

# CONVEX SOLUTIONS OF SYSTEMS ARISING FROM MONGE-AMPÈRE EQUATIONS

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*Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday*

## Abstract

We establish two criteria for the existence of convex solutions to a boundary value problem for weakly coupled systems arising from the Monge-Ampère equations. We shall use fixed point theorems in a cone.

**Key words and phrases:** Convex solutions, Monge-Ampère equations, cone.  
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## 1 Introduction

In this paper we consider the existence of convex solutions to the Dirichlet problem for the weakly coupled system

$$\begin{aligned} \left( (u_1'(t))^N \right)' &= Nt^{N-1}f(-u_2(t)) \quad \text{in } 0 < t < 1, \\ \left( (u_2'(t))^N \right)' &= Nt^{N-1}g(-u_1(t)) \quad \text{in } 0 < t < 1, \\ u_1'(0) = u_2'(0) &= 0, \quad u_1(1) = u_2(1) = 0, \end{aligned} \tag{1.1}$$

where  $N \geq 1$ . A nontrivial convex solution of (1.1) is negative on  $[0,1)$ . Such a problem arises in the study of the existence of convex radial solutions to the Dirichlet problem for the system of the Monge-Ampère equations

$$\begin{cases} \det D^2 u_1 = f(-u_2) & \text{in } B, \\ \det D^2 u_2 = g(-u_1) & \text{in } B, \\ u_1 = u_2 = 0 & \text{on } \partial B, \end{cases} \tag{1.2}$$

where  $B = \{x \in \mathbb{R}^N : |x| < 1\}$  and  $\det D^2 u_i$  is the determinant of the Hessian matrix  $(\frac{\partial^2 u_i}{\partial x_m \partial x_n})$  of  $u_i$ . For how to reduce (1.2) to (1.1), one may see Hu and the author [5].

The Dirichlet problem for a single unknown variable Monge-Ampère equations

$$\begin{cases} \det D^2 u = f(-u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \quad (1.3)$$

in general domains in  $\mathbb{R}^n$  may be found in Caffarelli, Nirenberg and Spruck [1]. Kutev [7] investigated the existence of strictly convex radial solutions of (1.3) when  $f(-u) = (-u)^p$ . Delano [3] treated the existence of convex radial solutions of (1.3) for a class of more general functions, namely  $\lambda \exp f(|x|, u, |\nabla u|)$ .

The author [10] and Hu and the author [5] showed that the existence, multiplicity and nonexistence of convex radial solutions of (1.3) can be determined by the asymptotic behaviors of the quotient  $\frac{f(u)}{u^N}$  at zero and infinity.

In this paper we shall establish the existence of convex radial solutions of the weakly coupled system (1.1) in superlinear and sublinear cases. First, introduce the notation

$$f_0 = \lim_{x \rightarrow 0^+} \frac{f(x)}{x^N}, \quad f_\infty = \lim_{x \rightarrow \infty} \frac{f(x)}{x^N},$$

and

$$g_0 = \lim_{x \rightarrow 0^+} \frac{g(x)}{x^N}, \quad g_\infty = \lim_{x \rightarrow \infty} \frac{g(x)}{x^N}.$$

We shall show that if (1.1) is superlinear, or  $f_0 = g_0 = 0$  and  $f_\infty = g_\infty = \infty$ , (1.1) is sublinear, or  $f_0 = g_0 = \infty$  and  $f_\infty = g_\infty = 0$ , then (1.1) has a convex solution.

Our main results are:

**Theorem 1.1** *Assume  $f, g : [0, \infty) \rightarrow [0, \infty)$  are continuous.*

(a). *If  $f_0 = g_0 = 0$  and  $f_\infty = g_\infty = \infty$ , then (1.1) has a convex solution.*

(b). *If  $f_0 = g_0 = \infty$  and  $f_\infty = g_\infty = 0$ , then (1.1) has a convex solution.*

## 2 Preliminaries

With a simple transformation  $v_i = -u_i, i = 1, 2$  (1.1) can be brought to the following equation

$$\begin{cases} \left( (-v_1'(r))^N \right)' = Nr^{N-1} f(v_2), & 0 < r < 1, \\ \left( (-v_2'(r))^N \right)' = Nr^{N-1} g(v_1), & 0 < r < 1, \\ v_i'(0) = v_i(1) = 0, & i = 1, 2. \end{cases} \quad (2.4)$$

Now we treat positive concave classical solutions of (2.4).

The following well-known result of the fixed point index is crucial in our arguments.

**Lemma 2.1** ([2, 4, 6]). *Let  $E$  be a Banach space and  $K$  a cone in  $E$ . For  $r > 0$ , define  $K_r = \{u \in K : \|x\| < r\}$ . Assume that  $T : \bar{K}_r \rightarrow K$  is completely continuous such that  $Tx \neq x$  for  $x \in \partial K_r = \{u \in K : \|x\| = r\}$ .*

(i) If  $\|Tx\| \geq \|x\|$  for  $x \in \partial K_r$ , then

$$i(T, K_r, K) = 0.$$

(ii) If  $\|Tx\| \leq \|x\|$  for  $x \in \partial K_r$ , then

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.1 to (2.4), let  $X$  be the Banach space  $C[0, 1] \times C[0, 1]$  and, for  $(v_1, v_2) \in X$ ,

$$\|(v_1, v_2)\| = \|v_1\| + \|v_2\|$$

where  $\|v_i\| = \sup_{t \in [0, 1]} |v_i(t)|$ . Define  $K$  to be a cone in  $X$  by

$$K = \{(v_1, v_2) \in X : v_i(t) \geq 0, t \in [0, 1], \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} v_i(t) \geq \frac{1}{4} \|v_i\|, i = 1, 2\}.$$

Also, define, for  $r$  a positive number,  $\Omega_r$  by

$$\Omega_r = \{(v_1, v_2) \in K : \|(v_1, v_2)\| < r\}.$$

Note that  $\partial\Omega_r = \{(v_1, v_2) \in K : \|(v_1, v_2)\| = r\}$ .

Let  $\mathbf{T} : K \rightarrow X$  be a map with components  $(T^1, T^2)$ , which are defined by

$$\begin{aligned} T^1(v_1, v_2)(r) &= \int_r^1 \left( \int_0^s N\tau^{N-1} f(v_2(\tau)) d\tau \right)^{\frac{1}{n}} ds, \quad r \in [0, 1], \\ T^2(v_1, v_2)(r) &= \int_r^1 \left( \int_0^s N\tau^{N-1} g(v_1(\tau)) d\tau \right)^{\frac{1}{n}} ds, \quad r \in [0, 1]. \end{aligned} \quad (2.5)$$

It is straightforward to verify that (2.4) is equivalent to the fixed point equation

$$\mathbf{T}(v_1, v_2) = (v_1, v_2) \quad \text{in } K.$$

Thus, if  $(v_1, v_2) \in K$  is a positive fixed point of  $\mathbf{T}$ , then  $(-v_1, -v_2)$  is a convex solution of (1.1). Conversely, if  $(u_1, u_2)$  is a convex solution of (1.1), then  $(-u_1, -u_2)$  is a fixed point of  $\mathbf{T}$  in  $K$ .

The following lemma is a standard result due to the concavity of  $u$ . We prove it here only for completeness.

**Lemma 2.2** *Let  $u \in C^1[0, 1]$  with  $u(t) \geq 0$  for  $t \in [0, 1]$ . Assume that  $u'(t)$  is nonincreasing on  $[0, 1]$ . Then*

$$u(t) \geq \min\{t, 1-t\} \|u\|, \quad t \in [0, 1],$$

where  $\|u\| = \sup_{t \in [0, 1]} u(t)$ . In particular,

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq \frac{1}{4} \|u\|.$$

PROOF Since  $u'(t)$  is nonincreasing, we have for  $0 \leq t_0 < t < t_1 \leq 1$ ,

$$u(t) - u(t_0) = \int_{t_0}^t u'(s) ds \geq (t - t_0)u'(t)$$

and

$$u(t_1) - u(t) = \int_t^{t_1} u'(s) ds \leq (t_1 - t)u'(t),$$

from which, we have

$$u(t) \geq \frac{(t_1 - t)u(t_0) + (t - t_0)u(t_1)}{t_1 - t_0}.$$

Considering the above inequality on  $[0, \sigma]$  and  $[\sigma, 1]$ , we have

$$u(t) \geq t\|u\| \quad \text{for } t \in [0, \sigma],$$

and

$$u(t) \geq (1 - t)\|u\| \quad \text{for } t \in [\sigma, 1],$$

where  $\sigma \in [0, 1]$  such that  $u(\sigma) = \|u\|$ . Hence,

$$u(t) \geq \min\{t, 1 - t\}\|u\|, \quad t \in [0, 1].$$

□

Lemma 2.3 can be verified by the standard procedures.

**Lemma 2.3** *Assume  $f, g : [0, \infty) \rightarrow [0, \infty)$  are continuous. Then  $\mathbf{T}(K) \subset K$  and  $\mathbf{T} : K \rightarrow K$  is a compact operator and continuous.*

Let

$$\Gamma = \frac{1}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} \left( \int_{\frac{1}{4}}^s N\tau^{N-1} d\tau \right)^{\frac{1}{N}} ds > 0.$$

**Lemma 2.4** *Let  $(v_1, v_2) \in K$  and  $\eta > 0$ . If*

$$f(v_2(t)) \geq (\eta v_2(t))^N \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

or

$$g(v_1(t)) \geq (\eta v_1(t))^N \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

then

$$\|\mathbf{T}(v_1, v_2)\| \geq \Gamma\eta\|v_2\|,$$

or

$$\|\mathbf{T}(v_1, v_2)\| \geq \Gamma\eta\|v_1\|,$$

respectively.

PROOF Note, from the definition of  $\mathbf{T}(v_1, v_2)$ , that  $T^i(v_1, v_2)(0)$  is the maximum value of  $T^i(v_1, v_2)$  on  $[0,1]$ . It follows that

$$\begin{aligned} \|\mathbf{T}(v_1, v_2)\| &\geq \sup_{t \in [0,1]} |T^1(v_1, v_2)(t)| \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \left( \int_{\frac{1}{4}}^s N\tau^{N-1} f(v_2(\tau)) d\tau \right)^{\frac{1}{N}} ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \left( \int_{\frac{1}{4}}^s N\tau^{N-1} (\eta v_2(\tau))^N d\tau \right)^{\frac{1}{N}} ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \left( \int_{\frac{1}{4}}^s N\tau^{N-1} \left(\frac{\eta}{4}\|v_2\|\right)^N d\tau \right)^{\frac{1}{N}} ds \\ &= \Gamma\eta\|v_2\|. \end{aligned}$$

Similarly,

$$\|\mathbf{T}(v_1, v_2)\| \geq \sup_{t \in [0,1]} |T^2(v_1, v_2)(t)| \geq \Gamma\eta\|v_1\|.$$

□

We define new functions  $\hat{f}(t), \hat{g}(t) : [0, \infty) \rightarrow [0, \infty)$  by

$$\hat{f}(t) = \max\{f(v) : 0 \leq v \leq t\}, \quad \hat{g}(t) = \max\{g(v) : 0 \leq v \leq t\}.$$

Note that  $\hat{f}_0 = \lim_{t \rightarrow 0} \frac{\hat{f}(t)}{t^N}$ ,  $\hat{f}_\infty = \lim_{t \rightarrow \infty} \frac{\hat{f}(t)}{t^N}$  and  $\hat{g}_0, \hat{g}_\infty$  can be defined similarly.

**Lemma 2.5** [9] *Assume  $f, g : [0, \infty) \rightarrow [0, \infty)$  are continuous. Then*

$$\hat{f}_0 = f_0, \quad \hat{f}_\infty = f_\infty,$$

and

$$\hat{g}_0 = g_0, \quad \hat{g}_\infty = g_\infty.$$

**Lemma 2.6** *Assume  $f, g : [0, \infty) \rightarrow [0, \infty)$  are continuous. Let  $r > 0$ . If there exists an  $\varepsilon > 0$  such that*

$$\hat{f}(r) \leq (\varepsilon r)^N, \quad \hat{g}(r) \leq (\varepsilon r)^N,$$

then

$$\|\mathbf{T}(v_1, v_2)\| \leq 2\varepsilon\|(v_1, v_2)\| \quad \text{for } (v_1, v_2) \in \partial\Omega_r.$$

PROOF From the definition of  $T$ , for  $(v_1, v_2) \in \partial\Omega_r$ , we have

$$\begin{aligned} \|\mathbf{T}(v_1, v_2)\| &= \sum_{i=1}^2 \sup_{t \in [0,1]} |T^i(v_1, v_2)(t)| \\ &\leq \left(\int_0^1 N\tau^{N-1} f(v_2(\tau)) d\tau\right)^{\frac{1}{N}} + \left(\int_0^1 N\tau^{N-1} g(v_1(\tau)) d\tau\right)^{\frac{1}{N}} \\ &\leq \left(\int_0^1 N\tau^{N-1} \hat{f}(r) d\tau\right)^{\frac{1}{N}} + \left(\int_0^1 N\tau^{N-1} \hat{g}(r) d\tau\right)^{\frac{1}{N}} \\ &\leq \left(\int_0^1 N\tau^{N-1} d\tau\right)^{\frac{1}{N}} \varepsilon r + \left(\int_0^1 N\tau^{N-1} d\tau\right)^{\frac{1}{N}} \varepsilon r \\ &\leq 2\varepsilon \|(u_1, u_2)\|. \end{aligned}$$

□

### 3 Proof of Theorem 1.1

PROOF Part (a). It follows from Lemma 2.5 that  $\hat{f}_0 = 0, \hat{g}_0 = 0$ . Therefore, we can choose  $r_1 > 0$  so that  $\hat{f}^i(r_1) \leq (\varepsilon r_1)^N, \hat{g}^i(r_1) \leq (\varepsilon r_1)^N$  where the constant  $\varepsilon > 0$  satisfies

$$\varepsilon < \frac{1}{2}.$$

We have by Lemma 2.6 that

$$\|\mathbf{T}(v_1, v_2)\| \leq 2\varepsilon \|(v_1, v_2)\| < \|(v_1, v_2)\| \quad \text{for } (v_1, v_2) \in \partial\Omega_{r_1}.$$

Now, since  $f_\infty = \infty, g_\infty = \infty$ , there is an  $\hat{H} > 0$  such that

$$f(v) \geq (\eta v)^N, g(v) \geq (\eta v)^N$$

for  $v \geq \hat{H}$ , where  $\eta > 0$  is chosen so that

$$\frac{1}{2}\Gamma\eta > 1.$$

Let  $r_2 = \max\{2r_1, 8\hat{H}\}$ . If  $(v_1, v_2) \in \partial\Omega_{r_2}$ , there exists one of  $i = 1$  or  $i = 2$  such that  $\sup_{t \in [0,1]} v_i \geq \frac{1}{2}r_2$ . Without loss of generality, assume that  $\sup_{t \in [0,1]} v_1 \geq \frac{1}{2}r_2$ . Then

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} v_1(t) \geq \frac{1}{4} \sup_{t \in [0,1]} v_1 \geq \frac{1}{8}r_2 \geq \hat{H},$$

which implies that

$$g(v_1(t)) \geq (\eta v_1(t))^N \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

It follows from Lemma 2.4 that

$$\|\mathbf{T}(v_1, v_2)\| \geq \Gamma\eta\|v_1\| > \frac{1}{2}\Gamma\eta r_2 \geq r_2 = \|(v_1, v_2)\|.$$

By Lemma 2.1,

$$i(\mathbf{T}, \Omega_{r_1}, K) = 1 \text{ and } i(\mathbf{T}, \Omega_{r_2}, K) = 0.$$

It follows from the additivity of the fixed point index that

$$i(\mathbf{T}, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = -1.$$

Thus,  $i(\mathbf{T}, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) \neq 0$ , which implies  $\mathbf{T}$  has a fixed point  $(v_1, v_2) \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$  by the existence property of the fixed point index. The fixed point  $(-v_1, -v_2) \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$  is the desired positive solution of (1.1).

Part (b). since  $f_0 = \infty, g_0 = \infty$ , there is an  $H > 0$  such that

$$f(v) \geq (\eta v)^N, g(v) \geq (\eta v)^N$$

for  $0 < v \leq H$ , where  $\eta > 0$  is chosen so that

$$\Gamma\eta > 1.$$

If  $(v_1, v_2) \in \partial\Omega_{r_1}$ , then

$$f(v_2(t)) \geq (\eta v_2)^N, \quad g(v_1(t)) \geq (\eta v_1)^N \text{ for } t \in [0, 1].$$

Lemma 2.4 implies that

$$\|\mathbf{T}(v_1, v_2)\| \geq \Gamma\eta\|(v_1, v_2)\| > \|(v_1, v_2)\| \text{ for } (v_1, v_2) \in \partial\Omega_{r_1}.$$

We now determine  $\Omega_{r_2}$ . It follows from Lemma 2.5 that  $\hat{f}_\infty = 0$  and  $\hat{g}_\infty = 0$ . Therefore there is an  $r_2 > 2r_1$  such that

$$\hat{f}^i(r_2) \leq (\varepsilon r_2)^N, \quad \hat{g}^i(r_2) \leq (\varepsilon r_2)^N,$$

where the constant  $\frac{1}{2} > \varepsilon > 0$ . Thus, we have by Lemma 2.6 that

$$\|\mathbf{T}(v_1, v_2)\| \leq 2\varepsilon\|(v_1, v_2)\| < \|(v_1, v_2)\| \text{ for } (v_1, v_2) \in \partial\Omega_{r_2}.$$

By Lemma 2.1,

$$i(\mathbf{T}, \Omega_{r_1}, K) = 0 \text{ and } i(\mathbf{T}, \Omega_{r_2}, K) = 1.$$

It follows from the additivity of the fixed point index that  $i(\mathbf{T}, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$ . Thus,  $\mathbf{T}$  has a fixed point  $(v_1, v_2)$  in  $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ . And  $(-v_1, -v_2)$  is the desired convex solution of (1.1).  $\square$

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