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Traveling Wave Phenomena in a Kermack–McKendrick SIR Model

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Abstract We study the existence and nonexistence of traveling waves of a general diffusive Kermack–McKendrick SIR model with standard incidence where the total population is not constant. The three classes, susceptible S, infected I and removed R, are all involved in the traveling wave solutions. We show that the minimum wave speed of traveling waves for the three-dimensional non-monotonic system can be derived from its linearizaion at the initial disease-free equilibrium. The proof in this paper is based on Schauder fixed point theorem and Laplace transform. Our study provides a promising method to deal with high dimensional epidemic models.

Keywords Traveling waves · SIR model · Schauder fixed point theorem · Laplace transform

Mathematics Subject Classification 92D30 · 35K57 · 34B40

1 Kermack–McKendrick SIR Model with Standard Incidence

Compartmental models describing the transmission of infectious diseases have been extensively studied in the literature, see e.g. [1,8]. The simple Kermack–McKendrik model [17]

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is the starting point for many epidemic models. In a closed community consisting of susceptible individuals ($S(t)$), infected individuals ($I(t)$) and removed individuals ($R(t)$), the deterministic susceptible-infected-removed (SIR) model is given as follows.

$$S' = -\tilde{\beta} SI, \tag{1.1}$$
$$I' = \tilde{\beta} SI - \gamma I, \tag{1.2}$$
$$R' = \gamma I, \tag{1.3}$$

where $\tilde{\beta}$ is the transmission coefficient, γ is the recovery rate. Adding the three equations together implies that the total population $N = S + I + R$ is constant.

It is reasonable to assume that the total population is fixed if the disease spreads quickly in the community and can be cleared within a short time. However, if the population has a significant change or the disease causes enough deaths to influence the population size, one may not have a constant total population (cf. [23]). To account for this, many researchers have proposed epidemic models with transmission coefficient taking the following form.

$$\tilde{\beta}(N) = \frac{C(N)}{N}, \tag{1.4}$$

where $N = S + I + R$ is the total population size and $C(N)$ is the adequate transmission rate. The mass-action incidence corresponds to the choice $C(N) = \beta N$ and standard incidence corresponds to $C(N) = \beta$, where β is a positive constant. In general, $C(N)$ is a non-decreasing function with respect to N. For example, Heesterbeek and Metz chose $C(N) = \frac{aN}{1+bN+\sqrt{1+2bN}}$ in [10]. Mena-Lorca and Hethcote [23] chose $C(N) = \lambda N^{\alpha}$ with $\alpha = 0.05$. Other types of $C(N)$ can be found in [1,24] and references therein.

Spatial structures play an important role in describing the spreading of communicable diseases. In designing effective prevention and control strategies, one should take into account various spatial factors such as immigration, vaccination, individual movements, border control and quarantine. This has been witnessed in recent global epidemic outbreaks of SARS, H1N1 and avian influenza [1,2]. Traveling wave solutions of spatial epidemic models represent the transition process of an outbreak from the initial disease-free equilibrium to another disease-free state; and describe the propagation of the pathogen as a wave with a fixed shape and a fixed speed. The study of traveling wave solutions provides important insight into the spatial patterns of invading diseases.

One of the key quantities in spatial models is the invasion speed of a species or infectious disease in a population. The spreading speed of a spatial model, which can be obtained through the limiting process of a sequence of solutions of the model; but independent of their initial conditions, is widely used to describe the invasion speed [19,33]. For cooperative systems, it is shown that the spreading speed is the same as the minimum wave speed [19,33]. An analogous result holds for the ray speed defined in terms of the spreading speed and certain direction [32]. For a large class of non-cooperative systems, one of the authors [28] proved that the spreading speed is the same as the minimum wave speed. For many other non-cooperative systems including the diffusive SIR models, it remains a challenging problem to establish a similar result. Nevertheless, one could investigate traveling wave solutions and calculate corresponding minimum wave speed. The formula of the minimum wave speed given below indicates that the infectious disease spreads linearly in time. Moreover, the minimum wave speed can be predicted quantitatively as a function of measurable parameters estimated from previous outbreaks of similar infectious diseases.

Several previous works have studied traveling wave solutions of basic diffusive SIR models. Källén [16] and Hosono and Ilyas [12] considered the existence of traveling wave solutions for the following diffusive epidemic model with mass-action incidence.

$$\partial_t S = d_1 \partial_{xx} S - \beta S I; \tag{1.5}$$

$$\partial_t I = d_2 \partial_{xx} I + \beta S I - \gamma I. \tag{1.6}$$

In particular, with the aid of the shooting technique and invariant manifold theory developed by Dunbar [6,7], Hosono and Ilyas [13] proved that if the basic reproduction number $R_0 := \beta S_{-\infty}/\gamma > 1$, then for each $c \geq c^* = 2\sqrt{d_2(\beta S_{-\infty} - \gamma)}$, system (1.5–1.6) has a traveling wave solution $(S(x+ct), I(x+ct))$ satisfying $S(-\infty) = S_{-\infty}$, $I(\pm\infty) = 0$, $S(\infty) < S_{-\infty}$. On the other hand, there is no traveling solution for (1.5–1.6) if $R_0 := \beta S_{-\infty}/\gamma \leq 1$. In a more recent work [29], Wang et al. proved a similar result for the following diffusive Kermack–McKendrick SIR model with standard incidence.

$$\partial_t S = d_1 \partial_{xx} S - \frac{\beta S I}{S + I}; \tag{1.7}$$

$$\partial_t I = d_2 \partial_{xx} I + \frac{\beta S I}{S + I} - \gamma I. \tag{1.8}$$

The above model reflects that if the infected individuals are removed from the population, they are no longer involved in the contact and disease transmission ([1]). It is also noted that the total population $S + I$ is not constant.

In this paper, we shall examine a more general diffusive Kermack–McKendrick SIR model with the assumption that some of the infective individuals will be removed from the population due to disease-induced death or quarantine, but others will be recovered and return in the community. The total population becomes $N = S + I + R$, where R is the number of recovered individuals. Because of mobility of individuals it is natural to assume that the total population is not fixed. For simplicity, we will concentrate on standard incidence rate (i.e. $C(N) = \beta$), and study the following diffusive Kermack–McKendrick SIR model.

$$\partial_t S = d_1 \partial_{xx} S - \frac{\beta S I}{N}, \tag{1.9}$$

$$\partial_t I = d_2 \partial_{xx} I + \frac{\beta S I}{N} - \gamma I - \delta I, \tag{1.10}$$

$$\partial_t R = d_3 \partial_{xx} R + \gamma I, \tag{1.11}$$

where $N = S+I+R$ is the total population. The constants d_1, d_2 and d_3 are the diffusion rates of the susceptible, infective and recovered individuals, respectively. $\gamma > 0$ is the recovery rate and $\delta \geq 0$ is the death/quarantine rate of infective individuals.

As the model focuses on the outbreak situation and ignores the natural death process, the model system (1.9–1.11) has infinitely many disease-free equilibria $(S, 0, R)$ with arbitrary $S \geq 0$, $R \geq 0$. If we consider the corresponding spatial-homogenous ordinary differential system and linearize it around the disease-free and recovered-free equilibrium $(S_{-\infty}, 0, 0)$, we obtain a simple linear equation for the infective individuals:

$$I'(t) = \beta I - (\gamma + \delta) I.$$

A standard application of next-generation method [5,25] gives an explicit formula for the basic reproduction number

$$R_0 = \frac{\beta}{\gamma + \delta}.$$

As we shall see later, the basic reproduction number is an important threshold parameter in the existence theorem of traveling wave solutions.

2 Traveling Wave Solutions

A traveling wave solution is a special type of solution with the form $(S(x+ct), I(x+ct), R(x+ct))$, which represents the transition process of an outbreak from the initial disease-free equilibrium $(S(-\infty), 0, R(-\infty))$ to another disease-free state $(S(\infty), 0, R(\infty))$. It is readily seen that the traveling wave solutions satisfy the following ordinary differential system.

$$cS' = d_1 S'' - \frac{\beta SI}{S+I+R}; \tag{2.1}$$

$$cI' = d_2 I'' + \frac{\beta SI}{S+I+R} - (\gamma+\delta)I; \tag{2.2}$$

$$cR' = d_3 R'' + \gamma I. \tag{2.3}$$

For cooperative systems, the minimum wave speed can be determined from linearization at low population densities [18,33,34]. Building on prior work, one of the authors [28] showed that, for cooperative systems and a large class of non-cooperative systems, the speeds of traveling wave solutions are simply the eigenvalues of the parameterized Jacobian matrix of its linearized system at the initial state and therefore the so-called minimum speed is simply the minimum of the eigenvalues. Lui [21] developed an analogous formula for recursion systems, which was later extended by Weinberger et al. [33] to cooperative systems of reaction-diffusion equations based on the time 1 maps. Here, we follow [28] and consider a system of reaction-diffusion equations

$$\mathbf{u}_t = \mathbf{d}\mathbf{u}_{xx} + \mathbf{f}(\mathbf{u}) \quad \text{for} \quad x \in \mathbb{R},\ t \geq 0, \tag{2.4}$$

where $\mathbf{u} = (u_1, \ldots, u_N)$, $\mathbf{d} = \text{diag}(d_1, d_2, \ldots, d_N)$ with $d_i > 0$ for $i = 1, \ldots, N$. The reaction function is given by

$$\mathbf{f}(u) = (f_1(u), f_2(u), \ldots, f_N(u)).$$

We are looking for a traveling wave solution of the form $\mathbf{u}(x+ct)$ with c being the traveling speed. Substituting $\mathbf{u}(x,t) = \mathbf{u}(x+ct)$ into (2.4) and letting $\xi = x+ct$, we obtain the following system.

$$\mathbf{d}\mathbf{u}''(\xi) - c\mathbf{u}'(\xi) + \mathbf{f}(\mathbf{u}(\xi)) = 0 \text{ for } \xi \in \mathbb{R}. \tag{2.5}$$

Now, we look for a solution of the form $(u_i) = \left(e^{\lambda\xi}\eta_\lambda^i\right)$, $\lambda > 0$, $\eta_\lambda = (\eta_\lambda^i) >> 0$ for the linearization of (2.5) at an initial equilibrium E, and arrive at the following system

$$\text{diag}\left(d_i\lambda^2 - c\lambda\right)\eta_\lambda + \mathbf{f}'(E)\eta_\lambda = 0,$$

which can be rewritten as the following eigenvalue problem.

$$\frac{1}{\lambda} A_\lambda \eta_\lambda = c\eta_\lambda, \tag{2.6}$$

where

$$A_\lambda = (a_\lambda^{i,j}) = \text{diag}(d_i\lambda^2) + \mathbf{f}'(E).$$

Let $\Psi(A_\lambda)$ be the spectral radius of A_λ for $\lambda \in [0, \infty)$, and define

$$\Phi(\lambda) := \frac{1}{\lambda}\Psi(A_\lambda) > 0.$$

Under the assumption that $\mathbf{f}'(E)$ has nonnegative off-diagonal elements and some other conditions, it was shown in [28] that $\Phi(\lambda)$ is a convex-like function and tends to ∞ at both limits of 0 and ∞. Therefore, $\Phi(\lambda)$ assumes the minimum over the domain $(0, \infty)$, which is the minimum speed of (2.4):

$$c^* = \inf_{\lambda>0} \Phi(\lambda) > 0.$$

For the SIR model (1.9–1.11), $\mathbf{f}$ is no longer cooperative and some of the off-diagonal elements of $\mathbf{f}'(E)$ may be negative. It remains an open question that under what additional conditions would $\Phi(\lambda)$ maintain convex-like properties.

Nevertheless, we can calculate the minimum wave speed of (1.9–1.11) from the largest eigenvalue of its linearized system at the initial equilibrium $E = (S_{-\infty}, 0, 0)$. From a biological perspective, we are interested in a traveling wave solution connecting $(S_{-\infty}, 0, 0)$ to another disease-free state $(S_\infty, 0, R_\infty)$. Now, we calculate the Jacobian of (1.9–1.11) at $(S_{-\infty}, 0, 0)$ as follows.

$$\mathbf{f}'(E) = \begin{pmatrix} 0 & -\beta & 0 \\ 0 & \beta-\gamma-\delta & 0 \\ 0 & \gamma & 0 \end{pmatrix}.$$

For $\lambda \geq 0$, three eigenvalues of the matrix

$$A_\lambda = \begin{pmatrix} d_1\lambda^2 & -\beta & 0 \\ 0 & d_2\lambda^2+\beta-\gamma-\delta & 0 \\ 0 & \gamma & d_3\lambda^2 \end{pmatrix}$$

are $d_1\lambda^2$, $d_2\lambda^2+\beta-\gamma-\delta$, $d_3\lambda^2$. The minimum wave speed can be defined as

$$c^* := \inf_{\lambda>0} \frac{d_2\lambda^2+\beta-\gamma-\delta}{\lambda} = 2\sqrt{d_2(\beta-\gamma-\delta)}.$$

The following theorem confirms that c^* is the cut-off speed for the existence of traveling solutions connecting the disease-free equilibrium $(S_{-\infty}, 0, 0)$ to another disease-free state $(S_\infty, 0, R_\infty)$. This implies that the minimum wave speed of (1.9–1.11) can be determined from its linearization at the initial disease-free equilibrium. Our main result is stated as follows.

Theorem 1 *Assume that the constants $d_i > 0$ with $i = 1, 2, 3$, $\beta > 0$, $\gamma > 0$ and $\delta \geq 0$. If*

$$d_3 < 2d_2, \tag{2.7}$$

then the minimum wave speed of (1.9–1.11) can be determined from its linearizaion at the initial disease-free equilibrium. Specifically, for any $S_{-\infty} > 0$, $R_0 := \beta/(\gamma+\delta) > 1$ and $c > c^* := 2\sqrt{d_2(\beta-\gamma-\delta)}$, there exist $S_\infty < S_{-\infty}$ and a traveling wave solution for (1.9–1.11) such that $S(-\infty) = S_{-\infty}$, $S(\infty) = S_\infty$, $I(\pm\infty) = 0$, $R(-\infty) = 0$ and $R(\infty) = \gamma(S_{-\infty} - S_\infty)/(\gamma+\delta)$. Furthermore, $S(x)$ is decreasing, $0 \leq I(x) \leq S_{-\infty} - S_\infty$ for $x \in \mathbb{R}$, $R(x)$ is increasing, and

$$\int_{-\infty}^{\infty} (\gamma+\delta)I(x)dx = \int_{-\infty}^{\infty} \frac{\beta S(x)I(x)}{S(x)+I(x)+R(x)}dx = c(S_{-\infty} - S_\infty). \tag{2.8}$$

On the other hand, if $c < c^*$ or $R_0 \leq 1$, then there does not exist a non-trivial and non-negative traveling wave solution for (1.9–1.11) such that $S(-\infty) = S_{-\infty}$, $S(\infty) < S_{-\infty}$, $I(\pm\infty) = 0$ and $R(-\infty) = 0$.

The technical condition (2.7) is similar to the first inequality of assumption (2.7) in [18]. It will be used in the construction of super-solutions and sub-solutions. The result also reveals that the basic reproduction number R_0 plays an essential role as in the corresponding spatial-homogenous ordinary differential system.

3 Existence of Traveling Wave Solutions

Throughout this section, we assume that $R_0 := \beta/(\gamma+\delta) > 1, c > c^* := 2\sqrt{d_2(\beta-\gamma-\delta)}$ and the inequality (2.7) is satisfied. It is noted that R_0 is the basic reproduction number for the ordinary differential system without diffusion [5,25]. Moreover, linearizing the equation for I at the point $(S_{-\infty}, 0, 0)$ gives the characteristic function

$$f(\lambda) := -d_2\lambda^2 + c\lambda - (\beta-\gamma-\delta). \tag{3.1}$$

We denote by

$$\lambda_0 := \frac{c - \sqrt{c^2 - 4d_2(\beta-\gamma-\delta)}}{2d_2} > 0 \tag{3.2}$$

the smaller positive root of the characteristic function $f(\lambda)$. It is readily seen that the inequality (2.7) implies

$$c - d_3\lambda_0 > 0. \tag{3.3}$$

Let α_1, α_2 and α_3 be three sufficiently large constants, we define the second-order differential operator D_i with $i = 1, 2, 3$ by

$$D_i h := -d_i h'' + ch' + \alpha_i h \tag{3.4}$$

for any $h \in C^2(\mathbb{R})$. Let

$$\lambda_i^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_i\alpha_i}}{2d_i} \qquad (\text{note that } \lambda_i^- < 0 < -\lambda_i^- < \lambda_i^+) \tag{3.5}$$

be the two roots of the function

$$f_i(\lambda) := -d_i\lambda^2 + c\lambda + \alpha_i. \tag{3.6}$$

Denote

$$\rho_i := d_i(\lambda_i^+ - \lambda_i^-) = \sqrt{c^2 + 4d_i\alpha_i}. \tag{3.7}$$

The inverse operator D_i^{-1} is given by the following integral representation

$$(D_i^{-1}h)(x) := \frac{1}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h(y)dy + \frac{1}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)}h(y)dy \tag{3.8}$$

for $h \in C_{\mu^-,\mu^+}(\mathbb{R})$ with $\mu^- > \lambda_i^-$ and $\mu^+ < \lambda_i^+$, where

$$C_{\mu^-,\mu^+}(\mathbb{R}) := \left\{ h \in C(\mathbb{R}) : \sup_{x\leq 0} |h(x)e^{-\mu^- x}| + \sup_{x\geq 0} |h(x)e^{-\mu^+ x}| < \infty \right\}. \tag{3.9}$$

It is readily seen from its integral representation in (3.8) that $D_i^{-1}h$ is differentiable and

$$(D_i^{-1}h)'(x) = \frac{\lambda_i^-}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h(y)dy + \frac{\lambda_i^+}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)}h(y)dy; \tag{3.10}$$

$$(D_i^{-1}h)''(x) = \frac{(\lambda_i^-)^2}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h(y)dy + \frac{(\lambda_i^+)^2}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)}h(y)dy - \frac{h(x)}{d_i}. \tag{3.11}$$

We choose

$$\alpha_1 > \beta, \quad \alpha_2 > \gamma + \delta \quad \text{and} \quad \alpha_3 > 0$$

be sufficiently large such that $|\lambda_i^-| = -\lambda_i^- > \lambda_0 > 0$ for $i = 1, 2, 3$. Given $\mu > \lambda_0 > 0$ such that $\mu < -\lambda_i^-$ for all $i = 1, 2, 3$, we have

$$\lambda_0 < \mu < -\lambda_i^- < \lambda_i^+, \quad i = 1, 2, 3$$

(see the definitions of λ_0 and λ_i^- in (3.2) and (3.5)) and

$$\lambda_i^- < -\mu < \mu < \lambda_i^+, \quad i = 1, 2, 3.$$

Now we can define the Banach space

$$B_\mu(\mathbb{R}, \mathbb{R}^n) := \underbrace{C_{-\mu,\mu}(\mathbb{R}) \times \cdots \times C_{-\mu,\mu}(\mathbb{R})}_{n} \tag{3.12}$$

equipped with the norm

$$|u|_\mu := \max_{1\le i\le n} \sup_{x\in\mathbb{R}} e^{-\mu|x|}|u_i(x)|, \tag{3.13}$$

where $u = (u_1, \cdots, u_n) \in B_\mu(\mathbb{R}, \mathbb{R}^n)$ with n being a positive integer. We then define a map $F = (F_1, F_2, F_3)$ on the space $B_\mu(\mathbb{R}, \mathbb{R}^3)$: given $u = (u_1, u_2, u_3) \in B_\mu(\mathbb{R}, \mathbb{R}^3)$, let

$$F_1(u_1, u_2, u_3) := D_1^{-1}[\alpha_1 u_1 - \beta u_1 u_2/(u_1 + u_2 + u_3)]; \tag{3.14}$$

$$F_2(u_1, u_2, u_3) := D_2^{-1}[\alpha_2 u_2 + \beta u_1 u_2/(u_1 + u_2 + u_3) - (\gamma + \delta)u_2]; \tag{3.15}$$

$$F_3(u_1, u_2, u_3) := D_3^{-1}[\alpha_3 u_3 + \gamma u_2]. \tag{3.16}$$

The following lemma shows that the fixed point of the map F is indeed a traveling wave solution.

Lemma 2 *Let $(S, I, R) \in B_\mu(\mathbb{R}, \mathbb{R}^3)$ be a fixed point of the map F, then (S, I, R) satisfies the traveling wave equations (2.1–2.3).*

Proof Set $h_1 := \alpha_1 S - \beta SI/(S + I + R)$. It follows from (3.7), (3.8), (3.10), (3.11) and the fact that $\lambda_1^\pm$ are the roots of $f_1(\lambda) = -d_1\lambda^2 + c + \alpha_1$ that

$$-d_1\big(D_1^{-1}h_1\big)'' + c\big(D_1^{-1}h_1\big)' + \alpha_1\big(D_1^{-1}h_1\big) = h_1.$$

Since (S, I, R) is a fixed point of F, it follows that $D_1^{-1}h_1 = S$. Thus, the above equation is the same as (2.1). Similarly, we can show that the other two Eqs. (2.2) and (2.3) are also satisfied. □

For $x \in \mathbb{R}$, we define super-solutions and sub-solutions as follows:

$$S_+(x) := S_{-\infty}; \tag{3.17}$$
$$S_-(x) := \max\left\{S_{-\infty}(1 - M_1 e^{\varepsilon_1 x}), 0\right\}; \tag{3.18}$$
$$I_+(x) := e^{\lambda_0 x}; \tag{3.19}$$
$$I_-(x) := \max\left\{e^{\lambda_0 x}(1 - M_2 e^{\varepsilon_2 x}), 0\right\}; \tag{3.20}$$
$$R_+(x) := \frac{\gamma}{c\lambda_0 - d_3\lambda_0^2} e^{\lambda_0 x}; \tag{3.21}$$
$$R_-(x) := \max\left\{\frac{\gamma}{c\lambda_0 - d_3\lambda_0^2} e^{\lambda_0 x}(1 - M_3 e^{\varepsilon_3 x}), 0\right\}, \tag{3.22}$$

where $M_1, M_2, M_3, \varepsilon_1, \varepsilon_2, \varepsilon_3$ are six positive constants to be determined in the following lemma. Its proof can be found in Appendix.

Lemma 3 *Given sufficiently large $M_1 > 0$, $M_2 > 0$, $M_3 > 0$ and sufficiently small $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, we have*

$$-\beta I_+ \geq -d_1 S_-'' + c S_-' \tag{3.23}$$

for $x \leq x_1 := -\varepsilon_1^{-1} \ln M_1$, and

$$\frac{\beta S_- I_-}{S_- + I_+ + R_+} - (\gamma + \delta) I_- \geq -d_2 I_-'' + c I_-' \tag{3.24}$$

for $x \leq x_2 := -\varepsilon_2^{-1} \ln M_2$, and

$$\gamma I_- \geq -d_3 R_-'' + c R_-' \tag{3.25}$$

for $x \leq x_3 := -\varepsilon_3^{-1} \ln M_3$.

With the aid of the super-solutions and sub-solutions, we are now ready to define a convex set Γ as

$$\Gamma := \left\{(S, I, R) \in B_\mu(\mathbb{R}, \mathbb{R}^3) :\ S_- \leq S \leq S_+ \ \&\ I_- \leq I \leq I_+ \&\ R_- \leq R \leq R_+\right\}. \tag{3.26}$$

Since $\mu > \lambda_0 > 0$, it is easily seen that Γ is uniformly bounded with respect to the norm $|\cdot|_\mu$ defined in (3.13). To prove invariance of the convex set Γ under the map F, we shall make use of the following results which were also proved in [29]. For completeness, its proof can be found in Appendix.

Lemma 4 ([29]) *Let $i = 1, 2, 3$. We have*

$$D_i^{-1}(D_i h) = h \tag{3.27}$$

for any $h \in C^2(\mathbb{R})$ such that $h, h', h'' \in C_{\mu^-,\mu^+}(\mathbb{R})$ with $\mu^- > \lambda_i^-$ and $\mu^+ < \lambda_i^+$. Let

$$g(x) := \max\{e^{\lambda x}(1 - M e^{\varepsilon x}), 0\}$$

for some $M > 0$, $\varepsilon > 0$ and λ such that $\lambda_i^- < \lambda < \lambda + \varepsilon < \lambda_i^+$, we have

$$D_i^{-1}(D_i g) \geq g. \tag{3.28}$$

Here $D_i g$ is understood as a piecewise defined function:

$$(D_i g)(x) = \begin{cases} f_i(\lambda) e^{\lambda x} - M f_i(\lambda + \varepsilon) e^{(\lambda+\varepsilon)x}, & x < x^* := -\varepsilon^{-1} \ln M; \\ 0, & x > x^* := -\varepsilon^{-1} \ln M. \end{cases}$$

Now we are ready to show that the convex set Γ defined in (3.26) is invariant under the map $F = (F_1, F_2, F_3)$ defined in (3.14), (3.15) and (3.16). Its proof can be found in Appendix.

Lemma 5 *The operator $F = (F_1, F_2, F_3)$ maps Γ into Γ, namely, for any $(S, I, R) \in B_\mu(\mathbb{R}, \mathbb{R}^3)$ such that $S_- \le S \le S_+$, $I_- \le I \le I_+$ and $R_- \le R \le R_+$, we have*

$$S_- \le F_1(S, I, R) \le S_+,$$

and

$$I_- \le F_2(S, I, R) \le I_+,$$

and

$$R_- \le F_3(S, I, R) \le R_+.$$

Before applying Schauder fixed point theorem, we shall verify that F is continuous and compact on Γ with respect to the norm $|\cdot|_\mu$ defined in (3.13). The proof is standard and can be found in Appendix.

Lemma 6 *The map $F = (F_1, F_2, F_3) : \Gamma \to \Gamma$ defined in (3.14), (3.15) and (3.16) is continuous and compact with respect to the norm $|\cdot|_\mu$ defined in (3.13).*

The following proposition gives the first part of our main theorem.

Proposition 7 *The map F has a fixed point $(S, I, R) \in \Gamma$ which satisfies the equations (2.1–2.3). As $x \to -\infty$, we have*

$$\begin{aligned} &S(x) \to S_{-\infty},\ I(x) \sim e^{\lambda_0 x},\ R(x) \sim \frac{\gamma e^{\lambda_0 x}}{c\lambda_0 - d_3\lambda_0^2}, \\ &S'(x),\ I'(x),\ R'(x),\ S''(x),\ I''(x),\ R''(x) \to 0. \end{aligned} \tag{3.29}$$

As $x \to \infty$, we have

$$\begin{aligned} &S(x) \to S_\infty < S_{-\infty},\ I(x) \to 0,\ R(x) \to \frac{\gamma(S_{-\infty} - S_\infty)}{\gamma + \delta}, \\ &S'(x),\ I'(x),\ R'(x),\ S''(x),\ I''(x),\ R''(x) \to 0. \end{aligned} \tag{3.30}$$

Moreover, $S(x)$ is decreasing, $0 \le I(x) \le S_{-\infty} - S_\infty$ for $x \in \mathbb{R}$, $R(x)$ is increasing, and

$$\int_{-\infty}^{\infty} (\gamma + \delta) I(x) dx = \int_{-\infty}^{\infty} \frac{\beta S(x) I(x)}{S(x) + I(x) + R(x)} dx = c(S_{-\infty} - S_\infty). \tag{3.31}$$

Proof The existence of fixed point follows from Lemmas 5, 6 and Schauder fixed point theorem. Namely, there exists $(S, I, R) \in B_\mu(R, \mathbb{R}^3)$ such that

$$S = F_1(S, I, R) = D_1^{-1}[\alpha_1 S - \beta SI/(S + I + R)]; \tag{3.32}$$

$$I = F_2(S, I, R) = D_2^{-1}[\alpha_2 I + \beta SI/(S + I + R) - (\gamma + \delta)I]; \tag{3.33}$$

$$R = F_3(S, I, R) = D_3^{-1}[\alpha_3 R + \gamma I]. \tag{3.34}$$

Since $S, I, R \in C_{-\mu,\mu}(\mathbb{R})$ and $\lambda_i^- < -\mu < \mu < \lambda_i^+$ for any $i = 1, 2, 3$, it is readily seen from (3.27) in Lemma 4 that

$$\begin{aligned} &D_1 S = \alpha_1 S - \beta SI/(S + I + R); \\ &D_2 I = \alpha_2 I + \beta SI/(S + I + R) - (\gamma + \delta)I; \\ &D_3 R = \alpha_3 R + \gamma I. \end{aligned}$$

Recalling the definition of D_i (with $i = 1, 2, 3$) in (3.4), we conclude that (S, I, R) satisfies the equations (2.1), (2.2) and (2.3). Since $S_- \le S \le S_+$, $I_- \le I \le I_+$ and $R_- \le R \le R_+$, we obtain from the definitions of $S_\pm$, $I_\pm$ and $R_\pm$ in (3.17–3.22) and the squeeze theorem that $S(x) \to S_{-\infty}$, $I(x) \sim e^{\lambda_0 x}$ and $R(x) \sim \gamma e^{\lambda_0 x}/(c\lambda_0 - d_3\lambda_0^2)$ as $x \to -\infty$. Furthermore, recall the integral representation (3.10) for the first derivative of $D_i^{-1}h$:

$$(D_i^{-1}h)'(x) = \frac{\lambda_i^-}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h(y)dy + \frac{\lambda_i^+}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)}h(y)dy$$

for any $h \in C_{-\mu,\mu}(\mathbb{R})$. We obtain from (3.32), (3.33), (3.34) and L'Hôpital's rule that $S'(x) \to 0$, $I'(x) \to 0$ and $R'(x) \to 0$ as $x \to -\infty$. Finally, from (2.1), (2.2) and (2.3), it follows that the second derivatives S'', I'' and R'' also vanish at $-\infty$. This gives (3.29).

Now we investigate asymptotic behaviors of S, I and R as $x \to \infty$. An integration of (2.1) from $-\infty$ to x gives

$$d_1 S'(x) = c[S(x) - S_{-\infty}] + \int_{-\infty}^{x} \frac{\beta S(y)I(y)}{S(y)+I(y)+R(y)}dy.$$

Since $S(x)$ is uniformly bounded, the integral on the right-hand side should be uniformly bounded; otherwise $S'(x) \to \infty$ as $x \to \infty$, which implies $S(x) \to \infty$ as $x \to \infty$, a contradiction. Thus, we obtain integrability of $\beta SI/(S+I+R)$ on $\mathbb{R}$, which together with the above equality implies that S' is uniformly bounded on $\mathbb{R}$. Note from (2.1) that

$$(e^{-cx/d_1}S')' = e^{-cx/d_1}(S'' - cS'/d_1) = e^{-cx/d_1}\beta SI/(S+I+R)/d_1.$$

Integrating the above equality from x to infinity gives

$$e^{-cx/d_1}S'(x) = -\int_{x}^{\infty} e^{-cy/d_1}\frac{\beta S(y)I(y)}{d_1[S(y)+I(y)+R(y)]}dy.$$

Hence, S is non-increasing. Furthermore, since S and I are non-trivial; see (3.29), the integral on the right-hand side of the above equality can not be identically zero, which implies $S'(x) < 0$ and $S(\infty) = S_\infty < S_{-\infty}$. We are now ready to study asymptotic behavior of $I(x)$ as $x \to \infty$. From (2.2), $I(-\infty) = 0$ and $I(x) \le I_+(x) = e^{\lambda_0 x}$, we have

$$I(x) = \frac{1}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)+R(y)}dy + \frac{1}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)+R(y)}dy, \tag{3.35}$$

where

$$\lambda^\pm := \frac{c \pm \sqrt{c^2 + 4d_2(\gamma+\delta)}}{2d_2}$$

and

$$\rho := d_2(\lambda^+ - \lambda^-) = \sqrt{c^2 + 4d_2(\gamma+\delta)}.$$

Remark that $\lambda^- < 0 < \lambda_0 < \lambda^+$ and $\lambda^\pm$ are the two roots of following equation

$$-d_2\lambda^2 + c\lambda + \gamma + \delta = 0.$$

We would also like to mention that the integral in (3.35) is well defined because of Lebesgue's dominated convergence theorem and uniform boundedness of $\beta SI/(S+I+R)$. Since

$\beta SI/(S+I+R)$ is integrable on $\mathbb{R}$, it follows from the integral equation (3.35) and Fubini's theorem that I is also integrable on $\mathbb{R}$, and

$$\int_{-\infty}^{\infty} I(x)dx = \frac{1}{\gamma+\delta}\int_{-\infty}^{\infty} \frac{\beta S(x)I(x)}{S(x)+I(x)+R(x)}dx. \tag{3.36}$$

Furthermore, since

$$\begin{aligned} I'(x) &= \frac{\lambda^-}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)+R(y)}dy \\ &\quad + \frac{\lambda^+}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)+R(y)}dy, \end{aligned}$$

we have from $\lambda^- < 0 < \lambda^+$, $\beta SI/(S+I+R) \le \beta I$ and $\rho = d_2(\lambda^+ - \lambda_-)$ that

$$|I'(x)| \le \frac{\beta}{d_2}\int_{-\infty}^{\infty} I(x)dx.$$

Since I' is uniformly bounded and $I \ge 0$ is integrable on $\mathbb{R}$, it is easily seen that $I(x) \to 0$ as $x \to \infty$; otherwise, we can find a number $\varepsilon > 0$, a sequence $x_n \to \infty$ and a number $\kappa > 0$ (since I' is uniformly bounded) such that $I(x) > \varepsilon$ for all $|x - x_n| < \kappa$, which contradicts the integrability of I on $\mathbb{R}$. By integrating (2.2) on the real line, it then follows from (3.29) and (3.36) that $I'(x) \to 0$ as $x \to \infty$ (noting that this can be also obtained from the integral representation of I' and L'Hôpital's rule). Again, from (2.2) we obtain $I''(x) \to 0$ as $x \to \infty$. Since $\beta SI/(S+I+R)$ is integrable on the real line, it is readily seen from (2.1) and (3.29) that S' is uniformly bounded, which in turn implies S'' is also uniformly bounded. Since $S' \le 0$ is integrable on $\mathbb{R}$, it can be shown that $S'(x) \to 0$ as $x \to \infty$. This, together with (2.1) gives $S''(x) \to 0$ as $x \to \infty$. Moreover, an integration of (2.1) on the real line yields

$$\int_{-\infty}^{\infty} \frac{\beta S(x)I(x)}{S(x)+I(x)+R(x)}dx = c(S_{-\infty} - S_{\infty}). \tag{3.37}$$

Solving the linear equation (2.3) gives

$$R(x) = \frac{\gamma}{c}\int_0^x I(y)dy + \frac{\gamma}{c}\int_x^0 e^{(c/d_3)(x-y)}I(y)dy + C_0 + C_1 e^{(c/d_3)x},$$

where C_0 and C_1 are constants of integration. Substituting x by $-\infty$, we obtain from $R(-\infty) = 0$ that

$$C_0 = \frac{\gamma}{c}\int_{-\infty}^{0} I(y)dy.$$

Furthermore, since $R(x) \le R_+(x) = \gamma e^{\lambda_0 x}/(c\lambda_0 - d_3\lambda_0^2)$ and $\lambda_0 < c/d_3$, we have $e^{-(c/d_3)x}R(x) \to 0$ as $x \to \infty$. Hence, it is readily seen that

$$C_1 = \frac{\gamma}{c}\int_0^{\infty} e^{-(c/d_3)y}I(y)dy.$$

Therefore, we obtain

$$R(x) = \frac{\gamma}{c}\int_{-\infty}^{x} I(y)dy + \frac{\gamma}{c}\int_x^{\infty} e^{(c/d_3)(x-y)}I(y)dy. \tag{3.38}$$

It follows from (3.36), (3.30), (3.37) and L'Hôpital's rule that

$$\lim_{x\to\infty} R(x) = \frac{\gamma}{c}\int_{-\infty}^{\infty} I(x)dx = \frac{\gamma}{\gamma+\delta}(S_{-\infty} - S_{\infty}).$$

Moreover, differentiating (3.38) once yields

$$R'(x) = \frac{\gamma}{d_3}\int_x^{\infty} e^{(c/d_3)(x-y)} I(y)dy > 0.$$

Note that $I(\infty) = 0$, we obtain from L'Hôpital's rule that

$$\lim_{x\to\infty} R'(x) = 0.$$

Consequently, it follows from (2.3) and $I(\infty) = 0$ that $R''(x) \to 0$ as $x \to \infty$. This proves (3.30).

Finally, we intend to prove the inequality $I(x) \le S_{-\infty} - S_{\infty}$ for all $x \in \mathbb{R}$. Since $I(x) \sim e^{\lambda_0 x}$ as $x \to -\infty$ and $I(x) \to 0$ as $x \to \infty$, we can define

$$J(x) := I(x) + \frac{\gamma+\delta}{c}\int_{-\infty}^{x} I(y)dy + \frac{\gamma+\delta}{c}\int_x^{\infty} e^{(c/d_2)(x-y)} I(y)dy. \tag{3.39}$$

It follows from (3.29), (3.30), (3.36), (3.37) and L'Hôpital's rule that

$$\lim_{x\to-\infty} J(x) = 0, \quad \lim_{x\to\infty} J(x) = \frac{\gamma+\delta}{c}\int_{-\infty}^{\infty} I(x)dx = S_{-\infty} - S_{\infty}.$$

Similarly, by differentiating (3.39) once, we obtain from the asymptotic formulas (3.29–3.30) and L'Hôpital's rule that

$$J'(x) = I'(x) + \frac{\gamma+\delta}{d_2}\int_x^{\infty} e^{(c/d_2)(x-y)} I(y)dy$$

and

$$\lim_{x\to-\infty} J'(x) = 0, \quad \lim_{x\to\infty} J'(x) = 0.$$

Furthermore, by differentiating (3.39) twice, it is readily seen from the differential equation for I in (2.2) that

$$-d_2 J'' + cJ' = -d_2 I'' + cI' + (\gamma+\delta)I = \beta SI/(S+I+R).$$

An integration of the above equation from x to ∞ gives

$$J'(x) = \frac{1}{d_2}\int_x^{\infty} e^{(c/d_2)(x-y)} \frac{\beta S(y)I(y)}{S(y)+I(y)+R(y)} dy > 0.$$

Here we have used the fact that $J'(\infty) = 0$. Since $J(\infty) = S_{-\infty} - S_{\infty}$, we obtain from the above inequality that $J(x) \le S_{-\infty} - S_{\infty}$ for all $x \in \mathbb{R}$. Since $I(x) \le J(x)$ by definition (3.39), it follows that $I(x) \le S_{-\infty} - S_{\infty}$ for all $x \in \mathbb{R}$. This ends the proof. □

4 Non-existence of Traveling Wave Solution

It is easily seen that the traveling wave solution (S, I, R) (if exists) of (2.1–2.3) satisfies the following integral equation (noting that $I(\pm\infty) = 0$)

$$I(x) = \frac{1}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)+R(y)}dy + \frac{1}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)+R(y)}dy, \tag{4.1}$$

where

$$\lambda^{\pm} := \frac{c \pm \sqrt{c^2 + 4d_2(\gamma+\delta)}}{2d_2}$$

and

$$\rho := d_2(\lambda^+ - \lambda^-) = \sqrt{c^2 + 4d_2(\gamma+\delta)}.$$

Remark that $\lambda^- < 0 < \lambda^+$ and $\lambda^{\pm}$ are the two roots of following equation

$$-d_2\lambda^2 + c\lambda + \gamma + \delta = 0.$$

Note that the integral in (4.1) is well defined because $\beta SI/(S + I + R)$ vanishes at infinity. By (4.1), the derivative of I has the following integral representation:

$$\begin{aligned} I'(x) = {} & \frac{\lambda^-}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)+R(y)}dy \\ & + \frac{\lambda^+}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)+R(y)}dy. \end{aligned}$$

An application of L'Hôpital's rule to the above equation yields $I'(\pm\infty) = 0$. Applying this to (2.2) gives $I''(\pm\infty) = 0$. We list the asymptotic behavior of I as below.

$$I(\pm\infty) = 0,\ \ I'(\pm\infty) = 0,\ \ I''(\pm\infty) = 0. \tag{4.2}$$

The following two propositions give the second statement in our main theorem.

Proposition 8 *If $R_0 := \beta/(\gamma + \delta) > 1$ and $c < c^* := 2\sqrt{d_2(\beta - \gamma - \delta)}$, then there does not exist a non-trivial and non-negative traveling wave solution of (2.1), (2.2) and (2.3) such that $S(-\infty) = S_{-\infty}$, $S(\infty) < S_{-\infty}$, $I(\pm\infty) = 0$ and $R(-\infty) = 0$.*

Proof We prove the statement by contradiction. Let (S, I, R) be a solution to (2.1), (2.2) and (2.3). Based on the argument at the beginning of this section, we have the asymptotic behavior of I as listed in (4.2). Since $\beta S(x)/[S(x) + I(x) + R(x)] \to \beta$ as $x \to -\infty$, there exists a number $\bar{x}$ such that

$$\beta S(x)/[S(x) + I(x) + R(x)] - \gamma - \delta > \sigma := (\beta - \gamma - \delta)/2 > 0$$

for all $x < \bar{x}$. Applying this to (2.2) yields

$$cI'(x) - d_2 I''(x) > \sigma I(x) \geq 0 \tag{4.3}$$

for all $x < \bar{x}$. Since $cI(x) - d_2 I'(x)$ is bounded as $x \to -\infty$ by (4.2), it follows that $cI'(x) - d_2 I''(x)$ is integrable at $-\infty$. Lebesgue's dominated convergence theorem and the above inequality implies that $I(x)$ is also integrable at $-\infty$. Define

$$K(x) := \int_{-\infty}^{x} I(y)dy.$$

An integration of (4.3) yields

$$\sigma K(x) \leq cI(x) - d_2 I'(x)$$

for all $x < \bar{x}$. A further integration of the above inequality, together with non-negativeness of I gives

$$\int_{-\infty}^{x} K(y)dy \leq (c/\sigma)K(x)$$

for all $x < \bar{x}$. Since K is non-decreasing, we have

$$\eta K(x-\eta) \leq \int_{x-\eta}^{x} K(y)dy \leq (c/\sigma)K(x)$$

for all $\eta > 0$ and all $x < \bar{x}$. Hence, there exists a large $\eta > 0$ such that

$$K(x-\eta) < K(x)/2$$

for all $x < \bar{x}$. Denote $\mu_0 := (\ln 2)/\eta > 0$ and let

$$L(x) := e^{-\mu_0 x} K(x).$$

It follows that

$$L(x-\eta) < L(x)$$

for all $x < \bar{x}$, which implies $L(x) = e^{-\mu_0 x} K(x)$ is bounded as $x \to -\infty$. On account of (4.2), it follows from (4.3) that

$$cI'(x) > d_2 I''(x), \ cI(x) > d_2 I'(x), \ cK(x) > d_2 I(x).$$

Hence, we conclude that $e^{-\mu_0 x} I(x)$, $e^{-\mu_0 x} I'(x)$ and $e^{-\mu_0 x} I''(x)$ are all bounded as $x \to -\infty$. In view of (4.2), they are actually uniformly bounded on the whole real line. Moreover, since $I(x)/[S(x)+I(x)+R(x)] \leq 1$ and $S(x)+I(x)+R(x) \to S_{-\infty}$ as $x \to -\infty$, $e^{-\mu_0 x} I(x)/[S(x)+I(x)+R(x)]$ is also uniformly bounded on $\mathbb{R}$. Noting that $R(-\infty) = 0$, we solve the linear equation (2.3) and obtain

$$R(x) = \frac{\gamma}{c}\int_{-\infty}^{x} I(y)dy + \frac{\gamma}{c}\int_{x}^{0} e^{(c/d_3)(x-y)} I(y)dy + C_1 e^{(c/d_3)x},$$

where C_1 is a constant of integration. Note that $e^{-\mu_0 x} I(x)$ is uniformly bounded as $x \to -\infty$. By choosing $\mu_1 > 0$ such that $\mu_1 < \min\{\mu_0, c/d_3\}$, we have for any $x < 0$,

$$\begin{aligned} e^{-\mu_1 x} R(x) &= \frac{\gamma}{c}\int_{-\infty}^{x} e^{-\mu_1(x-y)} e^{-\mu_1 y} I(y)dy \\ &\quad + \frac{\gamma}{c}\int_{x}^{0} e^{(c/d_3-\mu_1)(x-y)} e^{-\mu_1 y} I(y)dy + C_1 e^{(c/d_3-\mu_1)x} \\ &\leq \frac{\gamma}{c}\int_{-\infty}^{x} e^{-\mu_1 y} I(y)dy + \frac{\gamma}{c}\int_{x}^{0} e^{-\mu_1 y} I(y)dy + C_1 \\ &= \frac{\gamma}{c}\int_{-\infty}^{0} e^{-\mu_1 y} I(y)dy + C_1. \end{aligned}$$

Since $e^{-\mu_0 x} I(x)$ is uniformly bounded as $x \to -\infty$ and $\mu_1 < \mu_0$, it follows from the above inequality that $e^{-\mu_1 x} R(x)$ is uniformly bounded as $x \to -\infty$. Therefore, $e^{-\mu_1 x} R(x)/[S(x)+I(x)+R(x)]$ is uniformly bounded on $\mathbb{R}$.

Now, we can introduce two-side Laplace transform on the equation (2.2):

$$f(\mu)\int_{-\infty}^{\infty} e^{-\mu x} I(x)dx = -\int_{-\infty}^{\infty} e^{-\mu x} I(x)\frac{\beta[I(x)+R(x)]}{S(x)+I(x)+R(x)}dx,$$

where f is the characteristic function defined in (3.1). The integrals on both side of the above equality are well defined for any $\mu \in (0, \mu_0)$. Since $e^{-\mu_1 x} R(x)/[S(x)+I(x)+R(x)]$ and $e^{-\mu_0 x} I(x)/[S(x)+I(x)+R(x)]$ are uniformly bounded on the real line and $f(\mu)$ is always negative for all $\mu \in \mathbb{R}$ (noting that $c < c^* = 2\sqrt{d_2(\beta-\gamma-\delta)}$), the two Laplace integrals can be analytically continued to the whole right half plane; otherwise the integral on the left has a singularity at $\mu = \mu^* \in \mathbb{R}$ and it is analytic for all $\mu < \mu^*$ (cf. [3,31,35]). However, since $e^{-\mu_1 x}[I(x)+R(x)]/[S(x)+I(x)+R(x)]$ is uniformly bounded, the integral on the right is actually analytic for all $\mu < \mu^* + \mu_1$, a contradiction. Thus, the above equality holds for all $\mu > 0$ and can be rewritten as

$$\int_{-\infty}^{\infty} e^{-\mu x} I(x)\left\{f(\mu)+\frac{\beta[I(x)+R(x)]}{S(x)+I(x)+R(x)}\right\}dx = 0.$$

This again leads to a contradiction because $f(\mu)+\beta[I(x)+R(x)]/[S(x)+I(x)+R(x)] \to -\infty$ as $\mu \to \infty$, but $e^{-\mu x} I(x)$ is always non-negative for all $\mu \in \mathbb{R}$; see [3,31] for early ideas in different settings. Thus, we conclude the proof. □

Proposition 9 *If $R_0 := \beta/(\gamma+\delta) \le 1$, then there does not exist a non-trivial and non-negative traveling wave solution of (2.1), (2.2) and (2.3) such that $S(-\infty) = S_{-\infty}$, $S(\infty) < S_{-\infty}$, $I(\pm\infty) = 0$ and $R(-\infty) = 0$.*

Proof Again, we prove by contradiction. Let (S, I, R) be a solution to (2.1), (2.2) and (2.3). Based on the argument at the beginning of this section, we have the asymptotic behavior of I as listed in (4.2). If $R_0 := \beta/(\gamma+\delta) \le 1$, then $\frac{\beta S(x)I(x)}{S(x)+I(x)+R(x)} \le (\gamma+\delta)I(x)$ for all $x \in \mathbb{R}$. From (2.2) we have

$$\frac{d}{dx}\left[e^{-(c/d_2)x}\frac{d}{dx}I(x)\right] = -\frac{1}{d_2}e^{-(c/d_2)x}\left[\frac{\beta S(x)I(x)}{S(x)+I(x)+R(x)} - (\gamma+\delta)I(x)\right] \ge 0,$$

which implies that the function $e^{-(c/d_2)x}I'(x)$ is non-decreasing. Since $I'(\infty) = 0$ by (4.2) and $e^{-(c/d_2)x} \to 0$ as $x \to \infty$, it follows that $I'(x) \le 0$ for all $x \in \mathbb{R}$. Again from $I(\pm\infty) = 0$ in (4.2) we obtain $I(x) = 0$ for all $x \in \mathbb{R}$, a contradiction. □

One may prove the nonexistence results in a different way by analyzing the Jacobian matrix of the six-dimensional first-order linearized system of the traveling wave equation for (S, I, R, S', I', R') at the equilibrium with $S = S_{-\infty}$ and $I = R = S' = I' = R' = 0$. However, we prefer to use the technique of Laplace transformation which could be extended to the study of high-dimensional systems with delays.

5 Discussion

Broadly speaking, there are three types of interspecific interactions: predator-prey, competition, and mutualism. In general, mutualism gives rise to cooperative systems whose dynamics are better understood. In particular, the works by Lui [21] and Weinberger et al. [18,33,34] assure that the spreading speeds of cooperative systems can be determined by the corresponding linearized systems. Such a phenomenon is called the linear conjecture.

Unlike the cooperative systems, there have been counterexamples in competitive systems and predator-prey systems when the linear conjecture is not true; see [9,11,14]. It is known that some competition models can be converted into cooperative systems and therefore are linearly determinant with some appropriate assumptions [18,20,27]. For predator-prey systems, however, the problem of linear conjecture is more challenging [4,15]. In this paper, we show that the minimum traveling speed for the general diffusive model (1.9–1.11) with non-constant total population can be determined by its linearization at the initial disease-free equilibrium. This result may shed light on the linear conjecture for predator-prey systems.

Our method is mainly based on the Schauder fixed point theorem for equivalent non-monotone abstract operators. Similar ideas have been used in [29,31]. Related methods can also be found in other studies such as [20,22,26,30]. We remark that the general diffusive Kermack–McKendrick SIR model (1.9–1.11) involves three unknown variables S, I, R, and several significantly new ingredients have been introduced in our proof. Specifically, one of the challenging tasks is to construct and verify a suitable invariant convex set of three dimensions for the non-monotone operators. The approach in this paper provides a promising method to deal with high dimensional epidemic models. It would be difficult, if possible, to investigate traveling waves for high dimensional models with phase portrait analysis.

One might attempt to solve for R in terms of an integral of I and then reduce (2.1–2.3) to a two-dimensional system of S, I. It is unclear whether such approach can be used to prove the existence of traveling waves. However, it would make the model more complicated and prevent us from seeing biological meanings of R. The construction of a three-dimensional invariant set seems to be a better option for this problem as our results also involve nonexistence results of traveling waves, and more importantly, we provide an effective approach to deal with traveling waves of high dimensional epidemic models.

In our model, we have chosen the incidence function as standard incidence. We note that the existence of traveling wave solutions of our diffusive epidemic model is still determined by the basic reproduction number of the corresponding non-diffusive system, a phenomenon which has also been observed in the mass-action incidence case. In addition, the minimum wave speed of our diffusive epidemic model can be rewritten as $c^* = 2\sqrt{d_2(\gamma+\delta)(R_0-1)}$, which is analogous to the formula obtained for the model with mass-action incidence function [13]. It would be interesting to further investigate spatial epidemic models with more general incidence functions and find the conditions under which similar results on traveling wave solutions are still valid. We leave this problem for a future work.

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Appendix

Proof of Lemma 3

Proof In view of (3.18) and (3.19), the first inequality (3.23) is the same as

$$-\beta e^{\lambda_0 x} \geq S_{-\infty} e^{\varepsilon_1 x}\left(d_1 M_1 \varepsilon_1^2 - c M_1 \varepsilon_1\right)$$

for all $x \leq x_1 := -\varepsilon_1^{-1} \ln M_1$. The above inequality can be written as

$$M_1 \varepsilon_1 (c - d_1 \varepsilon_1) \geq \beta e^{(\lambda_0 - \varepsilon_1)x}.$$

Note that $x \le x_1 := -\varepsilon_1^{-1} \ln M_1$. It suffices to prove

$$M_1 \varepsilon_1 (c - d_1 \varepsilon_1) \ge \beta M_1^{-(\lambda_0 - \varepsilon_1)/\varepsilon_1},$$

which is obviously true if we choose $\varepsilon_1 > 0$ such that $\varepsilon_1 < \min\{\lambda_0, c/d_1\}$ and then let M_1 be sufficiently large.

Now we intend to prove the second inequality (3.24), which, by subtracting both sides by $(\beta - \gamma - \delta) I_-$, is the same as

$$-\frac{\beta I_-(I_+ + R_+)}{S_- + I_+ + R_+} \ge -d_2 I_-'' + c I_-' - (\beta - \gamma - \delta) I_- = -M_2 f(\lambda_0 + \varepsilon_2) e^{(\lambda_0 + \varepsilon_2)x},$$

where f is defined in (3.1) with λ_0 as its smaller root. Note that f is concave down, we can choose a sufficiently small $\varepsilon_2 \in (0, \varepsilon_1)$ such that $f(\lambda_0 + \varepsilon_2) > 0$. Then, we assume M_2 is sufficiently large that $x_2 < x_1$ holds. It suffices to show

$$M_2 f(\lambda_0 + \varepsilon_2) e^{(\lambda_0 + \varepsilon_2)x} \ge \frac{\beta I_+(I_+ + R_+)}{S_-},$$

which, in view of (3.18), (3.19) and (3.21), is equivalent to

$$M_2 f(\lambda_0 + \varepsilon_2) S_{-\infty} (1 - M_1 e^{\varepsilon_1 x}) \ge \frac{\beta \left(\gamma + c\lambda_0 - d_3 \lambda_0^2\right)}{c\lambda_0 - d_3 \lambda_0^2} e^{(\lambda_0 - \varepsilon_2)x}.$$

Noting that $x \le x_2 := -\varepsilon_2^{-1} \ln M_2$, we only need to prove

$$M_2 f(\lambda_0 + \varepsilon_2) S_{-\infty} (1 - M_1 M_2^{-\varepsilon_1/\varepsilon_2}) \ge \frac{\beta \left(\gamma + c\lambda_0 - d_3 \lambda_0^2\right)}{c\lambda_0 - d_3 \lambda_0^2} M_2^{-(\lambda_0 - \varepsilon_2)/\varepsilon_2},$$

which is true for large M_2 because as $M_2 \to \infty$, the left-hand side tends to infinity and the right-hand side vanishes (recall that $0 < \varepsilon_2 < \varepsilon_1 < \lambda_0$).

Finally, we are ready to verify the last inequality (3.25). First, since $c - d_3 \lambda_0 > 0$ in (3.3), we can choose $\varepsilon_3 \in (0, \varepsilon_2)$ so small that $c - d_3(\lambda_0 + \varepsilon_3) > 0$. In view of (3.20) and (3.22), the inequality (3.25) can be written as

$$\begin{aligned} &\gamma e^{\lambda_0 x} \left(1 - M_2 e^{\varepsilon_2 x}\right) \\ &\quad \ge \frac{\gamma}{c\lambda_0 - d_3 \lambda_0^2} \left\{ e^{\lambda_0 x} \left(c\lambda_0 - d_3 \lambda_0^2\right) - M_3 e^{(\lambda_0 + \varepsilon_3)x} \left[c(\lambda_0 + \varepsilon_3) - d_3 (\lambda_0 + \varepsilon_3)^2 \right] \right\}, \end{aligned}$$

which is equivalent to

$$\frac{c(\lambda_0 + \varepsilon_3) - d_3(\lambda_0 + \varepsilon_3)^2}{c\lambda_0 - d_3 \lambda_0^2} M_3 \ge M_2 e^{(\varepsilon_2 - \varepsilon_3)x}.$$

Note that $\varepsilon_3 < \varepsilon_2$ and $x \le x_3 := -\varepsilon_3^{-1} \ln M_3$, it suffices to prove the above inequality for $x = x_3$:

$$\frac{c(\lambda_0 + \varepsilon_3) - d_3(\lambda_0 + \varepsilon_3)^2}{c\lambda_0 - d_3 \lambda_0^2} M_3 \ge M_2 M_3^{-(\varepsilon_2 - \varepsilon_3)/\varepsilon_3}.$$

This is true for large M_3 because as $M_3 \to \infty$, the left-hand side tends to infinity and the right-hand side vanishes. This ends the proof of our lemma. □

Proof of Lemma 4

Proof It follows from the definitions of D_i and D_i^{-1} in (3.4) and (3.8) that

$$[D_i^{-1}(D_i h)](x) = \frac{1}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}[-d_i h''(y) + ch'(y) + \alpha_i h(y)]dy + \frac{1}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)}[-d_i h''(y) + ch'(y) + \alpha_i h(y)]dy.$$

Making use of integration by parts, we obtain

$$\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h'(y)dy = h(x) + \lambda_i^- \int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h(y)dy,$$

and

$$\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h''(y)dy = h'(x) + \lambda_i^- h(x) + (\lambda_i^-)^2 \int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h(y)dy.$$

Therefore, we have

$$\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}[-d_i h''(y) + ch'(y) + \alpha_i h(y)]dy = -d_i h'(x) + (-d_i\lambda_i^- + c)h(x).$$

Here we have used the fact that λ_i^- is a root of the function $f_i(\lambda) = -d_i\lambda^2 + c\lambda + \alpha_i$; see (3.5) and (3.6). Similarly, it can be shown that

$$\int_{x}^{\infty} e^{\lambda_i^+(x-y)}[-d_i h''(y) + ch'(y) + \alpha_i h(y)]dy = d_i h'(x) + (d_i\lambda_i^+ - c)h(x).$$

Applying the above two equalities to the expression of $D_i^{-1}(D_i h)$ gives

$$\left[D_i^{-1}(D_i h)\right](x) = \frac{d_i\left(\lambda_i^+ - \lambda_i^-\right)}{\rho_i}h(x) = h(x),$$

where in the last equality we have used the definition of ρ_i in (3.7). This proves (3.27).

Let $x^* := -\ln M/\varepsilon$ be the point where g is not differentiable. Recall from (3.5) and (3.6) that

$$f_i(k) := -d_i k^2 + ck + \alpha_i = d_i(k - \lambda_i^-)(\lambda_i^+ - k). \tag{6.1}$$

It is easily seen from (3.4) that

$$(D_i g)(x) = \begin{cases} f_i(\lambda)e^{\lambda x} - Mf_i(\lambda+\varepsilon)e^{(\lambda+\varepsilon)x}, & x < x^*, \\ 0, & x > x^*. \end{cases} \tag{6.2}$$

To prove (3.28), we will consider the two cases $x \le x^*$ and $x \ge x^*$ respectively. When $x \le x^*$, we have from (3.8) and (6.2) that

$$[D_i^{-1}(D_i g)](x) = f_i(\lambda)A(\lambda) - Mf_i(\lambda+\varepsilon)A(\lambda+\varepsilon), \tag{6.3}$$

where

$$A(k) := \frac{1}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)+ky}dy + \frac{1}{\rho_i}\int_{x}^{x^*} e^{\lambda_i^+(x-y)+ky}dy = \frac{e^{kx}(\lambda_i^+ - \lambda_i^-)}{\rho_i(k-\lambda_i^-)(\lambda_i^+ - k)} - \frac{e^{kx^*+\lambda_i^+(x-x^*)}}{\rho_i(\lambda_i^+ - k)}$$

for $k = \lambda$ or $\lambda + \varepsilon$. In view of (3.7) and (6.1), it follows from the above equality that

$$f_i(k)A(k) = e^{kx} - \frac{k - \lambda_i^-}{\lambda_i^+ - \lambda_i^-} e^{kx^* + \lambda_i^+(x - x^*)}.$$

Applying this to (6.3) and on account of $Me^{\varepsilon x^*} = 1$, we obtain

$$\begin{aligned}
\left[D_i^{-1}(D_i g)\right](x) &= \left[e^{\lambda x} - \frac{\lambda - \lambda_i^-}{\lambda_i^+ - \lambda_i^-} e^{\lambda x^* + \lambda_i^+(x - x^*)}\right] \\
&\quad - \left[Me^{(\lambda+\varepsilon)x} - \frac{\lambda + \varepsilon - \lambda_i^-}{\lambda_i^+ - \lambda_i^-} e^{\lambda x^* + \lambda_i^+(x - x^*)}\right] \\
&= \left[e^{\lambda x} - Me^{(\lambda+\varepsilon)x}\right] + \frac{\varepsilon}{\lambda_i^+ - \lambda_i^-} e^{\lambda x^* + \lambda_i^+(x - x^*)} \\
&\geq e^{\lambda x} - Me^{(\lambda+\varepsilon)x}.
\end{aligned}$$

This proves (3.28) for $x \leq x^*$. When $x \geq x^*$, we have from (3.8) and (6.2) that

$$\left[D_i^{-1}(D_i g)\right](x) = f_i(\lambda)B(\lambda) - Mf_i(\lambda + \varepsilon)B(\lambda + \varepsilon), \tag{6.4}$$

where

$$B(k) := \frac{1}{\rho_i} \int_{-\infty}^{x^*} e^{\lambda_i^-(x - y) + ky} dy = \frac{e^{kx^* + \lambda_i^-(x - x^*)}}{\rho_i(k - \lambda_i^-)}$$

for $k = \lambda$ or $\lambda + \varepsilon$. In view of (3.7) and (6.1), it follows from the above equality that

$$f_i(k)B(k) = \frac{\lambda_i^+ - k}{\lambda_i^+ - \lambda_i^-} e^{kx^* + \lambda_i^-(x - x^*)}.$$

Applying this to (6.4) and on account of $Me^{\varepsilon x^*} = 1$, we obtain

$$\begin{aligned}
[D_i^{-1}(D_i g)](x) &= \frac{\lambda_i^+ - \lambda}{\lambda_i^+ - \lambda_i^-} e^{\lambda x^* + \lambda_i^-(x - x^*)} - \frac{\lambda_i^+ - \lambda - \varepsilon}{\lambda_i^+ - \lambda_i^-} e^{\lambda x^* + \lambda_i^-(x - x^*)} \\
&= \frac{\varepsilon}{\lambda_i^+ - \lambda_i^-} e^{\lambda x^* + \lambda_i^-(x - x^*)} \\
&\geq 0.
\end{aligned}$$

This gives (3.28) in the case $x \geq x^*$. □

Proof of Lemma 5

Proof Throughout this proof, we will frequently use the inequalities $0 < S/(S+I+R) < 1$ and $0 < I/(S+I+R) < 1$. Since $\alpha_1 S - \beta SI/(S+I+R) \leq \alpha_1 S_+ = D_1 S_+$; see the definition of D_1 in (3.4), we obtain from (3.14) and (3.27) that

$$F_1(S, I, R) \leq D_1^{-1}(D_1 S_+) = S_+.$$

By (3.23) in Lemma 3, we have for $x \leq x_1$,

$$\alpha_1 S - \beta SI/(S+I+R) \geq \alpha_1 S_- - \beta I_+ \geq \alpha_1 S_- - d_1 S_-'' + cS_-' = D_1 S_-.$$

When $x \geq x_1$, it follows from $\alpha_1 > \beta$ (recalling the choice of α_1 in the paragraph after (3.11)) and $S_-(x) = 0$ that

$$\alpha_1 S - \beta SI/(S + I + R) \geq (\alpha_1 - \beta)S \geq 0 = D_1 S_-.$$

Coupling the above two inequalities and making use of (3.28) yield

$$F_1(S, I, R) \geq D_1^{-1}(D_1 S_-) \geq S_-.$$

Since $\alpha_2 > \gamma + \delta$ (by the choice of α_2) and λ_0 is a root of f defined in (3.1), we have

$$\begin{aligned}\alpha_2 I + \beta SI/(S + I + R) - (\gamma + \delta)I &\leq \alpha_2 I_+ + \beta I_+ - (\gamma + \delta)I_+ \\ &= \alpha_2 I_+ - d_2 I_+'' + cI_+' = D_2 I_+.\end{aligned}$$

In view of (3.27), we obtain from the above inequality that

$$F_2(S, I, R) \leq D_2^{-1}(D_2 I_+) = I_+.$$

By (3.24) in Lemma 3 and monotonicity of $\beta SI/(S + I + R)$ in S, we obtain

$$\begin{aligned}\alpha_2 I + \beta SI/(S + I + R) - (\gamma + \delta)I &\geq \alpha_2 I_- + \beta S_- I_-/(S_- + I_+ + R_+) - (\gamma + \delta)I_- \\ &\geq \alpha_2 I_- - d_2 I_-'' + cI_-' \\ &= D_2 I_-\end{aligned}$$

for $x \leq x_2$. When $x \geq x_2$, it is readily seen from $\alpha_2 > \gamma + \delta$ and $I_-(x) = 0$ that

$$\alpha_2 I + \beta SI/(S + I + R) - (\gamma + \delta)I \geq \alpha_2 I - (\gamma + \delta)I \geq 0 = D_2 I_-.$$

A combination of the above two inequalities and (3.28) yields

$$F_2(S, I, R) \geq D_2^{-1}(D_2 I_-) \geq I_-.$$

From the definitions of I_+ and R_+ in (3.19) and (3.21), we have

$$\alpha_3 R + \gamma I \leq \alpha_3 R_+ + \gamma I_+ = \alpha_3 R_+ + cR_+' - d_3 R_+'' = D_3 R_+.$$

Thus, it follows from (3.16) and (3.27) that

$$F_3(S, I, R) \leq D_3^{-1}(D_3 R_+) = R_+.$$

When $x \leq x_3$, we obtain from (3.25) that

$$\alpha_3 R + \gamma I \geq \alpha_3 R_- + \gamma I_- \geq \alpha_3 R_- + cR_-' - d_3 R_-'' = D_3 R_-.$$

When $x \geq x_3$, we have $R_-(x) = 0$ and

$$\alpha_3 R + \gamma I \geq 0 = D_3 R_-.$$

Hence, it follows from (3.28) that

$$F_3(S, I, R) \geq D_3^{-1}(D_3 R_-) \geq R_-.$$

This ends our proof of the lemma. □

Proof of Lemma 6

Proof Note that the standard incidence function $\beta SI/(S+I+R)$ has bounded partial derivatives with respect to S, I and R. For example, the partial derivative of $\beta SI/(S+I+R)$ with respect to S is $\beta(I+R)I/(S+I+R)^2$, which is bounded by β. Similarly, we can show that the partial derivatives with respect to I and R are also bounded by β. Therefore, for any $(S_1, I_1, R_1) \in \Gamma$ and $(S_2, I_2, R_2) \in \Gamma$, we have

$$\left|\frac{\beta S_1 I_1}{S_1+I_1+R_1} - \frac{\beta S_2 I_2}{S_2+I_2+R_2}\right| \leq \beta(|S_1-S_2|+|I_1-I_2|+|R_1-R_2|).$$

It is readily seen that

$$\begin{aligned}&\left|\left(\alpha_1 S_1 - \frac{\beta S_1 I_1}{S_1+I_1+R_1}\right) - \left(\alpha_1 S_2 - \frac{\beta S_2 I_2}{S_2+I_2+R_2}\right)\right|\\ &\quad\leq (\alpha_1+\beta)(|S_1-S_2|+|I_1-I_2|+|R_1-R_2|).\end{aligned}$$

Consequently, we obtain from the definition (3.14) that

$$\begin{aligned}&|F_1(S_1, I_1, R_1)(x) - F_1(S_2, I_2, R_2)(x)|e^{-\mu|x|}\\ &\quad\leq \frac{\alpha_1+\beta}{\rho_1}(|S_1-S_2|_\mu+|I_1-I_2|_\mu+|R_1-R_2|_\mu)C(x),\end{aligned}$$

where

$$C(x) := e^{-\mu|x|}\left[\int_{-\infty}^{x} e^{\lambda_1^-(x-y)+\mu|y|}dy + \int_{x}^{\infty} e^{\lambda_1^+(x-y)+\mu|y|}dy\right].$$

Here $S_1 - S_2 \in C_{-\mu,\mu}(\mathbb{R}) = B_\mu(\mathbb{R}, \mathbb{R})$ and $|S_1-S_2|_\mu = \sup_{x\in\mathbb{R}} e^{-\mu|x|}|S_1(x) - S_2(x)|$; see (3.12) and (3.13). To prove the continuity of F_1, it suffices to show that $C(x)$ is uniformly bounded for $x \in \mathbb{R}$. Since $\lambda_1^- < -\mu < \mu < \lambda_1^+$, applying L'Hôpital's rule to the above formula yields

$$C(-\infty) = \frac{1}{\mu+\lambda_1^+} - \frac{1}{\mu+\lambda_1^-}$$

and

$$C(\infty) = \frac{1}{\lambda_1^+-\mu} + \frac{1}{\mu-\lambda_1^-}.$$

Hence, we conclude that $C(x)$ is uniformly bounded on $\mathbb{R}$ and thus F_1 is a continuous map from Γ to $B_\mu(\mathbb{R}, \mathbb{R})$ with respect to the norm $|\cdot|_\mu$. Similarly, we can show that F_2 and F_3 are also continuous. Consequently, F is a continuous map on Γ with respect to the norm $|\cdot|_\mu$.

To prove the compactness of F, we shall make use of Arzela–Ascoli theorem and a standard diagonal process. Let $I_k := [-k, k]$ with $k \in \mathbb{N}$ be a compact interval on $\mathbb{R}$ and temporarily we regard Γ as a bounded subset of $C(I_k, \mathbb{R}^3)$ equipped with the maximum norm. Since F maps Γ into Γ, it is obvious that F is uniformly bounded. We will use the following two inequalities to show that F is equi-continuous. Namely, from the definition of F_i in (3.14–3.15) and integral representation for the derivative of D_i^{-1} in (3.10) we have for any $(S, I, R) \in \Gamma$,

$$\begin{aligned}|[F_1(S,I,R)]'(x)| &\le \frac{-\lambda_1^-\alpha_1 S_{-\infty}}{\rho_1}\int_{-\infty}^{x} e^{\lambda_1^-(x-y)}dy + \frac{\lambda_1^+\alpha_1 S_{-\infty}}{\rho_1}\int_{x}^{\infty} e^{\lambda_1^+(x-y)}dy\\ &= \frac{2\alpha_1 S_{-\infty}}{\rho_1},\end{aligned}$$

and

$$\begin{aligned}|[F_2(S,I,R)]'(x)| &\le \frac{-\lambda_2^-(\alpha_2+\beta-\gamma-\delta)}{\rho_2}\int_{-\infty}^{x} e^{\lambda_2^-(x-y)+\lambda_0 y}dy\\ &\quad + \frac{\lambda_2^+(\alpha_2+\beta-\gamma-\delta)}{\rho_2}\int_{x}^{\infty} e^{\lambda_2^+(x-y)+\lambda_0 y}dy\\ &= \frac{(\alpha_2+\beta-\gamma-\delta)e^{\lambda_0 x}}{\rho_2}\left(\frac{-\lambda_2^-}{\lambda_0-\lambda_2^-}+\frac{\lambda_2^+}{\lambda_2^+-\lambda_0}\right)\\ &= \frac{c\lambda_0+2\alpha_2}{\rho_2}e^{\lambda_0 x},\end{aligned}$$

and

$$\begin{aligned}|[F_3(S,I,R)]'(x)| &\le \frac{-\lambda_3^-\gamma(\alpha_3+c\lambda_0-d_3\lambda_0^2)}{\rho_3(c\lambda_0-d_3\lambda_0^2)}\int_{-\infty}^{x} e^{\lambda_3^-(x-y)+\lambda_0 y}dy\\ &\quad + \frac{\lambda_3^+\gamma(\alpha_3+c\lambda_0-d_3\lambda_0^2)}{\rho_3(c\lambda_0-d_3\lambda_0^2)}\int_{x}^{\infty} e^{\lambda_3^+(x-y)+\lambda_0 y}dy\\ &= \frac{e^{\lambda_0 x}\gamma(\alpha_3+c\lambda_0-d_3\lambda_0^2)}{\rho_3(c\lambda_0-d_3\lambda_0^2)}\left(\frac{-\lambda_3^-}{\lambda_0-\lambda_3^-}+\frac{\lambda_3^+}{\lambda_3^+-\lambda_0}\right)\\ &= \frac{\gamma(c\lambda_0+2\alpha_3)}{\rho_3(c\lambda_0-d_3\lambda_0^2)}e^{\lambda_0 x}\end{aligned}$$

Here we have made use of the facts that λ_0 defined in (3.2) is a root of f in (3.1) and $\lambda_i^{\pm}$ defined in (3.5) are the roots of f_i in (3.6). Let $\{u_n\}$ be a sequence of Γ, which can be also viewed as a bounded subset of $C(I_k)$ with $I_k := [-k, k]$. Since F is uniformly bounded and equi-continuous, by the Arzela–Ascoli theorem and the standard diagonal process, we can extract a subsequence $\{u_{n_k}\}$ such that $v_{n_k} := Fu_{n_k}$ converges in $C(I_k)$ for any $k \in \mathbb{N}$. Let v be the limit of v_{n_k}. It is readily seen that $v \in C(\mathbb{R}, \mathbb{R}^3)$. Furthermore, since $F(\Gamma) \subset \Gamma$ by Lemma 5 and Γ is closed, it follows that $v \in \Gamma$. Now we come back to the norm $|\cdot|_\mu$ defined in (3.13). Note that $\mu > \lambda_0 > 0$, it follows from (3.19) and (3.21) that $e^{-\mu|x|}I_+(x)$ and $e^{-\mu|x|}R_+(x)$ are uniformly bounded on $\mathbb{R}$. Thus, Γ is uniformly bounded with respect to the norm $|\cdot|_\mu$. Consequently, the norm $|v_{n_k} - v|_\mu$ is uniformly bounded for all $k \in \mathbb{N}$. Given any $\varepsilon > 0$, we can find an integer $M > 0$ independent of v_{n_k} such that

$$e^{-\mu|x|}|v_{n_k}(x) - v(x)| < \varepsilon$$

for any $|x| > M$ and $k \in \mathbb{N}$. Since v_{n_k} converges to v on the compact interval $[-M, M]$ with respect to the maximum norm, there exists $K \in \mathbb{N}$ such that

$$e^{-\mu|x|}|v_{n_k}(x) - v(x)| < \varepsilon$$

for any $|x| \le M$ and $k > K$. The above two inequalities imply that v_{n_k} converges to v with respect to the norm $|\cdot|_\mu$. This proves the compactness of the map F. □

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