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Convex solutions of boundary value problems

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Abstract

We establish two criteria for the existence of convex solutions for a boundary value problem arising from the study of the existence of convex radial solutions for the Monge–Ampère equations. We shall use fixed point theorems in a cone.

Keywords: Convex solutions; Monge–Ampère equations; Fixed index theorem

1. Introduction

In this paper we consider the existence of convex solutions for the boundary value problem

$$\begin{aligned} &\left(\left(u'(t)\right)^n\right)' = nt^{n-1} f\left(-u(t)\right) \quad \text{in } 0 < t < 1, \\ &u'(0) = 0, \qquad u(1) = 0, \end{aligned} \tag{1.1}$$

where $n \geqslant 1$. A nontrivial convex solution of (1.1) is negative on [0, 1). Such a problem occurs in the study of the existence of convex radial solutions for the Dirichlet problem of the Monge–Ampère equations

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$$\det D^2u = f(-u) \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B, \tag{1.2}$$

where $B = \{x \in \mathbb{R}^n \colon |x| < 1\}$. In fact, a convex radial solution of (1.2) can be viewed as a solution of (1.1). Kutev [7] obtained the existence of convex radial solutions of (1.2) with $f(-u) = (-u)^p$, $n \neq p > 0$ by reducing (1.2) to (1.1). We refer to [2,7] and references therein for further discussions regarding convex radial solutions of (1.2). Related results may be found in [3,6,8].

In this paper we shall apply a fixed point theorem in a cone to show the existence of convex solutions of (1.1) under superlinearity and sublinearity assumptions on the nonlinearity f. Our assumption for this paper is:

(H1) $f : [0, \infty) \to [0, \infty)$ is continuous.

In order to state our results, we introduce the notation

$$f_0 = \lim_{u \to 0} \frac{f(u)}{u^n} \quad \text{and} \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u^n}. \tag{1.3}$$

Our main results are:

Theorem 1.1. *Assume* (H1) *holds.*

(a) *If* $f_0 = 0$ *and* $f_\infty = \infty$, *then* (1.1) *has a nontrivial convex solution.*
(b) *If* $f_0 = \infty$ *and* $f_\infty = 0$, *then* (1.1) *has a nontrivial convex solution.*

2. Preliminaries

With a simple transformation $v = -u$, (1.1) can be brought to the following equation:

$$\begin{aligned} &\big(\big(-v'(t)\big)^n\big)' = nt^{n-1} f\big(v(t)\big) \quad \text{in } 0 < t < 1, \\ &v'(0) = 0, \qquad v(1) = 0. \end{aligned} \tag{2.1}$$

Now we treat positive concave classical solutions of (2.1).

The following well-known result of the fixed point index is crucial in our arguments.

Lemma 2.1. [1,4,5] *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{v \in K \colon \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{v \in K \colon \|x\| = r\}$.*

(i) *If $\|Tx\| \geqslant \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

(ii) *If $\|Tx\| \leqslant \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.1 to (2.1), let X be the Banach space $C[0,1]$ with $\|v\| = \sup_{t\in[0,1]} |v(t)|$.

Define the cone K by

$$K = \left\{ v \in X \colon\ v(t) \geqslant 0,\ \min_{1/4\leqslant t\leqslant 3/4} v(t) \geqslant \frac{1}{4}\|v\| \right\}.$$

Also, define, for r a positive number, Ω_r by

$$\Omega_r = \left\{ v \in K \colon\ \|v\| < r \right\}.$$

Note that $\partial\Omega_r = \{v \in K \colon\ \|v\| = r\}$.

Let the map $T : K \to X$ be defined by

$$Tv(t) = \int_t^1 \left(\int_0^s n\tau^{n-1} f\big(v(\tau)\big)\, d\tau \right)^{1/n} ds, \quad 0 \leqslant t \leqslant 1,$$

Thus, if $v \in K$ is a nonzero positive fixed point of T, then $-v$ is a nontrivial convex solution of (1.1). In view of Lemma 2.2, $-v$ is negative for $r \in [0, 1)$.

The following lemma is a simple consequence of the concavity of v.

Lemma 2.2. *Assume* (H1) *holds. Let* $v \in X$ *with* $v(t) \geqslant 0$ *for* $t \in [0,1]$. *If* $v'(t)$ *is nonincreasing on* $[0,1]$*, then*

$$v(t) \geqslant \min\{t, 1-t\}\|v\|, \quad t \in [0,1].$$

In particular,

$$\min_{1/4\leqslant t\leqslant 3/4} v(t) \geqslant \frac{1}{4}\|v\|.$$

Proof. Since $v'(t)$ is nonincreasing, we have for $0 \leqslant t_0 < t < t_1 \leqslant 1$,

$$v(t) - v(t_0) = \int_{t_0}^t v'(s)\, ds \geqslant (t - t_0)v'(t)$$

and

$$v(t_1) - v(t) = \int_t^{t_1} v'(s)\, ds \leqslant (t_1 - t)v'(t),$$

from which we have

$$v(t) \geqslant \frac{(t_1 - t)v(t_0) + (t - t_0)v(t_1)}{t_1 - t_0}.$$

Considering the above inequality on $[0, \sigma]$ and $[\sigma, 1]$, we have

$$v(t) \geqslant t\|v\| \quad \text{for } t \in [0, \sigma],$$

and

$$v(t) \geqslant (1-t)\|v\| \quad \text{for } t \in [\sigma, 1],$$

where $\sigma \in [0,1]$ such that $v(\sigma) = \|v\|$. Hence,

$$v(t) \geqslant \min\{t, 1-t\}\|v\|, \quad t \in [0,1]. \qquad \square$$

Lemma 2.3. *Assume* (H1) *holds. Then $T(K) \subset K$ and the map $T: K \to K$ is completely continuous.*

Proof. Lemma 2.2 implies that $T(K) \subset K$. It is not difficult to verify that T is compact and continuous. $\square$

Let

$$\Gamma = \int_{1/4}^{3/4} \left(n \int_{1/4}^{s} \tau^{n-1} \left(\frac{1}{4}\right)^n d\tau \right)^{1/n} ds > 0.$$

Lemma 2.4. *Assume* (H1) *holds and let $\eta > 0$. If $v \in K$ and $f(v(t)) \geqslant (v(t)\eta)^n$ for $t \in [1/4, 3/4]$, then*

$$\|Tv\| \geqslant \Gamma\eta\|v\|.$$

Proof. Note that from the definition of Tu that $Tv(0)$ is the maximum value of Tu on $[0,1]$. It follows that

$$\begin{aligned}
\|Tv\| &\geqslant \int_{1/4}^{3/4} \left(n \int_{1/4}^{s} \tau^{n-1} f\big(v(\tau)\big)\, d\tau \right)^{1/n} ds \\
&\geqslant \int_{1/4}^{3/4} \left(n \int_{1/4}^{s} \tau^{n-1} \big(v(\tau)\eta\big)^n\, d\tau \right)^{1/n} ds \\
&\geqslant \int_{1/4}^{3/4} \left(n \int_{1/4}^{s} \tau^{n-1} \left(\frac{1}{4}\eta\|v\|\right)^n d\tau \right)^{1/n} ds \\
&\geqslant \int_{1/4}^{3/4} \left(n \int_{1/4}^{s} \tau^{n-1} \left(\frac{1}{4}\right)^n d\tau \right)^{1/n} ds\,\eta\|v\| \\
&= \Gamma\eta\|v\|. \qquad \square
\end{aligned}$$

Define a new function

$$f^*(v) = \max_{0 \leqslant t \leqslant v} \big\{ f(t) \big\}.$$

Note that $f_0^* = \lim_{v \to 0} \frac{f^*(v)}{v^n}$ and $f_\infty^* = \lim_{v \to \infty} \frac{f^*(v)}{v^n}$.

Lemma 2.5. *Assume* (H1) *holds. Then* $f_0^* = f_0$ *and* $f_\infty^* = f_\infty$.

Proof. It is easy to see that $f_0^* = f_0$. For the second part, we consider two cases, (a) $f(v)$ is bounded and (b) $f(v)$ is unbounded. For the case (a), it follows that $f_\infty^* = 0 = f_\infty$. For the case (b), for any $\delta > 0$, let $M = \max_{0 \leqslant t \leqslant \delta}\{f(t)\}$ and

$$N_\delta = \min\bigl\{v\colon\ v \geqslant \delta,\ f(v) \geqslant M\bigr\} \geqslant \delta,$$

then

$$\max_{0 \leqslant t \leqslant N_\delta}\bigl\{f(t)\bigr\} = f(N_\delta).$$

Thus, for any $\delta > 0$, there exists a $N_\delta \geqslant \delta$ such that

$$f^*(v) = \max\Bigl\{\max_{0 \leqslant t \leqslant N_\delta}\bigl\{f(t)\bigr\},\ \max_{N_\delta \leqslant t \leqslant v}\bigl\{f(t)\bigr\}\Bigr\} = \max_{N_\delta \leqslant t \leqslant v}\bigl\{f(t)\bigr\} \quad \text{for } v > N_\delta.$$

Hence, it follows, from the definitions of f_∞ and f_∞^*, that $f_\infty^* = f_\infty$. □

Lemma 2.6. *Assume* (H1) *holds and let* $r > 0$. *If there exists an* $\varepsilon > 0$ *such that* $f^*(r) \leqslant \varepsilon^n r^n$, *then*

$$\|Tv\| \leqslant n^{1/n}\varepsilon\|v\| \quad \textit{for } v \in \partial\Omega_r.$$

Proof. From the definition of T, we have for $v \in \partial\Omega_r$,

$$\begin{aligned}
\|Tv\| &\leqslant \int_0^1 \left(\int_0^1 n\tau^{n-1} f\bigl(v(\tau)\bigr)\,d\tau\right)^{1/n} ds \\
&\leqslant \left(\int_0^1 n\tau^{n-1} f^*(r)\,d\tau\right)^{1/n} ds \\
&\leqslant \left(\int_0^1 n\tau^{n-1}\varepsilon^n r^n\,d\tau\right)^{1/n} ds \\
&\leqslant n^{1/n}\varepsilon r \\
&= n^{1/n}\varepsilon\|v\|. \qquad \square
\end{aligned}$$

3. Proof of Theorem 1.1

Part (a). It follows from Lemma 2.5 that $f_0^* = 0$. Therefore, we can choose $r_1 > 0$ so that $f^*(r_1) \leqslant \varepsilon^n r_1^n$, where the constant $\varepsilon > 0$ satisfies

$$n^{1/n}\varepsilon < 1.$$

We have by Lemma 2.6 that

$$\|Tv\| \leqslant n^{1/n}\varepsilon\|v\| < \|v\| \quad \text{for } v \in \partial\Omega_{r_1}.$$

Now, since $f_\infty = \infty$, there is an $\hat{H} > 0$ such that $f(v) \geqslant \eta^n v^n$ for $v \geqslant \hat{H}$, where $\eta > 0$ is chosen so that

$$\Gamma\eta > 1.$$

Let $r_2 = \max\{2r_1, 4\hat{H}\}$. If $v \in \partial\Omega_{r_2}$, then

$$\min_{1/4 \leqslant t \leqslant 3/4} v(t) \geqslant \frac{1}{4}\|v\| = \frac{1}{4} r_2 \geqslant \hat{H},$$

which implies that

$$f\big(v(t)\big) \geqslant \eta^n \big(v(t)\big)^n \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

It follows from Lemma 2.4 that

$$\|Tv\| \geqslant \Gamma\eta\|v\| > \|v\| \quad \text{for } v \in \partial\Omega_{r_2}.$$

By Lemma 2.1,

$$i(T, \Omega_{r_1}, K) = 1 \quad \text{and} \quad i(T, \Omega_{r_2}, K) = 0.$$

It follows from the additivity of the fixed point index that $i(T, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = -1$. Thus, $i(T, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) \neq 0$, which implies T has a fixed point $v \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ according to the existence property of the fixed point index. The fixed point $v \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ is the desired positive solution of (2.1).

Part (b). If $f_0 = \infty$, there is a $r_1 > 0$ such that $f(v) \geqslant \eta^n v^n$ for $0 \leqslant v \leqslant r_1$, where $\eta > 0$ is chosen so that $\Gamma\eta > 1$. If $v \in \partial\Omega_{r_1}$, then

$$f\big(v(t)\big) \geqslant \big(\eta v(t)\big)^n \quad \text{for } t \in [0, 1].$$

Lemma 2.4 implies that

$$\|Tv\| \geqslant \Gamma\eta\|v\| > \|v\| \quad \text{for } v \in \partial\Omega_{r_1}.$$

We now determine Ω_{r_2}. Since $f_\infty^* = f_\infty = 0$, there is a $r_2 > 2r_1$ such that $f^*(r_2) \leqslant \varepsilon^n r_2^n$, where the constant $\varepsilon > 0$ satisfies

$$\varepsilon(n)^{1/n} < 1.$$

Thus, we have by Lemma 2.6

$$\|Tv\| \leqslant \varepsilon(n)^{1/n}\|v\| < \|v\| \quad \text{for } v \in \partial\Omega_{r_2}.$$

By Lemma 2.1,

$$i(T, \Omega_{r_1}, K) = 0 \quad \text{and} \quad i(T, \Omega_{r_2}, K) = 1.$$

It follows from the additivity of the fixed point index that $i(T, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, T has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (2.1). □

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