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On the Existence of Positive Solutions for Semilinear Elliptic Equations in the Annulus*

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We study the existence of positive radial solutions of $\Delta u + g(|x|) f(u) = 0$ in annuli with Dirichlet (Dirichlet/Neumann) boundary conditions. We prove that the problems have positive radial solutions on any annulus if f is sublinear at 0 and ∞. © 1994 Academic Press, Inc.

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The existence and uniqueness of solutions for semilinear elliptic equations in general domains have been widely studied [1–4]. In this paper, we consider the existence of positive radial solutions of the equation

$$\Delta u + g(|x|) f(u) = 0, \qquad R_1 < |x| < R_2, \quad x \in R^N, \quad N \geqslant 2. \tag{1.1}$$

with one of the following sets of boundary conditions,

$$u = 0 \text{ on } |x| = R_1 \quad \text{and} \quad |x| = R_2, \tag{1.2a}$$

$$u = 0 \text{ on } |x| = R_1 \quad \text{and} \quad \partial u/\partial r = 0 \text{ on } |x| = R_2, \tag{1.2b}$$

$$\partial u/\partial r = 0 \text{ on } |x| = R_1 \quad \text{and} \quad u = 0 \text{ on } |x| = R_2, \tag{1.2c}$$

where $r = |x|$ and $\partial/\partial r$ denotes differentiation in the radial direction, and $0 < R_1 < R_2 < \infty$.

First, let $f_0 = \lim_{u \to 0} f(u)/u$ and $f_\infty = \lim_{u \to \infty} f(u)/u$.

For (1.1)–(1.2a), when $g(r) = 1$, Garaizar [6] proved the following result.

Assume f satisfies

(A-1)′ $f \in C[0, \infty)$, $f(u) > 0$ for $u > 0$ and $f(0) = 0.$,

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(A-2)′ $d_1 u^k \leqslant f(u) \leqslant d_2 u^k$ for large u, where $d_1, d_2 > 0$ and $k > -1$.

(A-3)′ $f_0 = 0$ and $k > 1$ or $f_0 = \infty$ and $k < 1$.

Then there is a radial positive solution to (1.1)–(1.2a) for any $R_1 < R_2$.

Bandle, Coffman, and Marcus [5] proved the theorem below.

Let $f \in C^1[0, \infty)$ and satisfy (A-1)′ and

(A-4)′ f is nondecreasing on $(0, \infty)$.

If f is superlinear at 0 and ∞, or $f_0 = 0$ and $f_\infty = \infty$, then (1.1)–(1.2) has a radial positive solution for any $R_1 < R_2$.

Also, it is remarked in [5] that (A-4)′ is not a necessary condition for existence. This was confirmed by Coffman and Marcus [8], Bandle and Kwong [9], Lin [7], and the author [10], who showed that $f_0 = 0$ and $f_\infty = \infty$ are sufficient to guarantee existence. Moreover, the approaches used in [5–9] are the shooting method combined with the Sturm comparion theorem and phase–plane method. The author [10] used the fixed point theorem in cones to prove the results.

There naturally arises a question whether or not the assumption that f is sublinear at 0 and ∞, or $f_0 = \infty$ and $f_\infty = 0$, implies (1.1)–(1.2) has a positive radial solution for any annulus. Obviously the previous results cannot deal with it. For example, let

$$f(u) = \begin{cases} u^{1/2}, & u \leqslant 1, \\ (-1 + \ln 2)(u-1) + 1, & 1 < u < 2, \\ \ln u, & u \geqslant 2. \end{cases}$$

Then $f_0 = \infty$ and $f_\infty = 0$. However, there does not exist $k > -1$ such that (A-2)′ holds for the function f. But in applications of the following theorem, it has a positive radial solution for any annulus.

We show that the answer to the question is yes by using fixed point techniques. The main result is as follows.

THEOREM 1. *Assume f satisfies*

(A-1) $f \in C[0, \infty)$, $f(u) \geqslant 0$ *for* $u \geqslant 0$.

(A-2) $g \in C[0, \infty)$, $g(r) \geqslant 0$ *for* $r \geqslant 0$ *and is not identically zero in any finite subinterval of* $(0, \infty)$.

If f is sublinear at 0 and ∞, or $f_0 = \infty$ and $f_\infty = 0$, then (1.1)–(1.2) *has a positive radial solution for any annulus* $R_1 < |x| < R_2$.

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In view of the spherical symmetry of $g(|x|)$, we seek a positive radial solution $u=u(r)$ to (1.1). Therefore we write (1.1)–(1.2) in the form

$$u''(r)+[(N-1)/r]\,u'(r)+g(r)\,f(u(r))=0, \qquad R_1<r<R_2, \tag{2.1}$$

$$u(R_1)=u(R_2)=0, \tag{2.2a}$$

$$u(R_1)=u'(R_2)=0, \tag{2.2b}$$

$$u'(R_1)=u(R_2)=0. \tag{2.2c}$$

Let $s=-\int_r^{R_2} 1/t^{N-1}\,dt$ and $v(s)=u(r(s))$, then (2.1)–(2.2) can be rewritten as

$$v''(s)+r^{2(N-1)}(s)\,g(r(s))\,f(v(s))=0, \qquad m<s<0,$$

$$v(m)=v(0)=0,$$

$$v(m)=v'(0)=0,$$

$$v'(m)=v(0)=0,$$

where $m=-\int_{R_1}^{R_2} 1/t^{N-1}\,dt$.

To be obvious, again let $t=(m-s)/m$ and $z(t)=v(s)$. Then (1.1)–(1.2) can also be written as

$$z''(t)+h(t)\,f(z(t))=0, \qquad 0<t<1, \tag{2.3}$$

$$z(0)=z(1)=0, \tag{2.4a}$$

$$z(0)=z'(1)=0, \tag{2.4b}$$

$$z'(0)=z(1)=0, \tag{2.4c}$$

where $h(t)=m^2r^{2(N-1)}(m(1-t))\,g(r(m(1-t)))$. It is easy to check that $h(t)$ also satisfies (A-2).

From now on, we concentrate on (2.3)–(2.4). Indeed, if we prove that there exists a positive solution to (2.3)–(2.4) for any $m\neq 0$, then (1.1)–(1.2) has a positive radial solution for any annulus. Hence, Theorem 1 is true. In what follows, we use the following.

FIXED POINT THEOREM [11]. *Let E be a Banach space, P a cone in E, Ω_1 and Ω_2 open subsets in E. Assume $0\in\Omega_1$, $\bar{\Omega}_1\subset\Omega_2$, and $A\colon P\cap(\bar{\Omega}_2\backslash\Omega_1)\to P$ a completely continuous operator. If*

$$\|Az\|\geqslant\|z\| \qquad \textit{for} \quad z\in P\cap\partial\Omega_1,$$

and

$$\|Az\| \leqslant \|z\| \qquad \text{for} \quad z \in P \cap \partial\Omega_2,$$

then A has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

First consider (2.3)–(2.4a). It is easy to check that (2.3)–(2.4a) is equivalent to the integral equation

$$z(t) = \int_0^1 k(t, s)\, h(s)\, f(z(s))\, ds \overset{\text{def}}{=} Az(t),$$

where $z(t) \in C[0, 1]$, and

$$k(t, s) = \begin{cases} t(1-s), & t \leqslant s \\ s(1-t), & t > s. \end{cases}$$

Denote $P = \{z(t) : z(t) \in C[0, 1],\ z(t) \geqslant 0,\ \min\{z(t), \frac{1}{4} \leqslant t \leqslant \frac{3}{4}\} \geqslant \|z\|/4\}$ (In this paper, only the sup norm is used.). It is obvious that P is a cone in $C[0, 1]$. Moreover, $AP \subset P$. Indeed, from $k(t, s) \leqslant s(1-s)$, it follows that for any $z(t) \in P$ one has

$$\|Az\| \leqslant \int_0^1 s(1-s)\, h(s)\, f(z(s))\, ds. \tag{2.5}$$

On the other hand, as $\frac{1}{4} \leqslant t \leqslant \frac{3}{4}$, one also has

$$k(t, s) \geqslant \begin{cases} (1-s)/4, & t \leqslant s \\ s/4, & t > s. \end{cases}$$

Hence,

$$k(t, s) \geqslant s(1-s)/4, \qquad \text{for} \quad \tfrac{1}{4} \leqslant t \leqslant \tfrac{3}{4}, \quad 0 \leqslant s \leqslant 1.$$

Therefore,

$$\min\{Az(t), \tfrac{1}{4} \leqslant t \leqslant \tfrac{3}{4}\} \geqslant \tfrac{1}{4} \int_0^1 s(1-s)\, h(s)\, f(z(s))\, ds.$$

In applications of (2.5) it follows that

$$\min\{Az(t), \tfrac{1}{4} \leqslant t \leqslant \tfrac{3}{4}\} \geqslant \|Az\|/4.$$

Note that $k(t, s) > 0$, and so we conclude that $Az \in P$. It is also easy to check that $A: P \to P$ is completely continuous.

From $f_0 = \infty$, there is an $H_1 > 0$ such that

$$f(z) \geqslant Mz, \qquad 0 < z \leqslant H_1.$$

where the constant $M > 0$ satisfies

$$M/4 \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)\, h(s)\, ds \geqslant 1.$$

Thus, for any $z(t) \in P$ and $\|z\| = H_1$, it follows that

$$Az(\tfrac{1}{2}) = \int_0^1 k(\tfrac{1}{2}, s)\, h(s)\, f(z(s))\, ds \geqslant \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)\, h(s)\, f(z(s))\, ds$$

$$\geqslant M/4 \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)\, h(s)\, ds\, \|z\| \geqslant \|z\|.$$

Therefore, let

$$\Omega_1 = \{z : z \in C[0, 1], \|z\| < H_1\};$$

then one has

$$\|Az\| \geqslant \|z\|, \qquad z \in P \cap \partial\Omega_1.$$

In what follows, we determine Ω_2. From $f_\infty = 0$, it follows that there is a $q > 0$ such that

$$f(z) \leqslant pz, \qquad \text{for any} \quad z \geqslant q.$$

where the constant $p > 0$ such that

$$p \int_0^1 s(1-s)\, h(s)\, ds \leqslant 1.$$

If f is bounded, say $f(z) \leqslant N$ for $z \in (0, \infty)$, where N is a positive constant, then we choose $H_2 > 2H_1$ such that for $z \in P$ and $\|z\| = H_2$,

$$Az(t) = \int_0^1 k(t, s)\, h(s)\, f(z(s))\, ds \leqslant \int_0^1 s(1-s)\, h(s)\, fz(s))\, ds$$

$$\leqslant N \int_0^1 s(1-s)\, h(s)\, ds \leqslant H_2.$$

If f is unbounded on $(0, \infty)$, H_2 is chosen so that $H_2 > \max\{2H_1, q\}$ and $f(z) \leqslant f(H_2)$ for $0 < z \leqslant H_2$. Then for $z \in P$ and $\|z\| = H_2$ we have

$$Az(t) = \int_0^1 k(t, s)\, h(s)\, f(z(s))\, ds \leqslant \int_0^1 s(1-s)\, h(s)\, f(H_2))\, ds$$

$$\leqslant p \int_0^1 s(1-s)\, h(s)\, ds\, H_2 \leqslant H_2.$$

In either case, put

$$\Omega_2 = \{z : z \in C[0, 1],\ \|z\| < H_2\};$$

then

$$\|Az\| \leqslant \|z\| \qquad \text{for} \quad z \in P \cap \partial\Omega_2.$$

Now applying the fixed point theorem, we conclude that A has a fixed point z in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $H_1 \leqslant \|z\| \leqslant H_2$. It is also a solution to (2.3)–(2.4a). Because of the properties of $k(t, s)$, it follows that $z(t) > 0$, $0 < t < 1$. Therefore (1.1)–(1.2a) has a positive radial solution.

This completes the proof of the first part of Theorem 1.

Next we consider (2.3)–(2.4b) and (2.3)–(2.4c). It is easy to check that (2.3)–(2.4b) and (2.3)–(2.4c) are equivalent to the integral equations $z(t) = \int_0^1 k_1(t, s)\, h(s)\, f(z(s))\, ds = A_1 z(t)$ and $z(t) = \int_0^1 k_2(t, s)\, h(s)\, f(z(s))\, ds = A_2 z(t)$, respectively, where $z(t) \in C[0, 1]$ and

$$k_1(t, s) = \begin{cases} t, & t \leqslant s \\ s, & t > s, \end{cases} \qquad k_2(t, s) = \begin{cases} 1 - s, & t \leqslant s \\ 1 - t, & t > s. \end{cases}$$

For A_1 let P_1 be the cone

$$P_1 = \{z(t) : z \in C[0, 1],\ z(t) \geqslant 0,\ \min\{z(t), \tfrac{1}{2} \leqslant t \leqslant 1\} \geqslant \|z\|/2\}.$$

For A_2 let P_2 be the cone

$$P_2 = \{z(t) : z \in C[0, 1],\ z(t) \geqslant 0,\ \min\{z(t), 0 \leqslant t \leqslant \tfrac{1}{2}\} \geqslant \|z\|/2\}.$$

By the method above, the fixed point theorem is used to prove that both (1.1)–(1.2b) and (1.1)–(1.2c) have positive solutions.

To sum up, we complete the proof of Theorem 1.

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