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Positive Solutions for Nonlinear Eigenvalue Problems

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1. INTRODUCTION

We are concerned with determining values of λ (eigenvalues), for which there exist positive solutions of the boundary value problem

$$u'' + \lambda a(t) f(u) = 0, \qquad 0 < t < 1, \tag{1λ}$$

$$u(0) = u(1) = 0, \tag{2}$$

where

(A) f: $[0, \infty) \to [0, \infty)$ is continuous,

(B) a: $[0, 1] \to [0, \infty)$ is continuous and does not vanish identically on any subinterval, and

(C) $f_0 = \lim_{x \to 0^+}(f(x)/x)$ and $f_\infty = \lim_{x \to \infty}(f(x)/x)$ exist.

We remark that, if $u(t)$ is a nonnegative solution of (1λ), (2), then $u(t)$ is concave on $[0, 1]$.

Boundary value problems (1λ), (2) describe many phenomena in the applied mathematical sciences, which can be found in the theory of nonlinear diffusion generated by nonlinear sources, in thermal ignition of

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gases, and in concentration in chemical or biological problems, where only positive solutions are meaningful; see, for example, [7, 11–13]. Also, the problem (1λ), (2) has been dealt with by Fink [8] and Fink, Gatica, and Hernandez [9] in modeling the one-dimensional case of the Dirichlet problem, when $a(t)$ satisfies certain integrability conditions. The results of [9] were generalized to nth order problems in [3].

For our motivation, we consider methods of solutions as (1λ), (2) arises in applications involving nonlinear elliptic problems in annular regions; see [1, 2, 6, 10, 15]. For the case when $\lambda = 1$ and f is either superlinear ($f_0 = 0$ and $f_\infty = \infty$), or f is sublinear ($f_0 = \infty$ and $f_\infty = 0$), Erbe and Wang [5] obtained solutions that are positive with respect to a cone and which lie in an annular type region. The methods of [5] were then extended to higher order boundary value problems in [4].

It is not required in this work that f be either sublinear or superlinear, yet, as in [5, 4], the arguments presented here for obtaining solutions of (1λ), (2), for certain λ, involve concavity properties of solutions, which are employed in defining a cone on which a positive integral operator is defined. A Krasnosel'skii fixed point theorem [14] is applied to yield positive solutions of (1λ), (2), for λ belonging to an open interval.

In Section 2, we present some properties of Green's functions that are used in defining a positive operator. We also state the Krasnosel'skii fixed point theorem. In Section 3, we give an appropriate Banach space and construct a cone to which we apply the fixed point theorem yielding solutions of (1λ), (2), for an open interval of eigenvalues.

2. SOME PRELIMINARIES

In this section, we state the above mentioned Krasnosel'skii fixed point theorem. We will apply this fixed point theorem to a completely continuous integral operator, whose kernel, $G(t, s)$, is the Green's function for

$$-y'' = 0, \tag{3}$$

$$y(0) = y(1) = 0. \tag{4}$$

In particular

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases} \tag{5}$$

from which

$$G(t,s) > 0 \qquad \text{on } (0,1) \times (0,1), \tag{6}$$

$$G(t,s) \le G(s,s) = s(1-s), \qquad 0 \le t, s \le 1, \tag{7}$$

and it is shown in [5] that

$$G(t,s) \geq \frac{1}{4}G(s,s) = \frac{1}{4}s(1-s), \qquad \frac{1}{4} \leq t \leq \frac{3}{4}, 0 \leq s \leq 1. \quad (8)$$

We will apply the following fixed point theorem to obtain solutions of (1λ), (2), for certain λ.

THEOREM 1. *Let $\mathscr{B}$ be a Banach space, and let $\mathscr{P} \subset \mathscr{B}$ be a cone in $\mathscr{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathscr{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T: \mathscr{P} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \mathscr{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \leq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathscr{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. SOLUTIONS IN THE CONE

In this section, we will apply Theorem 1 to the eigenvalue problem (1λ), (2). We note that $u(t)$ is a solution of (1λ), (2) if, and only if,

$$u(t) = \lambda \int_0^1 G(t,s)a(s)f(u(s))\, ds, \qquad 0 \leq t \leq 1.$$

For our constructions, let $\mathscr{B} = C[0,1]$, with norm, $\|x\| = \sup_{0 \leq t \leq 1}|x(t)|$. Define a cone, $\mathscr{P}$, by

$$\mathscr{P} = \left\{ x \in \mathscr{B} \,|\, x(t) \geq 0 \text{ on } [0,1], \quad \text{and} \quad \min_{1/4 \leq t \leq 3/4} x(t) \geq \frac{1}{4}\|x\| \right\}.$$

Also, let the number $\tau \in [0,1]$ be defined by

$$\int_{1/4}^{3/4} G(\tau,s)a(s)\, ds = \max_{0 \leq t \leq 1} \int_{1/4}^{3/4} G(t,s)a(s)\, ds. \quad (9)$$

THEOREM 2. *Assume that conditions* (A), (B), *and* (C) *are satisfied. Then, for each λ satisfying*

$$\frac{4}{\left(\int_{1/4}^{3/4} G(\tau,s)a(s)\, ds\right) f_\infty} < \lambda < \frac{1}{\left(\int_0^1 s(1-s)a(s)\, ds\right) f_0}, \quad (10)$$

there exists at least one solution of (1λ), (2) *in* $\mathscr{P}$.

Proof. Let λ be given as in (10). Now, let $\varepsilon > 0$ be chosen such that

$$\frac{4}{\left(\int_{1/4}^{3/4} G(\tau, s) a(s)\, ds\right)(f_\infty - \varepsilon)} \le \lambda \le \frac{1}{\left(\int_0^1 s(1-s) a(s)\, ds\right)(f_0 + \varepsilon)}.$$

Define an integral operator $T: \mathscr{P} \to \mathscr{B}$ by

$$Tu(t) = \lambda \int_0^1 G(t, s) a(s) f(u(s))\, ds, \qquad u \in \mathscr{P}. \tag{11}$$

We seek a fixed point of T in the cone $\mathscr{P}$.

Notice from (6) that, for $u \in \mathscr{P}$, $Tu(t) \ge 0$ on $[0, 1]$. Also, for $u \in \mathscr{P}$, we have from (7) that

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t, s) a(s) f(u(s))\, ds \\ &\le \lambda \int_0^1 s(1-s) a(s) f(u(s))\, ds, \end{aligned}$$

so that

$$\|Tu\| \le \lambda \int_0^1 s(1-s) a(s) f(u(s))\, ds. \tag{12}$$

And next, if $u \in \mathscr{P}$, we have by (8) and (12),

$$\begin{aligned} \min_{1/4 \le t \le 3/4} Tu(t) &= \min_{1/4 \le t \le 3/4} \lambda \int_0^1 G(t, s) a(s) f(u(s))\, ds \\ &\ge \frac{\lambda}{4} \int_0^1 s(1-s) a(s) f(u(s))\, ds \\ &\ge \frac{1}{4} \|Tu\|. \end{aligned}$$

As a consequence, $T: \mathscr{P} \to \mathscr{P}$. In addition, standard arguments show that T is completely continuous.

Now, turning to f_0, there exists an $H_1 > 0$ such that $f(x) \le (f_0 + \varepsilon)x$, for $0 < x \le H_1$. So, choosing $u \in \mathscr{P}$ with $\|u\| = H_1$, we have from (7)

$$\begin{aligned} Tu(t) &\le \lambda \int_0^1 s(1-s) a(s) f(u(s))\, ds \\ &\le \lambda \int_0^1 s(1-s) a(s) (f_0 + \varepsilon) u(s)\, ds \\ &\le \lambda \int_0^1 s(1-s) a(s)\, ds (f_0 + \varepsilon) \|u\| \\ &\le \|u\|. \end{aligned}$$

Consequently, $\|Tu\| \le \|u\|$. So, if we set

$$\Omega_1 = \{x \in \mathscr{B} \mid \|x\| < H_1\},$$

then

$$\|Tu\| \le \|u\|, \qquad \text{for } u \in \mathscr{P} \cap \partial\Omega_1. \tag{13}$$

Next, considering f_∞, there exists an $\overline{H}_2 > 0$ such that $f(x) \ge (f_\infty - \varepsilon)x$, for all $x \ge \overline{H}_2$. Let $H_2 = \max\{2H_1, 4\overline{H}_2\}$ and let

$$\Omega_2 = \{x \in \mathscr{B} \mid \|x\| < H_2\}.$$

If $u \in \mathscr{P}$ with $\|u\| = H_2$, then $\min_{1/4 \le t \le 3/4} u(t) \ge \frac{1}{4}\|u\| \ge \overline{H}_2$, and

$$\begin{aligned} Tu(\tau) &\ge \lambda \int_{1/4}^{3/4} G(\tau, s) a(s) f(u(s))\, ds \\ &\ge \lambda \int_{1/4}^{3/4} G(\tau, s) a(s)(f_\infty - \varepsilon) u(s)\, ds \\ &\ge \frac{\lambda}{4} \int_{1/4}^{3/4} G(\tau, s) a(s)\, ds (f_\infty - \varepsilon)\|u\| \\ &\ge \|u\|. \end{aligned}$$

Thus, $\|Tu\| \ge \|u\|$. Hence,

$$\|Tu\| \ge \|u\|, \qquad \text{for } u \in \mathscr{P} \cap \partial\Omega_2. \tag{14}$$

Applying (i) of Theorem 1 to (13) and (14) yields that T has a fixed point $u(t) \in \mathscr{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. As such, $u(t)$ is a desired solution of (1λ), (2) for the given λ. The proof is complete. ∎

THEOREM 3. *Assume that conditions* (A), (B), *and* (C) *are satisfied. Then, for each* λ *satisfying*

$$\frac{4}{\left(\int_{1/4}^{3/4} G(\tau, s) a(s)\, ds\right) f_0} < \lambda < \frac{1}{\left(\int_0^1 s(1-s) a(s)\, ds\right) f_\infty}, \tag{15}$$

there exists at least one solution of (1λ), (2) *in* $\mathscr{P}$.

Proof. Let λ be given as in (15), and choose $\varepsilon > 0$ such that

$$\frac{4}{\left(\int_{1/4}^{3/4} G(\tau, s) a(s)\, ds\right)(f_0 - \varepsilon)} \le \lambda \le \frac{1}{\left(\int_0^1 s(1-s) a(s)\, ds\right)(f_\infty + \varepsilon)}.$$

Let T be the cone preserving, completely continuous operator that was defined by (11).

Beginning with f_0, there exists an $H_1 > 0$ such that $f(x) \geq (f_0 - \varepsilon)x$, for $0 < x \leq H_1$. So, for $u \in \mathscr{P}$ and $\|u\| = H_1$, we have

$$\begin{aligned} Tu(\tau) &= \lambda \int_0^1 G(\tau, s) a(s) f(u(s))\, ds \\ &\geq \lambda \int_{1/4}^{3/4} G(\tau, s) a(s) f(u(s))\, ds \\ &\geq \lambda \int_{1/4}^{3/4} G(\tau, s) a(s) (f_0 - \varepsilon) u(s)\, ds \\ &\geq \frac{\lambda}{4} \int_{1/4}^{3/4} G(\tau, s) a(s)\, ds (f_0 - \varepsilon) \|u\| \\ &\geq \|u\|. \end{aligned}$$

Thus, $\|Tu\| \geq \|u\|$. So, if we let

$$\Omega_1 = \{x \in \mathscr{B} \mid \|x\| < H_1\},$$

then

$$\|Tu\| \geq \|u\|, \qquad \text{for } u \in \mathscr{P} \cap \partial\Omega_1. \tag{16}$$

It remains to consider f_∞. There exists an $\overline{H}_2 > 0$ such that $f(x) \leq (f_\infty + \varepsilon)x$, for all $x \geq \overline{H}_2$. There are the two cases, (a) f is bounded, and (b) f is unbounded.

For case (a), suppose $N > 0$ is such that $f(x) \leq N$, for all $0 < x < \infty$. Let $H_2 = \max\{2H_1, N\lambda\int_0^1 s(1-s)a(s)\, ds\}$. Then, for $u \in \mathscr{P}$ with $\|u\| = H_2$, we have

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t, s) a(s) f(u(s))\, ds \\ &\leq \lambda N \int_0^1 s(1-s) a(s)\, ds \\ &\leq \|u\|, \end{aligned}$$

so that $\|Tu\| \leq \|u\|$. So, if

$$\Omega_2 = \{x \in \mathscr{B} \mid \|x\| < H_2\},$$

then

$$\|Tu\| \leq \|u\|, \qquad \text{for } u \in \mathscr{P} \cap \partial\Omega_2. \tag{17}$$

For case (b), let $H_2 > \max\{2H_1, \overline{H}_2\}$ be such that $f(x) \le f(H_2)$, for $0 < x \le H_2$. Choosing $u \in \mathscr{P}$ with $\|u\| = H_2$,

$$\begin{aligned} Tu(t) &\le \lambda \int_0^1 s(1-s)a(s)f(u(s))\,ds \\ &\le \lambda \int_0^1 s(1-s)a(s)f(H_2)\,ds \\ &\le \lambda \int_0^1 s(1-s)a(s)\,ds(f_\infty + \varepsilon)H_2 \\ &= \lambda \int_0^1 s(1-s)a(s)\,ds(f_\infty + \varepsilon)\|u\| \\ &\le \|u\|, \end{aligned}$$

and so $\|Tu\| \le \|u\|$. For this case, if we let

$$\Omega_2 = \{x \in \mathscr{B} \mid \|x\| < H_2\},$$

then

$$\|Tu\| \le \|u\|, \qquad \text{for } u \in \mathscr{P} \cap \partial\Omega_2. \tag{18}$$

Thus, in either of the cases, an application of part (ii) of Theorem 1 yields a solution of (1λ), (2) which belongs to $\mathscr{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. This completes the proof. ∎

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