fully investigated by Atkinson and Reuter [5], Brown and Carr [9], and Diekmann [13]. We refer to Mollison [43] for an excellent survey for the progress in traveling waves of the Kendall's model until 1977. Non-local interaction arises naturally in epidemic models when age structures are considered (cf. [23]). Using the method of upper and lower solutions, Ducrot [16] proved the existence of traveling wave solutions to the following scalar age-structure model:

$$\partial_t u + \partial_a u = \Delta u + u(1-u)\pi(a),$$

where $\pi(a) = \exp[\int_0^a \delta(a')da']$ with δ being age-dependent death rate. Later, Ducrot and Magal [17] studied the traveling wave solutions to a more general epidemic model with age-of-infection structure. In [52], Ruan and Xiao proved existence of traveling wave solutions to the host-vector disease model with specific exponential kernels (i.e., $G(t) = c^2te^{-ct}$ or $G(t) = ce^{-ct}$):

$$\partial_t u = d\partial_{xx}u - au + b(1-u)\int_{-\infty}^t G(t-s)u(s,x)ds.$$

See also [60] for a result with the general kernel, and Pan [47] for the case with non-local diffusion. Recently, Wang and Wu [62] obtained the existence (and non-existence) of traveling wave solutions for a general class of diffusive Kermack–McKendrick models with nonlocal and delayed disease transmission and Gan et al. [24] and Yang et al. [69] both considered the case with birth and death processes incorporated, using the idea of partial quasi-monotonicity developed in Huang and Zou [28]. We remark that in the work of [63], the Schauder fixed point theorem was used to obtain the existence of wave solutions for models with stage structure. We believe our current work can be extended to general epidemic models with structured populations.

Much has been done for the endemic rather than outbreak Kermack–McKendrick model that involves demography. For instance, Abual-Rub [3] studied traveling waves of diffusive Kermack–McKendrick model with logistic birth, constant vaccination and no diffusion on susceptible. Djebali [15] proved an existence theorem of traveling wave solutions to a diffusive Kermack–McKendrick model with external sources.

Wave solutions for epidemic models involving animal hosts and vectors have been intensively studied in the past. For the spread of rabies: in [31], Källén et al. formulated a simple model for the spatial spread and control of rabies; Murray et al. [46] studied traveling waves of such models where the rabid foxes are assumed to wonder randomly; Bosch et al. [7] calculated numerically traveling velocity of rabies; and Murray and Seward [45] studied the spatial speed of rabies in foxes with immunity. Other studies include [25] for measles epidemics; [11] for Lyme disease; [56] and [41] for dengue; and [1] and [2] for Hantavirus infection in deer mice.

Capasso et al. [10] proposed a simple cholera epidemic model as follows

$$\begin{aligned}\partial_t u_1 &= d_1\Delta u_1 - a_{11}u_1 + a_{12}u_2,\\ \partial_t u_2 &= d_2\Delta u_2 - a_{22}u_2 + g(u_1).\end{aligned}$$

When $d_2 = 0$, the space dimension is one and g is monotone, Zhao and Wang [72] established the existence of Fisher type traveling wave solutions by using the method of upper and lower solutions. By applying phase plane techniques and spectrum analysis, Xu and Zhao [67] established existence, uniqueness and exponential stability of bistable monotone traveling waves. They later [68] obtained existence threshold of traveling wave solutions by applying the theory of monotone traveling waves developed by Thieme and Zhao [58]. Their results were then extended by Wu and Liu [65] to the non-monotone case by constructing two auxiliary monotone equations. If latency is considered and takes a special form, namely, $g(u)$ replaced by $pu(t-\tau)e^{-au(t-\tau)}$, Yang et al. [70] proved the stability of traveling fronts by using weighted energy method and the comparison principle. When there are two distinct human habitats without migration, Mukhopadhyay and Bhattacharyya [44] established the existence of traveling wave solutions by constructing upper and lower solutions. For the non-autonomous case (i.e., g depends also on time t), Liang et al. [38] developed a theory of traveling wave based on some abstract results of Liang and Zhao [39]. These abstract results were also used by Weng and Zhao [64] to prove the existence of traveling waves to a multi-type SIS model, offering an answer to an open problem presented by Rass and Radcliffe [48]. Later, Zhang and Zhao [71] applied the abstract theory in [39] to obtain existence theorem of traveling wave solutions to a spatial discrete SIS model.

It is interesting to note that it is not always the case that traveling wave solutions exist for a large continuous set of wave speeds. Wylie et al. [66] provided an example for which traveling waves connecting two stable equilibria can exist only for a discrete set of wave speeds. It is also encouraging to note there has been substantial progress towards numerical computations of traveling wave in epidemic models: for example, in [36], Kuperman and Wio conducted some numerical analysis on the traveling waves of epidemiological models with spatial dependence; Sazanov et al. [53, 54] considered numerically traveling wave solutions for a one-dimensional lattice of SIR model; Faddy and Slorach [20] provided bounds on the velocity of traveling fronts for an epidemic model with spatial connected colonies; and Renshaw [49] later derived some recurrence relationships between the coefficients of traveling wave solutions which are quite useful for numerical calculations.

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REFERENCES

[1] G. Abramson and V. M. Kenkre, *Spatiotemporal patterns in hantavirus infection*, Phys. Rev. E, **66** (2002), 011912, 5 pp.

[2] G. Abramson, V. M. Kenkre, T. L. Yates and R. R. Parmenter, *Traveling waves of infection in the hantavirus epidemics*, Bull. Math. Biol., **65** (2003), 519–534.

[3] M. S. Abual-Rub, *Vaccination in a model of an epidemic*, Int. J. Math. Math. Sci., **23** (2000), 425–429.

[4] S. Ai and W. Huang, *Traveling waves for a reaction-diffusion system in population dynamics and epidemiology*, Proc. Roy. Soc. Edinburgh Sect. A, **135** (2005), 663–675.

[5] C. Atkinson and G. E. H. Reuter, *Deterministic epidemic waves*, Math. Proc. Camb. Phil. Soc., **80** (1976), 315–330.

[6] M. S. Bartlett, *Measles periodicity and community size (with discussion)*, J. Roy. Stat. Soc. A, **120** (1957), 48–70.

[7] F. van den Bosch, J. A. J. Metz and O. Diekmann, *The velocity of spatial population expansion*, J. Math. Biol., **28** (1990), 529–565.

[8] F. Brauer and C. Castillo-Chávez, "Mathematical Models in Population Biology and Epidemiology," Texts in Applied Mathematics, **40**, Springer-Verlag, New York, 2001.

[9] K. Brown and J. Carr, *Deterministic epidemic waves of critical velocity*, Math. Proc. Camb. Phil. Soc., **81** (1977), 431–433.

[10] V. Capasso and L. Maddalena, *A nonlinear diffusion system modelling the spread of oro-faecal diseases*, in "Nonlinear Phenomena in Mathematical Sciences" (ed. V. Lakshmikantham), Academic Press, New York, 1981.

[11] T. Caraco, S. Glavanakov, G. Chen, J. E. Flaherty, T. K. Ohsumi and B. K. Szymanski, *Stage-structured infection transmission and a spatial epidemic: A model for lyme disease*, Am. Nat., **160** (2002), 348–359.

[12] J. Carr and A. Chmaj, *Uniqueness of traveling waves for nonlocal monostable equations*, Proc. Amer. Math. Soc., **132** (2004), 2433–2439.

[13] O. Diekmann, *Thresholds and traveling waves for the geographical spread of an infection*, J. Math. Biol., **6** (1978), 109–130.

[14] O. Diekmann and H. Kaper, *On the bounded solutions of a nonliear convolution equation*, Nonlinear Analysis, **2** (1978), 721–737.

[15] S. Djebali, *Traveling front solutions for a diffusive epidemic model with external sources*, Annales de la Faculté des Sciences de Toulouse Sér. 6, **10** (2001), 271–292.

[16] A. Ducrot, *Travelling wave solutions for a scalar age-structured equation*, Discrete Contin. Dyn. Syst. Ser. B, **7** (2007), 251–273.

[17] A. Ducrot and P. Magal, *Travelling wave solutions for an infection-age structured model with diffusion*, Proc. R. Soc. Edin. Sect. A, **139** (2009), 459–482.

[18] S. R. Dunbar, *Traveling wave solutions of diffusive Lotka-Volterra equations*, J. Math. Biol., **17** (1983), 11–32.

[19] S. R. Dunbar, *Traveling wave solutions of diffusive Lotka-Volterra equations: A heteroclinic connection in* R^4, Trans. Amer. Math. Soc., **286** (1984), 557–594.

[20] M. J. Faddy and I. H. Slorach, *Bounds on the velocity of spread of infection for a spatially connected epidemic process*, J. Appl. Probab., **17** (1980), 839–845.

[21] J. Fang and J. Wu, *Monotone traveling waves for delayed Lotka-Volterra competition systems*, Discrete Contin. Dyn. Syst. Ser. A, **32** (2012), 3043–3058.

[22] R. Fisher, *The wave of advance of advantageous genes*, Ann. of Eugenics, **7** (1937), 355–369.

[23] W. E. Fitzgibbon, M. E. Parrott and G. F. Webb, *Diffusion epidemic models with incubation and crisscross dynamics*, Math. Biosci., **128** (1995), 131–155.

[24] Q. Gan, R. Xu and P. Yang, *Travelling waves of a delayed SIRS epidemic model with spatial diffusion*, Nonlinear Anal. Real World Appl., **12** (2011), 52–68.

[25] B. T. Grenfell, O. N. Bjornstad and J. Kappey, *Travelling waves and spatial hierarchies in measles epidemics*, Nature, **414** (2001), 716–723.

[26] Y. Hosono and B. Ilyas, *Existence of traveling waves with any positive speed for a diffusive epidemic model*, Nonlinear World, **1** (1994), 277–290.

[27] Y. Hosono and B. Ilyas, *Traveling waves for a simple diffusive epidemic model*, Math. Models Methods Appl. Sci., **5** (1995), 935–966.

[28] J.-H. Huang and X.-F. Zou, *Travelling wave solutions in delayed reaction diffusion systems with partial monotonicity*, Acta Mathematicae Applicatae Sinica Engl. Ser., **22** (2006), 243–256.

[29] W. Huang, *Traveling waves for a biological reaction-diffusion model*, J. Dynam. Differential Equations, **16** (2004), 745–765.

[30] A. Källén, *Thresholds and travelling waves in an epidemic model for rabies*, Nonlinear Anal., **8** (1984), 851–856.

[31] A. Källén, P. Arcuri and J. D. Murray, *A simple model for the spatial spread and control of rabies*, J. Theor. Biol., **116** (1985), 377–393.

[32] W. O. Kermack and A. G. McKendrick, *A contribution to the mathematical theory of epidemics*, Proc. R. Soc. Lond. B, **115** (1927), 700–721.

[33] D. G. Kendall, *Discussion on Professor Bartlett's paper*, J. Roy. Stat. Soc. A, **120** (1957), 64–67.

[34] D. G. Kendall, *Mathematical models of the spread of infection*, in "Mathematics and Computer Science in Biology and Medicine," Medical Research Council, London, (1965), 213–225.

[35] C. R. Kennedy and R. Aris, *Traveling waves in a simple population model involving growth and death*, Bull. Math. Biol., **42** (1980), 397–429.

[36] M. N. Kuperman and H. S. Wio, *Front propagation in epidemiological models with spatial dependence*, Physica A, **272** (1999), 206–222.

[37] W.-T. Li, G. Lin and S. Ruan, *Existence of travelling wave solutions in delayed reaction-diffusion systems with applications to diffusion-competition systems*, Nonlinearity, **19** (2006), 1253–1273.

[38] X. Liang, Y. Yi and X.-Q. Zhao, *Spreading speeds and traveling waves for periodic evolution systems*, J. Differential Equations, **231** (2006), 57–77.

[39] X. Liang and X.-Q. Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications*, Comm. Pure Appl. Math., **60** (2007), 1–40; Erratum: **61** (2008), 137–138, MR2361306.

[40] S. Ma, *Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem*, J. Differential Equations, **171** (2001), 294–314.

[41] N. A. Maidana and H. M. Yang, *Describing the geographic spread of dengue disease by traveling waves*, Math. Biosci., **215** (2008), 64–77.

[42] D. Mollison, *Possible velocities for a simple epidemic*, Adv. Appl. Prob., **4** (1972), 233–257.

[43] D. Mollison, *Spatial contact models for ecological and epidemic spread*, J. Roy. Stat. Soc. B, **39** (1977), 283–326.

[44] B. Mukhopadhyay and R. Bhattacharyya, *Existence of epidemic waves in a disease transmission model with two-habitat population*, Internat. J. Systems Sci., **38** (2007), 699–707.

[45] J. D. Murray and W. L. Seward, *On the spatial spread of rabies among foxes with immunity*, J. Theor. Biol., **156** (1992), 327–348.

[46] J. D. Murray, E. A. Stanley and D. L. Brown, *On the spatial spread of rabies among foxes*, Proc. R. Soc. Lond. B, **229** (1986), 111–150.

[47] S. Pan, *Traveling wave fronts in an epidemic model with nonlocal diffusion and time delay*, Int. Journal of Math. Analysis, **2** (2008), 1083–1088.

[48] L. Rass and J. Radcliffe, "Spatial Deterministic Epidemics," Math. Surveys Monogr., **102**, Amer. Math. Soc., Providence, RI, 2003.

[49] E. Renshaw, *Waveforms and velocities for models of spatial infection*, J. Appl. Probab., **18** (1981), 715–720.

[50] S. Ruan, *Spatial-temporal dynamics in nonlocal epidemiological models*, in "Mathematics for Life Science and Medicine" (eds. Y. Iwasa, K. Sato and Y. Takeuchi), Biol. Med. Phys. Biomed. Eng., Springer, Berlin, (2007), 97–122.

[51] S. Ruan and J. Wu, *Modeling spatial spread of communicable diseases involving animal hosts*, in "Spatial Ecology," Chapman & Hall/CRC, Boca Raton, FL, (2009), 293–316.

[52] S. Ruan and D. Xiao, *Stability of steady states and existence of travelling waves in a vector-disease model*, Proc. Roy. Soc. Edinburgh Sect. A, **134** (2004), 991–1011.

[53] I. Sazonov and M. Kelbert, *Travelling waves in a network of SIR epidemic nodes with an approximation of weak coupling*, Math. Med. Biol., **28** (2011), 165–183.

[54] I. Sazonov, M. Kelbert and M. B. Gravenor, *The speed of epidemic waves in a one-dimensional lattice of SIR models*, Mathematical Modelling of Natural Phenomena, **3** (2008), 28–47.

[55] H. L. Smith and X.-Q. Zhao, *Traveling waves in a bio-reactor model*, Nonlinear Anal. Real World Appl., **5** (2004), 895–909.

[56] L. T. Takahashi, N. A. Maidana, W. C. Ferreira, Jr., P. Pulino and H. M. Yang, *Mathematical models for the Aedes aegypti dispersal dynamics: Travelling waves by wing and wind*, Bull. Math. Biol., **67** (2005), 509–528.

[57] H. Thieme, *Density-dependent regulation of spatially distributed populations and their asymptotic speed of spread*, J. Math. Biol., **8** (1979), 173–187.

[58] H. Thieme and X.-Q. Zhao, *Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models*, J. Differential Equations, **195** (2003), 430–470.

[59] P. van den Driessche and J. Watmough, *Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission*, Math. Biosci., **180** (2002), 29–48.

[60] H. Wang, *On the existence of traveling waves for delayed reaction-diffusion equations*, J. Differential Equations, **247** (2009), 887–905.

[61] X.-S. Wang, J. Wu and Y. Yang, *Richards model revisited: Validation by and application to infection dynamics*, submitted.

[62] Z.-C. Wang and J. Wu, *Traveling waves of a diffusive Kermack-McKendrick epidemic model with non-local delayed transmission*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **466** (2010), 237–261.

[63] Z.-C. Wang and J. Wu, *Traveling waves in a bio-reactor model with stage-structure*, J. Math. Anal. Appl., **385** (2012), 683–692.

[64] P. Weng and X.-Q. Zhao, *Spreading speed and traveling waves for a multi-type SIS epidemic model*, J. Differential Equations, **229** (2006), 270–296.

[65] S.-L. Wu and S.-Y. Liu, *Existence and uniqueness of traveling waves for non-monotone integral equations with application*, J. Math. Anal. Appl., **365** (2010), 729–741.

[66] J. Wylie, H. Huang and R. M. Miura, *Systems of coupled diffusion equations with degenerate nonlinear source terms: Linear stability and traveling waves*, Discrete Contin. Dyn. Syst., **23** (2009), 561–569.

[67] D. Xu and X.-Q. Zhao, *Bistable waves in an epidemic model*, J. Dynamics and Differential Equations, **16** (2004), 679–707.

[68] D. Xu and X.-Q. Zhao, *Asymptotic speed of spread and traveling waves for a nonlocal epidemic model*, Discrete Contin. Dyn. Syst. Ser. B, **5** (2005), 1043–1056.

[69] J. Yang, S. Liang and Y. Zhang, *Travelling waves of a delayed SIR epidemic model with nonlinear incidence rate and spatial diffusion*, PLoS ONE, **6** (2011), e21128.

[70] Y.-R. Yang, W.-T. Li and S.-L. Wu, *Exponential stability of traveling fronts in a diffusion epidemic system with delay*, Nonlinear Anal. Real World Appl., **12** (2011), 1223–1234.

[71] F. Zhang and X.-Q. Zhao, *Spreading speed and travelling waves for a spatially discrete SIS epidemic model*, Nonlinearity, **21** (2008), 97–112.

[72] X.-Q. Zhao and W. Wang, *Fisher waves in an epidemic model*, Discrete Contin. Dyn. Syst. Ser. B, **4** (2004), 1117–1128.

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