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Modeling boyciana–fish–human interaction with partial differential algebraic equations

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ABSTRACT

Under the influence of human population distribution, the boyciana–fish ecological system is considered. First, the system can be described as a nonlinear partial differential algebraic equations system (PDAEs) with Neumann boundary conditions and ratio-dependent functional response. Second, we examine the system's persistence properties: the loacl stabilities of positive steady states, the absorbtion region and the global stability. And the proposed approach is illustrated by numerical simulation. Finally, by using the realistic data collected in the past fourteen years, the PDAEs parameter optimization model is built to predict the boyciana population.

1. Introduction

In Northern China, Beidaihe wetland is located in the junction of three big ecosystems that are forest, ocean and wetland. Beidaihe wetland is one of the important channels for Far East migratory birds. In September and October, there are about 400 species of birds which migrate to Beidaihe wetland park. At the same time, as an international tourist city, Beidaihe attracts a large number of tourists to travel from May to October. Due to the increasing human activities, many living and breeding areas are polluted, leading to the decrease of the bird population.

Boyciana is one of the most sensitive species in Beidaihe wetland system. It is the first-grade state-protected animal in China. Boyciana was widely distributed in Northeast Asia. In 1986, the experts found that 2729 boyciana moved through Beidaihe wetland region. However, in recent decades, the human's activities made boyciana's predatory object quantity to reducing. The habitat environment of boyciana was also destroyed. Many environmental factors influence the spatiotemporal distribution of boyciana such as hidden factor, water factor, vegetation factor and food factor are directly or indirectly related to human activities [1].

Despite a rich literature on the spatiotemporal research of ecosystem [2], the human's interference in the ecological spatiotemporal process is rarely studied. Thus, the goal of the theoretical ecology model is to study how the interactions between boyciana and their food-wetland fish with human social behavior influence.

Mathematically, reaction-diffusion equation can be used to model the spatiotemporal distribution and abundance of organisms [3–6]. In recent decades, the role of the reaction-diffusion effect in maintaining bio-diversity has received a great deal of attention in the literature on ecology conservation [7]. Empirical evidence suggests that the spatial scale can influence population interactions. The major classes of spatial models are those that treat space as a continuum and describe the distribution of populations in terms of densities. A typical form of reaction-diffusion population model is

$$\frac{\partial x}{\partial t} = D\Delta x + x f(z, x)$$

where $x(t, z)$ is the population densities vector at time t and space point z, D is the diffusion constant matrix, Δ is the Laplace operator with respect to the spatial variable z, and $f(z, x)$ is the growth function vector. Such an ecological model was first considered by Skellam [8], and the reaction-diffusion biological models were also studied by Fisher [9] and Kolmogoroff [10].

For the predator–prey type reaction-diffusion biological models, in reference to the functional response models, the traditional used function are Lotka–Volterra [11], Allee effect [12], Holloing type [13], Bedding-DeAngelis [14], ratio-dependent [15] etc. All the predator–prey models cannot be directly applied in the human interference model (2.1). The developed system includes three interaction species: boyciana, fish and human.

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In this purpose, the ratio-dependent reaction-diffusion system model [15] is modified in which the incorporating one prey and two competing predator species was considered. In [15], we replace one competing predator species by human. Specially, the human influence part is some degenerated Fisher population model of elliptic type. Although the global attractor and persistence of the parabolic equations system was discussed in [15] by comparison principle theory. While the model (2.1) involves some elliptic equation, the new methods are proposed on the persistence property. The diffusion-driven instability or turing instability, which has attracted the attention of some investigators, is also discussed in this study by using the qualitative theory.

The aim of this work is to propose the qualitative analysis about three species (boyciana, fish and human) interrelated spatiotemporal ecological wetland model expressed in term of PDAEs. And as an application, it illustrates the numerical simulation and prediction to show the effective of the results.

The remaining part of this paper is organized as follows. In Section 2, the PDAEs model is built on the ecological system. In Section 3, by developed PDE energy estimation method, the persistence property of the system (2.1) is examined. In addition, the numerical simulations are carried out to show the effective of the proposed results. In Section 4, under some reasonable assumptions, with the data collected from the wetland conservation, the PDAEs parameter optimal model is carried to predict the boyciana population in the future.

Notations: $\mathbb{N}_0$ is natural number set. Ω is a bounded plane domain with the boundary $\partial\Omega$. $\|\cdot\|$ denotes the Euclidean norm for vectors. For a symmetric matrix M, $M > (<)0$ means that it is positive (negative) definite. I is the identity matrix. The superscript T is used for the transpose. Matrices, if not explicitly stated, are assumed to have compatible dimensions. For the convenience, the following Hilbert space is defined:

$$\mathrm{H}_2(\Omega) \triangleq \{x : \Omega \times [0, +\infty) \to \mathbb{R}^n \text{ and } \|x\| < \infty\}$$

with inner product and L_2-norm respectively defined by

$$\langle x, y\rangle \triangleq \int_\Omega x^T y \mathrm{d}\mathbf{z}, \ \|x\| \triangleq \left\{\int_\Omega \|x\|^2 \mathrm{d}\mathbf{z}\right\}^{1/2}.$$

2. Boyciana–fish–human model

In a protected environment, human, boyciana and fish are chosen as the research objects. The spatiotemporal dynamics between boyciana and fish with human activity affect is

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = d_1\Delta x_1 + x_1\left(1 - x_1 - \dfrac{cx_2}{x_1+\alpha x_2} - h_1x_3\right), \mathbf{z}\in\Omega, t>0, \\ \dfrac{\partial x_2}{\partial t} = d_2\Delta x_2 + x_2\left(-d + \dfrac{mx_1}{x_1+\alpha x_2} - h_2x_3\right), \mathbf{z}\in\Omega, t>0, \\ 0 = \Delta x_3 + rx_3(1-x_3), \mathbf{z}\in\Omega, \\ \dfrac{\partial x_1}{\partial \mathbf{n}} = \dfrac{\partial x_2}{\partial \mathbf{n}} = \dfrac{\partial x_3}{\partial \mathbf{n}} = 0, \mathbf{z}\in\partial\Omega, t>0, \\ x_1(0,\mathbf{z}) = x_1^0(\mathbf{z}) \ge 0, x_2(0,\mathbf{z}) = x_2^0(\mathbf{z}) \ge 0, \mathbf{z}\in\Omega. \end{cases} \tag{2.1}$$

Here, the state variables are $x_1 = x_1(t,\mathbf{z}), x_2 = x_2(t,\mathbf{z}), x_3 = x_3(\mathbf{z})$, Ω is an idealized rectangular domain (See Fig. 1), $\mathbf{z} = [z_1, z_2] \in \Omega$ is the spatial coordinate, $\mathbf{n}$ is the outward unit normal vector of the boundary $\partial\Omega$, the coefficients c, α, m, d, h_1, h_2 and r are positive constants. The initial value $x_1^0(\mathbf{z}), x_2^0(\mathbf{z})$ are non-negative smooth functions which are not identically zero.

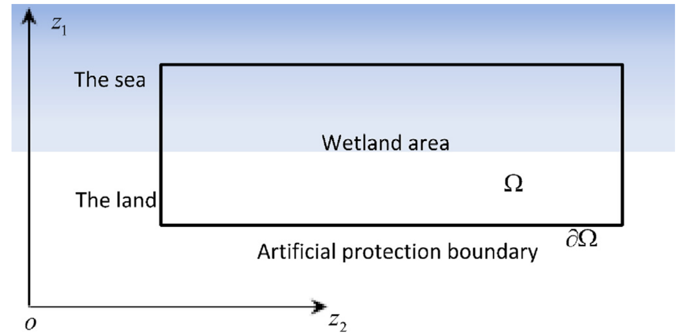


Fig. 1. The idealized spatial domain is a rectangular domain with the sea oriented direction z_1 and coast line direction z_2. The wetland conservation is closed with no flux boundary conditions imposed.

2.1. Ecological description

The system (2.1) describes the population dynamics of boyciana–fish system with humans interference, which is dispersed by diffusion in the habitat area Ω (See Fig. 1).

$x_1(t,\mathbf{z}), x_2(t,\mathbf{z})$ represent the population densities of fish and boyciana at time $t > 0$ and spatial position $\mathbf{z}\in\Omega$ respectively. $x_3(\mathbf{z})$ stands for the human density. $x_1^0(\mathbf{z}), x_2^0(\mathbf{z})$ are the initial population distribution of fish and bird respectively. Obviously, these two initial values are depended on the spatial the geographical position $\mathbf{z}$. In this point of view, this is more realistic to describe spatial population distribution than the ODEs system.

The Neumann boundary conditions

$$\frac{\partial x_1}{\partial \mathbf{n}} = \frac{\partial x_2}{\partial \mathbf{n}} = \frac{\partial x_3}{\partial \mathbf{n}} = 0, \mathbf{z}\in\partial\Omega$$

mean that (2.1) is self-contained and has no population flux across the boundary $\partial\Omega$, so that $\partial\Omega$ acts as a perfect barrier to dispersal.

The interaction between fish and boyciana is based on two ratio-dependent functional response functions

$$\frac{cx_1x_2}{x_1+\alpha x_2}, \frac{mx_1x_2}{x_1+\alpha x_2}$$

where c is the capturing rate (or catching efficiency) of the boyciana, m is the conversion rate. Fish population follows the logistic growth in the absence of boyciana and human. d is the death rate of boyciana.

When the distribution of the individuals is not uniform and depends on different spatial locations, the standard method to describe the spatial effects is to introduce the diffusion terms. Therefore, the diffusion coefficient matrix about the fish and the boyciana is introduced, that is

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\Delta x \tag{2.2}$$

where $x = [x_1(t,\mathbf{z}), x_2(t,\mathbf{z})]^T$, $\Delta = \Sigma_{i=1}^2 \partial_{z_i}^2$ is the Laplace operator. It is well known that the appearance of the spatial dispersal makes the dynamics and behaviors of the boyciana–fish system even more complicated.

Specifically, for a wetland ecosystem, the influence of human can be regarded as an invasive species and not be affected by other species. Therefore, mathematically, $-h_1x_3$ and $-h_2x_3$ are added in the first two equations of (2.1) which represent the human's interferences on fish and boyciana, respectively.

The third equation of (2.1) is derived from the well-known Fisher equation

$$\frac{\partial x_3(t,\mathbf{z})}{\partial t} = \Delta x_3(t,\mathbf{z}) + rx_3(t,\mathbf{z})(1-x_3(t,\mathbf{z})), \mathbf{z}\in\Omega, t>0. \tag{2.3}$$

where the nonlinear function $rx_3(1-x_3)$ is referred to as a logistic nonlinearity. Since the local human population distribution can reach a time independent dynamic balance in a short time. Thus (2.3) degenerates to the following elliptic equation

$$0=\Delta x_3(\mathbf{z})+r_3x_3(\mathbf{z})(1-x_3(\mathbf{z})), \mathbf{z}\in\Omega. \tag{2.4}$$

2.2. PDAEs description

By [16], the system (2.1) can be rewritten as the following matrix form

$$E\frac{\partial x}{\partial t}=D\Delta x+f(x) \tag{2.5}$$

subject to the boundary conditions (BCs)

$$\frac{\partial x_\iota(t,\mathbf{z})}{\partial\mathbf{n}}=0, \mathbf{z}\in\partial\Omega, \iota\in\vartheta_{BC} \tag{2.6}$$

and the initial conditions (ICs)

$$x_\kappa(0,\mathbf{z})=x_\kappa^0(\mathbf{z}), \mathbf{z}\in\Omega, \kappa\in\vartheta_{IC} \tag{2.7}$$

where $x=(x_1,x_2,x_3)^T$, $\vartheta_{BC}=\{1,2,3\}$, $\vartheta_{IC}=\{1,2\}$ and

$$E=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}, D=\mathrm{diag}(D_1,1), D_1=\begin{pmatrix}d_1&0\\0&d_2\end{pmatrix},$$

$$f(x)=\begin{pmatrix}x_1\left(1-x_1-\frac{cx_2}{x_1+\alpha x_2}-h_1x_3\right)\\ x_2\left(-b+\frac{mx_1}{x_1+\alpha x_2}-h_2x_3\right)\\ rx_3(1-x_3)\end{pmatrix}.$$

The applications and mathematically research on PDAEs have attracted increasing attention to academics [17–19]. In view of the latest literature in reaction-diffusion system research, in most of these system, the derivative coefficient matrix D is invertible. In other words, they are the parabolic type nonlinear partial differential equations system (PDEs). Most of them can be analyzed with the existing qualitative theory [20] directly.

However, from the above PDAEs system (2.5)–(2.7), the time derivative coefficient matrix

$$E=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

is singular. Mathematically, it is a generalization of the classical parabolic PDEs system with E as a unit matrix. Because some theoretical results [21,22] cannot be directly applied in this parabolic–elliptic type, researches in the singular case are relatively scarce. The identifiability and stability properties of some PDAEs have been studied [16] with singular system theory.

On the other hand, in the field of control, this system is also called singular distributed parameter system (SDPS). If the diffusion term (2.2) is not include in this system, it changes to a singular system or generalized state-space system [23].

3. Local and global stability

In this section, the focus lies in the steady state solutions of (2.1). Since the semi-linear elliptic part of (2.1) is independent of time, the following parabolic subsystem is the main concern:

$$\begin{cases}\dfrac{\partial x_1}{\partial t}=d_1\Delta x_1+x_1\left(1-x_1-\dfrac{cx_2}{x_1+\alpha x_2}-h_1x_3\right), \mathbf{z}\in\Omega, t>0,\\ \dfrac{\partial x_2}{\partial t}=d_2\Delta x_2+x_2\left(-d+\dfrac{mx_1}{x_1+\alpha x_2}-h_2x_3\right), \mathbf{z}\in\Omega, t>0,\\ \dfrac{\partial x_1}{\partial\mathbf{n}}=\dfrac{\partial x_2}{\partial\mathbf{n}}=0, \mathbf{z}\in\partial\Omega, t>0,\\ x_1(0,\mathbf{z})=x_1^0(\mathbf{z})\ge0, x_2(0,\mathbf{z})=x_2^0(\mathbf{z})\ge0, \mathbf{z}\in\Omega.\end{cases} \tag{3.1}$$

where x_3 satisfies the elliptic boundary problem

$$\begin{cases}0=\Delta x_3(\mathbf{z})+rx_3(\mathbf{z}), \mathbf{z}\in\Omega,\\ \dfrac{\partial x_3(\mathbf{z})}{\partial\mathbf{n}}=0, \mathbf{z}\in\partial\Omega.\end{cases} \tag{3.2}$$

In comparison with the autonomous system in [14], as external input variables, human influence functions h_1x_3, h_2x_3 are added into the nonlinear parabolic subsystem (3.1). This leads to different dynamic behavior with human population distribution $x_3(\mathbf{z})$.

In order to clearly present the human population distribution, the property of $x_3(\mathbf{z})$ is first considered in Section 3.1. In Section 3.1, it also shows that the human spatial distribution $x_3(\mathbf{z})$ is directly related to the intrinsic growth distribution parameter r. In consequence, this lead us to investigate the persistence property of (3.1) with parameter r in Sections 3.2–3.4.

3.1. Studied on the human distribution subsystem

Consider the human distribution subsystem in (2.1)

$$\begin{cases}0=\Delta x_3(\mathbf{z})+rx_3(\mathbf{z})(1-x_3(\mathbf{z})), \mathbf{z}\in\Omega,\\ \dfrac{\partial x_3(\mathbf{z})}{\partial\mathbf{n}}=0, \mathbf{z}\in\partial\Omega.\end{cases} \tag{3.3}$$

The solution of (3.3) represents the spatial distribution of the human population which has direct influence on the populations of boyciana and their food. With the existing results in elliptic PDE theory [21,24], the solution of the system (3.3) could be not unique. Specially, $x_3(\mathbf{z})$ depends not only on the choice of parameter r but also on the shape of the domain Ω [21]. The following discusses the existence and uniqueness of the solution of (3.3) and gives some ecological description on the positive solution.

First of all, it is obvious that the system (3.3) has a coupled upper and lower constant solutions

$$\bar{x}_3(\mathbf{z})=1, \underline{x}_3(\mathbf{z})=0, \quad \mathbf{z}\in\Omega.$$

The trivial solution $\bar{x}_3(\mathbf{z})=1$ can be interpreted as human population density reaches the wetland ecology system capacity limit. In other words, the wetland area is facing the human's over development threat. On the other hand, the solution $\underline{x}_3(\mathbf{z})=0$ means that the wetland ecology system is in a human-free environment. However, under the realistic circumstance, human population density has decreasing property along z_1 axis

(see Fig. 1) which is perpendicular to the coastline. Therefore, what the study is interested in is the nontrivial solution of Neumann problem (3.3).

Theorem 3.1 (Uniqueness of nontrivial positive solution)**.** *Let $\lambda_1(\Omega)$ be the first eigenvalue of the following eigenvalue problem*

$$\begin{cases}0=\Delta x_3(\mathbf{z})+rx_3(\mathbf{z}), \mathbf{z}\in\Omega,\\ \dfrac{\partial x_3(\mathbf{z})}{\partial\mathbf{n}}=0, \mathbf{z}\in\partial\Omega.\end{cases} \tag{3.4}$$

If $r>\lambda_1(\Omega)$ holds, then (3.3) has a unique nontrivial positive solution $x_3^p(\mathbf{z})$ such that $0\le x_3^p(\mathbf{z})\le1$ and

$$\lim_{r\searrow\lambda_1(\Omega)}x_3^p(\mathbf{z})=0, \quad \int_\Omega x_3^p(\mathbf{z})\mathrm{d}\mathbf{z}\le(r-\lambda_1(\Omega))|\Omega|, \tag{3.5}$$

where $\searrow$ represents the right limit, $|\Omega|$ is the measure of Ω. Additionally, if $\partial\Omega\in C^1$ then

$$\lim_{r\to\infty}x_3^p=1, \quad \mathbf{z}\in\overline{\Omega'}. \tag{3.6}$$

where $\overline{\Omega'}\subset\Omega$ is closed in Ω.

Proof. Noticing that $\bar{x}_3, \underline{x}_3$ are a couple of upper and lower solutions of (3.3), by directly using the comparison principle [24], there exists a nonconstant positive solution $x_3^p(\mathbf{z})$ satisfies $0\le x_3^p\le1$.

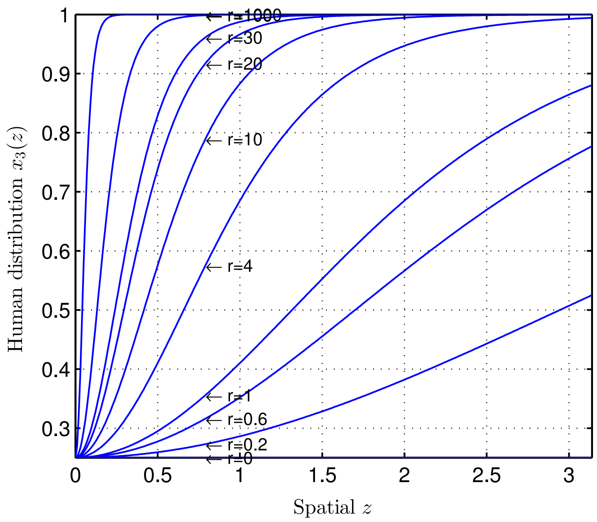


Fig. 2. Human distribution curve with different parameter values r, $\Omega = [0, \pi]$.

For the uniqueness of the nontrivial positive solutions, [25, Section 3.5.3] shows that (4.1) has the unique positive solution $x_3^p(\mathbf{z})$ if and only if

$$r > \lambda_1(\Omega)$$

where $\lambda_1(\Omega)$ is the first eigenvalue of (3.4). This completes the proof. □

By the above discussion, three nonnegative solutions of (3.3) are obtained

$$\underline{x}_3 = 0, \bar{x}_3 = 1, x_3^p(\mathbf{z}).$$

These correspond to three different kinds of population distribution: human-free, population limit and normal non-uniform distribution. The biological interpretation of Theorem 3.1 is that if the wetland system has the humans carrying capacity 1 then over all of Ω the total population would be $|\Omega|$. So the integral inequality (3.5) shows that the total population reduces with r approach $\lambda_1(\Omega)$. The limit equality (3.6) shows if the spatial scale of Ω as measured by r is sufficiently large then the population density in Ω will be close to its carrying capacity $\bar{x}_3$ on Ω except for a relatively narrow strip near the boundary $\partial\Omega$ (See Fig. 2). To guarantee the existence and uniqueness of the nontrivial positive solution, it assumes $r > \lambda_1(\Omega)$ throughout this study.

3.2. Local stability and instability of human free model ($r \searrow \lambda_1(\Omega)$)

From Theorem 3.1,

$$\lim_{r \searrow \lambda_1(\Omega)} x_3(\mathbf{z}) = 0$$

holds. By substituting human distribution $x_3 = 0$ into (3.1), it reaches

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = d_1 \Delta x_1 + x_1\left(1 - x_1 - \dfrac{cx_2}{x_1 + \alpha x_2}\right), \mathbf{z} \in \Omega, t > 0, \\ \dfrac{\partial x_2}{\partial t} = d_2 \Delta x_2 + x_2\left(-d + \dfrac{mx_1}{x_1 + \alpha x_2}\right), \mathbf{z} \in \Omega, t > 0, \\ \dfrac{\partial x_1}{\partial \mathbf{n}} = \dfrac{\partial x_2}{\partial \mathbf{n}} = 0, \mathbf{z} \in \partial\Omega, t > 0, \\ x_1(0, \mathbf{z}) = x_1^0(\mathbf{z}) \geq 0, x_2(0, \mathbf{z}) = x_2^0(\mathbf{z}) \geq 0, \mathbf{z} \in \Omega. \end{cases} \tag{3.7}$$

From the ecological point of view, the system (3.7) represents that the wetland system is in the original ecological environment situation. Now, mathematically the local stability property of the system (3.7) is studied.

The positive constant equilibrium of the system (3.7) is the positive solution of the following nonlinear equations

$$\begin{cases} f_1(x_1, x_2) := x_1\left(1 - x_1 - \dfrac{cx_2}{x_1 + \alpha x_2}\right) = 0, \\ f_2(x_1, x_2) := x_2\left(-d + \dfrac{mx_1}{x_1 + \alpha x_2}\right) = 0. \end{cases} \tag{3.8}$$

By simple analysis for (3.8), it has

Theorem 3.2 (Existence of positive equilibrium)**.** *When $1 - \frac{\alpha}{c} < \frac{d}{m} < 1$, the system (3.7) has a unique positive equilibrium $e_1 := (u_1^*, \upsilon_1^*)$ with*

$$u_1^* = 1 - \frac{c(m-d)}{m\alpha} = 1 - \frac{c}{\alpha}\left(1 - \frac{d}{m}\right),$$
$$\upsilon_1^* = \frac{m(1-u_1^*)u_1^*}{cd} = \frac{1}{\alpha}\left(\frac{m}{d} - 1\right)u_1^*$$

In addition, $0 < u_1^* < 1$, $\upsilon_1^* > 0$; $u_1^* \geq 1$ and $\upsilon_1^* \leq 0$ when $\frac{d}{m} \geq 1$.

Now, with the linearized method of dynamic system and the eigenvalue theory of PDE, the local asymptotic stability of the positive constant steady state e_1 is discussed.

Considering the linearizing system of (3.7) at e_1

$$\frac{\partial x}{\partial t} = \mathscr{L}x \tag{3.9}$$

where the linear operator $\mathscr{L}$ be defined by

$$\mathscr{L} = D_1\Delta + \bar{f}_J \tag{3.10}$$

with $\bar{f}_J$ is the Jacobian matrix of $\bar{f} = (f_1, f_2)^T$ at e_1.

Let

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$$

be the eigenvalues of the operator $-\Delta$ on Ω with Neumann BC. The corresponding eigenfunctions are represented by $\phi_n (n \in \mathbb{N}_0)$. Thus, $\lambda_n, \phi_n (n \in \mathbb{N}_0)$ satisfy

$$\begin{cases} -\Delta\phi = \lambda\phi \quad \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Then the function sequence $\{\phi_n\}_{n=0}^{\infty}$ forms an orthonormal base of $\mathrm{H}^2(\Omega)$. It should be noticed that the eigenvector w.r.t. $\lambda_0 = 0$ is $\phi_0 = \text{const}$. The corresponding solution is trivial which cannot influence the stability of the system. Therefore, it is concerned with the following infinity dimensional ODE systems (see [16] for detail):

$$\dot{X}_n = \mathscr{L}_n X_n, n \in \mathbb{N}_0. \tag{3.12}$$

where $X_n = \langle x, \phi_n \rangle = \int_\Omega x(t, \mathbf{z})\phi(\mathbf{z})\mathrm{d}\mathbf{z}$,

$$\mathscr{L}_n = -D_1\lambda_n + \bar{f}_J. \tag{3.13}$$

Substituted $\bar{f}_J$ with e_1, it follows with

$$\mathscr{L}_n = \begin{pmatrix} -\dfrac{m^2\alpha + cd^2 - cm^2 + d_1\lambda_n\alpha m^2}{m^2\alpha} & -c\dfrac{d^2}{m^2} \\ \dfrac{(m-d)^2}{m\alpha} & -\dfrac{\lambda_n d_2 m + md - d^2}{m} \end{pmatrix}. \tag{3.14}$$

Let $\mathrm{Dt}(e_1)$, $\mathrm{Tr}(e_1)$ denote the determinant and the trace of the matrix $\mathscr{L}_n$ respectively. Then

$$\begin{cases} \mathrm{Dt}(e_1) = \dfrac{m^2 d\alpha - cd^3 - md^2\alpha - m^2 dc + 2md^2 c}{m^2\alpha} \\ \quad + \dfrac{\lambda_n d^2 cd_2 + \lambda_n m^2 \alpha d_2 - \lambda_n m^2 cd_2 + \lambda_n^2 m^2 \alpha d_1 d_2 + \lambda_n m^2 d\alpha d_1 - \lambda_n md^2 \alpha d_1}{m^2\alpha}, \\ \mathrm{Tr}(e_1) = -\dfrac{m^2\alpha\lambda_n(d_1+d_2) + m^2\alpha + d^2 c - m^2 c}{m^2\alpha}. \end{cases} \tag{3.15}$$

The characteristic equation of the matrix $\mathscr{L}_n$ is

$$\mu^2 + \mu\mathrm{Tr}(e_1) + \mathrm{Dt}(e_1) = 0.$$

From Theorem 3.2, when $1 - \frac{\alpha}{c} < \frac{d}{m} < 1$, there stands $m > d$.

If $c \le \alpha$, one can obtain

$$m^2 d\alpha - m^2 dc \ge 0, \quad 2md^2 c - cd^3 - md^2\alpha > 0,$$

then the first part of $\mathrm{Dt}(e_1)$ satisfies

$$\frac{m^2 d\alpha - cd^3 - md^2\alpha - m^2 dc + 2md^2 c}{m^2\alpha} > 0.$$

In the same way,

$$\lambda_n m^2 \alpha d_2 - \lambda_n m^2 cd_2 \ge 0$$

and

$$m^2 d\alpha d_1 - \lambda_n md^2 \alpha d_1 > 0$$

imply

$$\frac{\lambda_n d^2 cd_2 + \lambda_n m^2 \alpha d_2 - \lambda_n m^2 cd_2 + \lambda_n^2 m^2 \alpha d_1 d_2 + \lambda_n m^2 d\alpha d_1 - \lambda_n md^2 \alpha d_1}{m^2\alpha} > 0.$$

Then $\mathrm{Tr}(e_1) < 0$, $\mathrm{Dt}(e_1) > 0$ hold for any $\lambda_n, n \in \mathbb{N}_0$.

If $c > \alpha$, by the monotonicity of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, through some directly computing, when $\lambda_1(d_1+d_2) - \frac{c}{\alpha}\frac{m-d}{m}\frac{d}{m} > 0$, there stand $\mathrm{Tr}(e_1) < 0$ and $\mathrm{Dt}(e_1) > 0$. In summary, the following conclusion is reached

Theorem 3.3 (Local stability). *Under the condition $1 - \frac{\alpha}{c} < \frac{d}{m} < 1$,*

S1) When $c \le \alpha$, the positive equilibrium e_1 is locally asymptotically stable for any diffusion coefficients $d_1, d_2 > 0$.

S2) When $c > \alpha$, the positive equilibrium e_1 is locally asymptotically stable, if and only if

$$\lambda_1(d_1+d_2) - \frac{c}{\alpha}\frac{m-d}{m}\frac{d}{m} > 0$$

where λ_1 is the first eigenvalue of S-L problem (3.11).

From a biological point of view, Theorem 3.3 implies that if there is no human interference ($x_3 = 0$), when the capturing rate c is low with the high conversion rate m, boyciana population x_2 and their fish food x_1 can maintain the stable coexistence situation. In addition, the diffusion term $\lambda_1(d_1+d_2)$ in the inequality shows that better diffusion effect helps maintain the balance of the ecosystem. The above described circumstances consistent with experience.

Define the domains I, II and III in the (c, α, m, d) space respectively by

$$\mathrm{I} = \left\{c \le \alpha, 0 < \frac{d}{m} < 1\right\}, \tag{3.16}$$

$$\mathrm{II} = \left\{c > \alpha, 1 - \frac{\alpha}{c} < \frac{d}{m} < 1, \frac{c}{\alpha}\frac{m-d}{m}\frac{d}{m} \le \lambda_1(d_1+d_2)\right\}, \tag{3.17}$$

$$\mathrm{III} = \left\{c > \alpha, 1 - \frac{\alpha}{c} < \frac{d}{m} < 1, \frac{c}{\alpha}\frac{m-d}{m}\frac{d}{m} > \lambda_1(d_1+d_2)\right\}. \tag{3.18}$$

We assert that if $(c, \alpha, m, d) \in$ I, II, then the positive equilibrium e_1 of system (3.7) is locally asymptotically stable, else if (c, α, m, d) is in III then e_1 is unstable. The following is the numerical illustration.

For the numerical evaluation of the human free system (3.7), fix the system parameters as

$$c = 1.000, \alpha = 0.5000, d = 0.9000, m = 1.000. \tag{3.19}$$

By Theorem 3.3, (3.19) fulfill the positive condition

$$1 - \frac{\alpha}{c} < \frac{d}{m} < 1.$$

The corresponding equilibrium is $e_1 \approx (0.8000, 0.1778)$.

Simulation results with $d_1 = d_2 = 1.000$ are exemplarily depicted in Fig. 3. In Fig. 3a, the spatiotemporal response surface of fish $x_1(t, z)$ and boyciana $x_2(t, z)$ are depicted on the top part. And also, the time evolution of both species are studied on the bottom part. Fig. 3b illustrates the spatial distribution curves of boyciana and fish at discrete time points. With these, the equilibrium e_1 shows the stable property. On the other hand, in Fig. 4c and b, by decreasing the diffusion coefficients d_1, d_2 to $d_1 = d_2 = 0.01$, the numerical results are shown with unstable property.

Obviously, due to Theorem 3.3, if $d_1 = d_2 = 1$ then

$$\lambda_1(d_1+d_2) - \frac{c}{\alpha}\frac{m-d}{m}\frac{d}{m} \approx 1.820 > 0.$$

e_1 is locally stable. If $d_1 = d_2 = 0.01$ then

$$\lambda_1(d_1+d_2) - \frac{c}{\alpha}\frac{m-d}{m}\frac{d}{m} \approx -0.160 < 0.$$

e_1 is unstable. The simulation results correspond with the theoretical conclusion.

3.3. Overdevelopment model ($r \nearrow +\infty$)

From Theorem 3.1,

$$\lim_{r\nearrow\infty} x_3(\mathbf{z}) = 1$$

holds. By substituting human distribution $x_3 = 1$ into (3.1), one gets

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = d_1\Delta x_1 + x_1\left(1 - x_1 - \dfrac{cx_2}{x_1+\alpha x_2} - h_1\right), \mathbf{z} \in \Omega, t > 0, \\ \dfrac{\partial x_2}{\partial t} = d_2\Delta x_2 + x_2\left(-d + \dfrac{mx_1}{x_1+\alpha x_2} - h_2\right), \mathbf{z} \in \Omega, t > 0, \\ \dfrac{\partial x_1}{\partial \mathbf{n}} = \dfrac{\partial x_2}{\partial \mathbf{n}} = 0, \mathbf{z} \in \partial\Omega, t > 0, \\ x_1(0, \mathbf{z}) = x_1^0(\mathbf{z}) \ge 0, x_2(0, \mathbf{z}) = x_2^0(\mathbf{z}) \ge 0, \mathbf{z} \in \Omega. \end{cases} \tag{3.20}$$

The system (3.20) represents the wetland system is in the overdevelopment of resources situation.

Analogously, the positive constant equilibrium of (3.20) satisfies

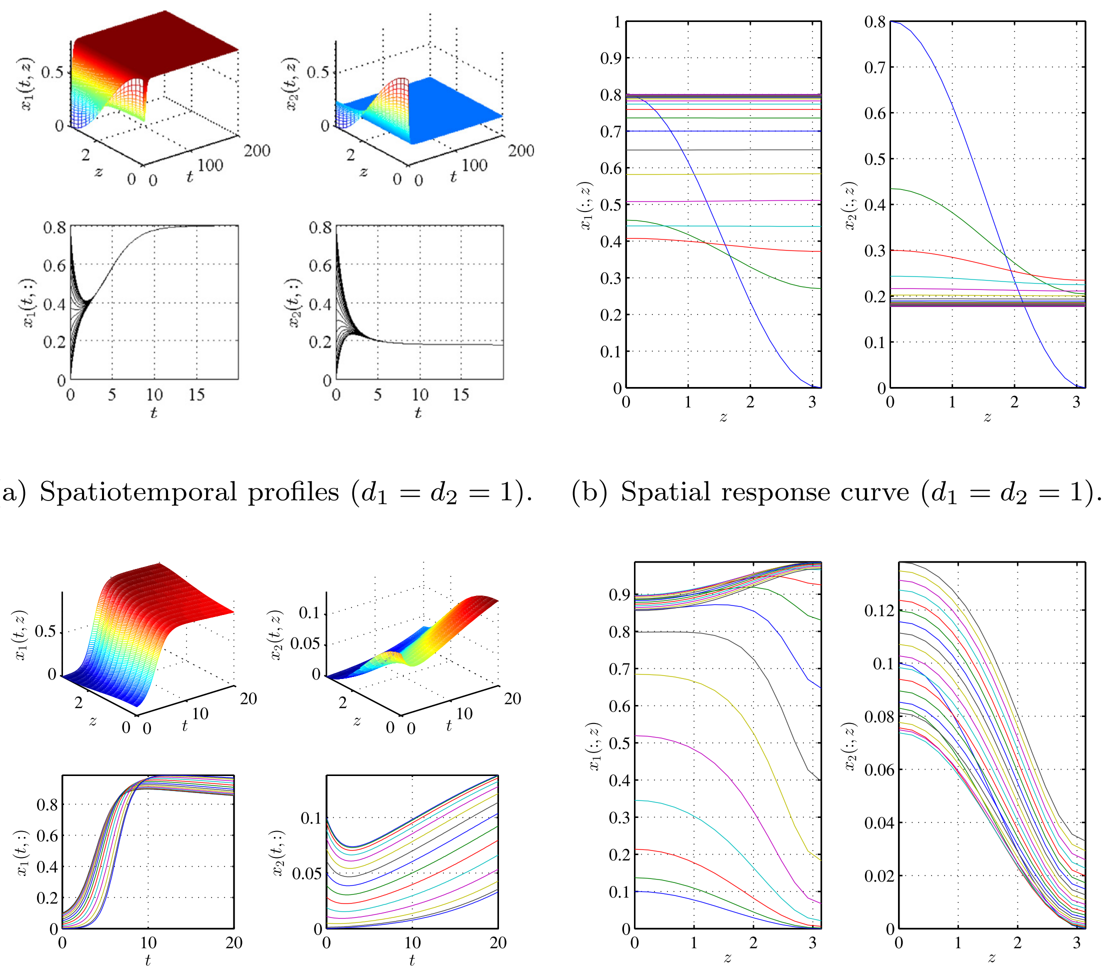
$$\begin{cases} x_1\left(1 - x_1 - \dfrac{cx_2}{x_1+\alpha x_2} - h_1\right) = 0, \\ x_2\left(-d + \dfrac{mx_1}{x_1+\alpha x_2} - h_2\right) = 0. \end{cases} \tag{3.21}$$

By some simple analysis for (3.21), it reaches

Theorem 3.4 (Existence of positive equilibrium). *When $1 - \frac{\alpha}{c}(1 - h_1) < \frac{d+h_2}{m} < 1$, the system (3.20) has a unique positive equilibrium $e_2 := (u_2^*, \upsilon_2^*)$ with*

$$u_2^* = 1 - h_1 - \frac{c(m-d-h_2)}{m\alpha} = 1 - h_1 - \frac{c}{\alpha}\left(1 - \frac{d+h_2}{m}\right),$$

$$\upsilon_2^* = \frac{m(1-h_1-u_2^*)u_2^*}{c(d+h_2)} = \frac{1}{\alpha}\left(\frac{m}{d+h_2} - 1\right)u_2^*.$$



(a) Spatiotemporal profiles ($d_1 = d_2 = 1$). (b) Spatial response curve ($d_1 = d_2 = 1$).

(c) Spatiotemporal profiles ($d_1 = d_2 = 0.01$). (d) Spatial response curve ($d_1 = d_2 = 0.01$).

Fig. 3. Numerical results for the local stable property of e_1 with $\Omega = [0, \pi]$. (a),(c): The spatiotemporal profiles (the top part) and the corresponding time evolution curves (the bottom part) of $x_1(t, z)$ and $x_2(t, z)$; (b),(d):Spatial response curve about $x_1(t, z)$ and $x_2(t, z)$ at differential discrete time points.

In addition, $0 < u_2^* < 1 - h_1, \upsilon_2^* > 0$; $u_2^* \geq 1$ *and* $\upsilon_2^* \leq 0$ *when* $\frac{d+h_2}{m} \geq 1$.

In a similar manner, the local asymptotically stability at e_2 can be investigated by linearization method. The concerned infinity dimensional ODEs system matrix is

$$\mathscr{L}_n^* = \begin{pmatrix} -\dfrac{m^2\alpha + cd^2 - cm^2 + d_1\lambda_n\alpha m^2}{m^2\alpha} - h_1 & -c\dfrac{d^2}{m^2} \\ \dfrac{(m-d)^2}{m\alpha} & -\dfrac{\lambda_n d_2 m + md - d^2}{m} - h_2 \end{pmatrix} \tag{3.22}$$

Noticing that

$$\mathscr{L}_n^* = \mathscr{L}_n - \lambda_n \mathrm{diag}(h_1, h_2),$$

if the determinant and the trace of the matrix $\mathscr{L}_n^*$ are denoted as $\mathrm{Dt}(u_2^*), \mathrm{Tr}(u_2^*)$ respectively, then

$$\begin{cases} \mathrm{Dt}(u_2^*) = \mathrm{Dt}(u_1^*) - (a_{11}\lambda_n h_1 + a_{22}\lambda_n h_2) + h_1 h_2 \lambda_n^2, \\ \mathrm{Tr}(u_2^*) = \mathrm{Tr}(u_2^*) - (h_1 + h_2)\lambda_n. \end{cases} \tag{3.23}$$

where a_{11}, a_{22} are the diagonal elements of $\mathrm{Dt}(u_1^*)$.

From Theorem 3.3,

$$a_{11} < 0, a_{22} < 0,$$

if $\lambda_1(d_1 + d_2) - \frac{c}{\alpha}\frac{m-d}{m}\frac{d}{m} > 0$ is provided. Thus by the nonnegativity of λ_n, it reaches

$$\mathrm{Dt}(u_2^*) > 0, \mathrm{Tr}(u_2^*) < 0.$$

By summarizing, the following conclusion is reached

Theorem 3.5. *Under the condition* $1 - \frac{\alpha}{c}(1 - h_1) < \frac{d+h_2}{m} < 1$,

S3) When $c \leq \alpha$, *the positive equilibrium* e_2 *is locally asymptotically stable for any diffusion coefficients* d_1, $d_2 > 0$ *and interference coefficients* h_1, $h_2 > 0$.

S4) When $c > \alpha$, *the positive equilibrium* e_2 *is locally asymptotically stable, if*

$$\lambda_1(d_1 + d_2) - \frac{c}{\alpha}\frac{m-d}{m}\frac{d}{m} > 0$$

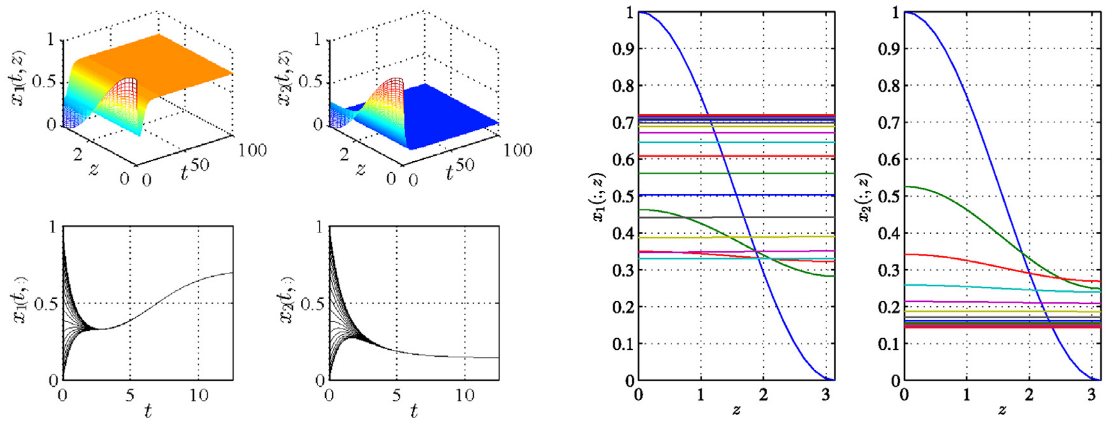
is provided. Here λ_1 *is the first eigenvalue of S-L problem (3.11).*

From Theorem 3.5, the diffusion coefficients d_1, d_2 also determine the stability of the equilibrium e_2. In addition, the interference coefficients h_1, h_2 makes the equilibrium e_2 more smaller, even change to zeros. If

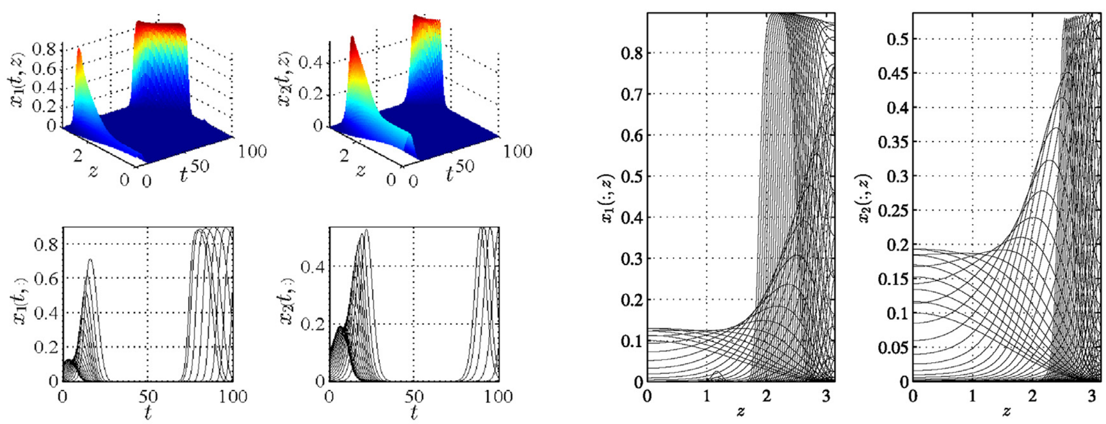
$$h_2 > \frac{m\alpha}{c}h_1,$$

then the positive condition

$$1 - \frac{\alpha}{c}(1 - h_1) < \frac{d + h_2}{m} < 1$$



(a) Spatiotemporal profiles ($d_1 = d_2 = 1$). (b) Spatial response curve ($d_1 = d_2 = 1$).



(c) Spatiotemporal profiles ($d_1 = d_2 = 0.01$). (d) Spatial response curve ($d_1 = d_2 = 0.01$).

Fig. 4. Numerical results for the local stable property of e_2 with $\Omega = [0, \pi]$. (a),(c): The spatiotemporal profiles (the top part) and the corresponding time evolution curves (the bottom part) of $x_1(t, z)$ and $x_2(t, z)$; (b),(d):Spatial response curve about $x_1(t, z)$ and $x_2(t, z)$ at differential discrete time points.

implies

$$1 - \frac{\alpha}{c} < \frac{d}{m} < 1.$$

It is means that the steady state e_1 can become unstable with human influence.

For the numerical illustration, considering Theorem 3.5, following from (3.19), the system's parameters are chosen as

$$\begin{aligned} &c = 1.000, \alpha = 0.5000, d = 0.9000, \\ &m = 1.000, h_1 = 0.1, h_2 = 0.01. \end{aligned} \tag{3.24}$$

where $h_1 = 0.1$ and $h_2 = 0.01$ are humans interference coefficients. The other parameters of (3.56) are the same as (3.19). By directly computing, one can see that the above parameters fulfill the positive condition

$$1 - \frac{\alpha}{c}(1 - h_1) < \frac{d + h_2}{m} < 1.$$

The equilibrium is $e_2 \approx (0.7200, 0.1424)$.

Simulation results in the (z, t) domain are exemplarily depicted in Fig. 4. Fig. 4(a) and (b) illustrates the stable property of e_2. Fig. 4(c) and (d) is the unstable case. It is worth noting that in Fig. 4(b) the steady densities of boyciana and fish are both decreasing more than the previous human-free model in Fig. 3(b). This corresponds with the theoretical result.

3.4. Nontrivial human distribution model ($\lambda_1(\Omega) < r < +\infty$)

From the ecological point of view, the above two subsections describe two ecological situations: human free and over development ecological models. Now, for Beidaihe wetland system, human, boyciana and the wetland fish are coexistent in wetland environment. A more realistic situation is described by (2.1) with the nontrivial human distribution function $x_3^p(\mathbf{z})$. As described in Theorem 3.1, $x_3^p(\mathbf{z})$ is the nonconstant spatial function. The steady state e_3 satisfies the nonlinear elliptic PDEs

$$\begin{cases} 0 = d_1 \Delta x_1 + x_1 \left(1 - x_1 - \dfrac{c x_2}{x_1 + \alpha x_2} - h_1 x_3\right), \mathbf{z} \in \Omega, \\ 0 = d_2 \Delta x_2 + x_2 \left(-d + \dfrac{m x_1}{x_1 + \alpha x_2} - h_2 x_3\right), \mathbf{z} \in \Omega, \\ 0 = \Delta x_3 + r x_3 (1 - x_3), \mathbf{z} \in \Omega, \\ \dfrac{\partial x_1}{\partial \mathbf{n}} = \dfrac{\partial x_2}{\partial \mathbf{n}} = \dfrac{\partial x_3}{\partial \mathbf{n}} = 0, \mathbf{z} \in \partial\Omega. \end{cases} \tag{3.25}$$

where

$$\lambda_1(\Omega) < r < +\infty.$$

Obviously, under this situation, a similar situation in front of the equilibrium point can't be expected. The linearized method cannot be applied in this case.

3.4.1. Invariant domain

Proposition 3.6 ([21]Comparison principle)**.** *Consider scalar nonlinear parabolic equation about $x = x(t, \mathbf{z})$*

$$Px = f(\mathbf{z}, t, x), (\mathbf{z}, t) \in \mathscr{D} \tag{3.26}$$

where $\mathscr{D} = \Omega \times (0, T) \subset \mathbb{R}^n \times \mathbb{R}_+$ is a bounded domain with smooth boundary $\partial\Omega$. Here

$$-Px = \frac{\partial x}{\partial t} - \sum_{j,k=1}^{n} a_{j,k} \frac{\partial^2 x}{\partial z_j \partial z_k}$$

is uniformly parabolic in $\mathscr{D}$. Finally we assume that f is C^1 in x and Hölder continuous in $\mathbf{z}$ and t. Let x and y are C^2 functions of $\mathbf{z}$ in Ω, C^1 functions of t on $[0, T]$, and consider the following three conditions:

$$\begin{cases} Px - f(\mathbf{z}, t, x) \le Py - f(\mathbf{z}, t, y), (\mathbf{z}, t) \in \mathscr{D}, \\ x(\mathbf{z}, 0) \ge y(\mathbf{z}, 0), \mathbf{z} \in \Omega, \\ \dfrac{\partial x}{\partial \mathbf{n}} + \beta x \ge \dfrac{\partial y}{\partial \mathbf{n}} + \beta y, (\mathbf{z}, t) \in \partial\Omega \times (0, T), \end{cases} \tag{3.27}$$

Then $x(\mathbf{z}, t) \ge y(\mathbf{z}, t)$ for all $(\mathbf{z}, t) \in \mathscr{D}$.

Theorem 3.7. *If $1 - \frac{\alpha}{c} < \frac{d}{m} < 1$, $1 - \frac{\alpha}{c}(1 - h_1) < \frac{d+h_2}{m} < 1$, then for all the nonnegative solution of (2.1) $x(t, \mathbf{z}), \mathbf{z} \in \overline{\Omega}$, the following inequalities stand*

$$\limsup_{t\to\infty} x_1(t, \mathbf{z}) \le \bar{x}_1 := 1, \tag{3.28}$$

$$\limsup_{t\to\infty} x_2(t, \mathbf{z}) \le \bar{x}_2 := \frac{1}{\alpha} \frac{m - d}{d}, \tag{3.29}$$

$$\liminf_{t\to\infty} x_1(t, \mathbf{z}) \ge \underline{x}_1 := 1 - h_1 - \frac{c}{\alpha}, \tag{3.30}$$

$$\liminf_{t\to\infty} x_2(t, \mathbf{z}) \ge \underline{x}_2 := \frac{1}{\alpha} \frac{m - (d + h_2)}{d + h_2}\left(1 - h_1 - \frac{c}{\alpha}\right). \tag{3.31}$$

Consequently, the domain given by

$$\mathscr{A} := \left[1 - h_1 - \frac{c}{\alpha}, 1\right] \times \left[\frac{1}{\alpha} \frac{m - (d + h_2)}{d + h_2}\left(1 - h_1 - \frac{c}{\alpha}\right), \frac{1}{\alpha}\left(\frac{m - d}{d}\right)\right]$$

is a positively invariant region for global solution of system (2.1).

Proof. Let x_1, x_2 be a solution of (2.1). Then from the first equation of (2.1) one can observe that x_1 satisfies

$$\begin{cases} \dfrac{\partial x_1}{\partial t} \le d_1 \Delta x_1 + x_1(1 - x_1), \mathbf{z} \in \Omega, t > 0, \\ \dfrac{\partial x_1}{\partial \mathbf{n}} = 0, \mathbf{z} \in \partial\Omega, t > 0, \\ x_1(0, \mathbf{z}) = x_1^0(\mathbf{z}) \ge 0, \mathbf{z} \in \Omega. \end{cases} \tag{3.32}$$

In view of the comparison principle Proposition 3.6, one can get that

$$\limsup_{t\to\infty} x_1(t, \mathbf{z}) \le 1.$$

Thus the second equation of (2.1) yields that x_2 satisfies

$$\begin{cases} \dfrac{\partial x_2}{\partial t} \le d_2 \Delta x_2 + x_2(-d + \dfrac{m}{1 + \alpha x_2}), \mathbf{z} \in \Omega, t > 0, \\ \dfrac{\partial x_2}{\partial \mathbf{n}} = 0, \mathbf{z} \in \partial\Omega, t > 0, \\ x_2(0, \mathbf{z}) = x_2^0(\mathbf{z}) \ge 0, \mathbf{z} \in \Omega. \end{cases} \tag{3.33}$$

Noticing that $\bar{x}_2 = \frac{1}{\alpha} \frac{m-d}{d}$ satisfies

$$\begin{cases} \dfrac{\partial \bar{x}_2}{\partial t} = d_2 \Delta \bar{x}_2 + \bar{x}_2\left(-d + \dfrac{m}{1 + \alpha \bar{x}_2}\right), \mathbf{z} \in \Omega, t > 0, \\ \dfrac{\partial \bar{x}_2}{\partial \mathbf{n}} = 0, \mathbf{z} \in \partial\Omega, t > 0. \end{cases} \tag{3.34}$$

Hence the inequality (3.29) holds.

In addition, from the first equation of (2.1) we can also get

$$\begin{cases} \dfrac{\partial x_1}{\partial t} \ge d_1 \Delta x_1 + x_1\left(1 - \dfrac{c}{\alpha} - h_1 - x_1\right), \mathbf{z} \in \Omega, t > 0, \\ \dfrac{\partial x_1}{\partial \mathbf{n}} = 0, \mathbf{z} \in \partial\Omega, t > 0, \\ x_1(0, \mathbf{z}) = x_1^0(\mathbf{z}) \ge 0, \mathbf{z} \in \Omega. \end{cases} \tag{3.35}$$

It follows that the inequality (3.30) holds. From the second equation of (2.1) we have

$$\begin{cases} \dfrac{\partial x_2}{\partial t} \ge d_2 \Delta x_2 + x_2\left(-d + \dfrac{m\underline{x}_1}{\underline{x}_1 + \alpha x_2} - h_2\right), \mathbf{z} \in \Omega, t > 0, \\ \dfrac{\partial x_2}{\partial \mathbf{n}} = 0, \mathbf{z} \in \partial\Omega, t > 0, \\ x_2(0, \mathbf{z}) = x_2^0(\mathbf{z}) \ge 0, \mathbf{z} \in \Omega. \end{cases} \tag{3.36}$$

Therefore, (3.31) holds. This completes the proof. □

Remark 3.8. For the system (2.1), the reaction function vector is

$$\bar{f}(x) = \begin{pmatrix} x_1\left(1 - x_1 - \frac{cx_2}{x_1 + \alpha x_2} - h_1 x_3^p(\mathbf{z})\right) \\ x_2\left(-d + \frac{mx_1}{x_1 + \alpha x_2} - h_2 x_3^p(\mathbf{z})\right) \end{pmatrix}. \tag{3.37}$$

Since the Jacobian matrix of $\bar{f}$ is

$$L = \begin{pmatrix} 1 - c\alpha\left(\frac{x_2}{x_1 + \alpha x_2}\right)^2 - 2x_1 - h_1 x_3^p & -c\left(\frac{x_1}{x_1 + \alpha x_2}\right)^2 \\ m\alpha\left(\frac{x_2}{x_+ \alpha x_2}\right)^2 & m\left(\frac{x_1}{x_1 + \alpha x_2}\right)^2 - d - h_2 x_3^p \end{pmatrix}, \tag{3.38}$$

$\bar{f}$ is a mixed quasi-monotone function vector in $\mathbb{R}_+^2 = \{(x_1, x_2) | x_1 \ge 0, x_2 \ge 0\}$. Therefore the above theorem implies that $\bar{x} := (\bar{x}_1, \bar{x}_2)$ and $\underline{x} :=(\underline{x}_1, \underline{x}_2)$ are a pair of coupled upper and lower solutions of (2.1). Consequently, by [20, Chapter 8 Theorem 3.3], there exists a solution $x(t, \mathbf{z})$ of (2.1) with

$$\underline{x} \le x(t, \mathbf{z}) \le \bar{x}.$$

From the first two inequalities (3.28), (3.29) of the above theorem, we observe that if $m < d$, then $\lim_{t\to\infty} x_2 = 0$ uniformly on $\overline{\Omega}$. Ecologically, the boyciana population will tends toward extinction. The last two inequalities (3.30), (3.31) give sufficient conditions such that the positive solution of (3.20) has the persistence property. That is, we provide some necessary conditions on parameters such that the boyciana and fish always coexist with humans influence

$$h_1 < 1 - \frac{c}{\alpha}, h_2 < m - d.$$

This shows that it is reasonable to expect the persistence of boyciana and fish when there is a suitable weak humans influence.

3.4.2. Energy estimation

In this subsection, the PDE energy estimation theory is extended to the PDAEs (2.1). A developed Lyapunov energy function is proposed to investigate the stability of the system (2.1). The following lemma will be needed.

Lemma 3.9 (Poincare inequality [21])**.** *Let $x \in W_2^2(\Omega)$, then if μ_1 is the smallest positive eigenvalue of $-\Delta$ on Ω (with the appropriate boundary conditions) the following Poincaré inequalities hold:*

$$\|\nabla x\|^2 \geq \mu_1\|x-\bar{x}\|^2,\ \|\Delta x\|^2 \geq \mu_1\|\nabla x\|^2\ if\ \frac{dx}{d\mathbf{n}}=0\ on\ \partial\Omega, \tag{3.39}$$

$$\|\nabla x\|^2 \geq \mu_1\|x\|^2\ if\ x=0\ on\ \partial\Omega, \tag{3.40}$$

where $\bar{x}=\frac{1}{|\Omega|}\int_\Omega x\mathrm{d}\mathbf{z}$.

Theorem 3.10. *Assume that* $x(t,\mathbf{z})=(x_1(t,\mathbf{z}),x_2(t,\mathbf{z}),x_3(\mathbf{z}))^T$ *is a bounded solution of (2.1). Assume that spectral radius of the Jacobian matrix (2.1) is* ρ, μ_1 *is the smallest positive eigenvalue of* $-\Delta$ *on* Ω, $d_m=\min\{d_1,d_2\}$ *and*

$$\delta=d_m\mu_1-(\rho+h_M)>0.$$

where $h_M=\max\{h_1,h_2\}$. *Let* $\breve{x}=(x_1(t,\mathbf{z}),x_2(t,\mathbf{z}))^T$, *then*

$$E(t)=\int_\Omega\sum_{i=1}^2\frac{\partial\breve{x}^T}{\partial z_i}\frac{\partial\breve{x}}{\partial z_i}\mathrm{d}\mathbf{z}=\int_\Omega\|\nabla\breve{x}\|^2\mathrm{d}\mathbf{z}\leq c_1\exp(-\delta t)+c_2, \tag{3.41}$$

$$\int_\Omega\|\breve{x}(t,\mathbf{z})-\breve{x}_M(t)\|\mathrm{d}\mathbf{z}\leq c_3\exp(-\delta t)+c_4 \tag{3.42}$$

hold for positive constant c_1, c_2 *and* c_3, c_4. *Here,*

$$\breve{x}_M(t)=\frac{1}{|\Omega|}\int_\Omega\breve{x}(t,\mathbf{z})\mathrm{d}\mathbf{z}$$

is the spatial average function.

Proof. The energy integral (Lyapunov function)

$$E(t)=\frac{1}{2}\int_\Omega\|\nabla\breve{x}\|^2\mathrm{d}\mathbf{z}=\frac{1}{2}\int_\Omega\sum_{i=1}^2\left[\left(\frac{\partial x_i}{\partial z_1}\right)^2+\left(\frac{\partial x_i}{\partial z_2}\right)^2\right]\mathrm{d}\mathbf{z} \tag{3.43}$$

is introduced. By computing the derivative of $E(t)$, one get

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=\int_\Omega\sum_{i=1}^2\left[\frac{\partial x_i}{\partial z_1}\frac{\partial^2x_i}{\partial z_1\partial t}+\frac{\partial x_i}{\partial z_2}\frac{\partial^2x_i}{\partial z_2\partial t}\right]\mathrm{d}\mathbf{z} \tag{3.44}$$

Noticing that $\breve{x}$ satisfies the system (2.1), it follows from above

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=\int_\Omega\sum_{i=1}^2\left(\frac{\partial\breve{x}^T}{\partial z_i}\frac{\partial}{\partial z_i}(D_1\Delta x+\bar{f}(x))\right)\mathrm{d}\mathbf{z} \tag{3.45}$$

where $D_1=\mathrm{diag}(d_1,d_2)$, $\bar{f}=(f_1,f_2)^T$ is the nonlinear reaction vector function. Applying for divergence theorem taking into account the BCs, the above equation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)=-\int_\Omega\Delta\breve{x}^TD_1\Delta\breve{x}\mathrm{d}\mathbf{z}+\int_\Omega\sum_{i=1}^2\frac{\partial\breve{x}^T}{\partial z_i}\frac{\partial\bar{f}}{\partial z_i}\mathrm{d}\mathbf{z}=I_1+I_2. \tag{3.46}$$

For the first integral I_1, the following estimation holds

$$I_1\leq-d_m\int_\Omega\|\Delta\breve{x}\|^2\mathrm{d}\mathbf{z}\leq-d_m\mu_1\int_\Omega\|\nabla\breve{x}\|^2\mathrm{d}\mathbf{z}. \tag{3.47}$$

The second integral I_2 can be estimated as

$$I_2=\int_\Omega\sum_{j=1}^2\sum_{i=1}^2\left\{\frac{\partial x_j}{\partial z_i}\left(\frac{\partial f_j}{\partial x_1}\frac{\partial x_1}{\partial z_i}+\frac{\partial f_j}{\partial x_2}\frac{\partial x_2}{\partial z_i}+\frac{\partial f_j}{\partial x_3}\frac{\partial x_3}{\partial z_i}\right)\right\}\mathrm{d}\mathbf{z} \tag{3.48}$$

or

$$I_2=\int_\Omega\nabla\breve{x}^T\frac{\partial\bar{f}}{\partial\breve{x}}\nabla\breve{x}\mathrm{d}\mathbf{z}+\int_\Omega\left(\frac{\partial f_1}{\partial x_3}\nabla x_1\cdot\nabla x_3+\frac{\partial f_2}{\partial x_3}\nabla x_2\cdot\nabla x_3\right)\mathrm{d}\mathbf{z} \tag{3.49}$$

where $\frac{\partial\bar{f}}{\partial\breve{x}}$ is the Jacobi matrix $\bar{f}$ w.r.t. $\breve{x}$. Thus

$$I_2\leq\rho\int_\Omega\|\nabla\breve{x}\|^2\mathrm{d}\mathbf{z}+h_1\int_\Omega\|\nabla x_1\|^2\mathrm{d}\mathbf{z}+h_2\times\int_\Omega\|\nabla x_2\|^2\mathrm{d}\mathbf{z}+(h_1+h_2)\int_\Omega\|\nabla x_3\|^2\mathrm{d}\mathbf{z}. \tag{3.50}$$

According to Lemma 3.9, (3.47) and (3.50) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)\leq(-d_m\mu_1+\rho+h_M)E(t)+(h_1+h_2)\int_\Omega\|\nabla x_3\|^2\mathrm{d}\mathbf{z}. \tag{3.51}$$

Noticing that

$$\Delta x_3(\mathbf{z})+rx_3(\mathbf{z})(1-x_3(\mathbf{z}))=0,\mathbf{z}\in\Omega$$

stands with the Neumann boundary conditions. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)\leq(-d_m\mu_1+\rho+h_M)E(t)+r(h_1+h_2)\int_\Omega x_3^2(1-x_3)\mathrm{d}\mathbf{z}. \tag{3.52}$$

From the assumption $d_m\mu_1>\rho+h_M$, with Gronwall inquality, we obtain that

$$E(t)\leq(E(0)-q)\exp(-\delta t)+q, \tag{3.53}$$

where $\delta=d_m\mu_1-(\rho+h_M)>0$,

$$q=\frac{r(h_1+h_2)\int_\Omega x_3^2(1-x_3)\mathrm{d}\mathbf{z}}{\delta}.$$

Therefore, (3.41) holds. By Lemma 3.9, (3.41) implies (3.42). This completes the proof. □

Our proposed result generalized the energy estimation result on common parabolic type PDEs. From above theorem, the value of $E(t)$ is asymptotically decreasing tends to a small value q if the following condition is provided

$$\frac{r(h_1+h_2)\int_\Omega x_3^2(1-x_3)\mathrm{d}\mathbf{z}}{d_m\mu_1-(\rho+h_M)}\to0.$$

If we consider two special cases $x_3(\mathbf{z})\equiv0$ and $x_3(\mathbf{z})\equiv1$, then $q=0$ and the system (2.1) is translated into PDEs system. Immediately, we get the following corollary.

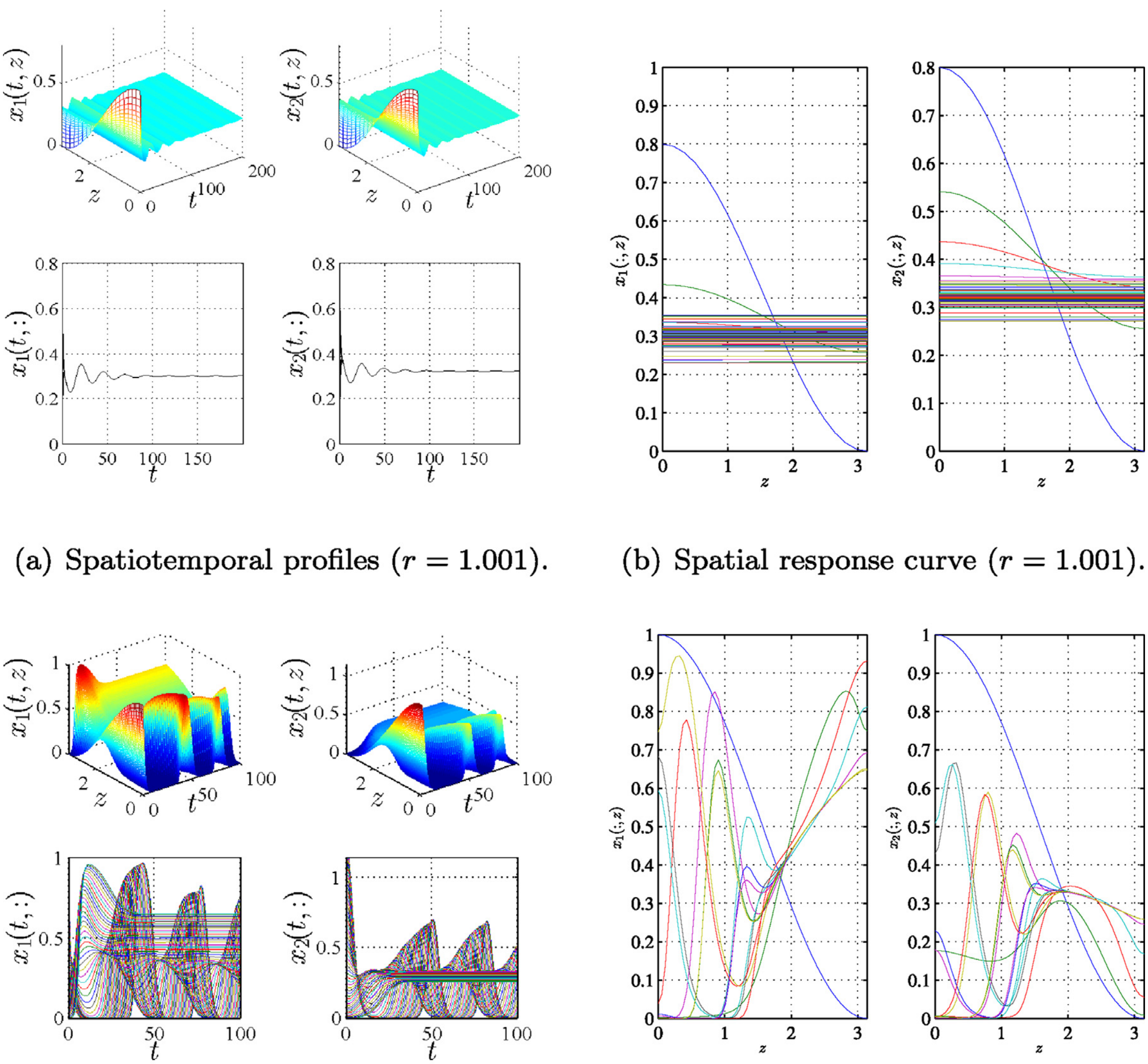
Corollary 3.11. *Under the conditions described in Theorem 3.10, If* $x_3(\mathbf{z})\equiv0$ *or* 1, *then for any bounded solutions of (2.1) , the following conclusions hold*

$$\int_\Omega\sum_{i=1}^2\frac{\partial\breve{x}^T}{\partial z_i}\frac{\partial\breve{x}}{\partial z_i}\mathrm{d}\mathbf{z}=\int_\Omega\|\nabla\breve{x}\|^2\mathrm{d}\mathbf{z}\leq c_1\exp(-\delta_1t), \tag{3.54}$$

$$\int_\Omega\|\breve{x}-\breve{x}_M(t)\|\mathrm{d}\mathbf{z}\leq c_2\exp(-\delta_1t) \tag{3.55}$$

where c_1, c_2 *are all independent positive constants;* $\breve{x}_M(t)$ *is the spatial average functions of* $\breve{x}(t)$ *respectively. i.e. the state variable vector* $\breve{x}$ *generated by the system (2.1) is exponentially stable and asymptotically converge to their spatial average, respectively.*

Ecologically, $E(t)$ represents the spatial mobility integral of boyciana and fish. It can be seen as the summation of oscillation amplitudes of two species in the domain. Theorem 3.10 shows that in order to avoid the inference of human behaviors (for example: human fishing, water pollution), boyciana and wetland fish must carry on the unceasing migration. This means the wetland system is in an unstable state. However, if we reduced the human distribution density (for example, set up human restrict area) and enhance



(a) Spatiotemporal profiles ($r = 1.001$). (b) Spatial response curve ($r = 1.001$).

(c) Spatiotemporal profiles ($r = 100$). (d) Spatial response curve ($r = 100$).

Fig. 5. Numerical results for the steady state $\Omega = [0, \pi]$. (a),(c): The spatiotemporal profiles (the top part) and the corresponding time evolution curves (the bottom part) of $x_1(t, z)$ and $x_2(t, z)$; (b),(d):Spatial response curve about $x_1(t, z)$ and $x_2(t, z)$ at differential discrete time points.

the spatial diffusion capacity (for example, avoid river pollution), the wetland ecological system can tend to a stable state.

3.4.3. Illustrative example

As an effective demonstration of our previous theoretic analysis, we simulate the steady state for the boyciana–fish–human system. In order to show human's interference on the stability of ecological system, r is taken as the parameter variable of human population distribution. Therefore, in (2.1), except for r, all the other parameters are fixed by

$$d_1 = d_2 = 0.01, c = 1.000, \alpha = 0.5000, \tag{3.56}$$

$$d = 0.3000, m = 1.000, h_1 = 0.01, h_2 = 0.3. \tag{3.57}$$

As shown in Fig. 5, when $r = 1.001$, x_1 and x_2 are numerically converge to the desired steady state as $t \to \infty$. When $r = 100$, x_1, x_2 can't maintain the steady state. These simulation results indicate that $r \to +\infty$ may lead to the turning instable property of the PDAEs system (2.1).

4. Model fitting on real data

In this section, as application of our boyciana–fish–human model, the proposed PDAEs model is evaluated by model parameter fitting with the actual bird data set. First, the actual observation data is dealt with. Then, one dimension space PDAEs model is built. And the PDEs fitting technique provided in [26] is used to search for parameters to best fit the real data.

4.1. Treatment on actual boyciana and fish data

With the help of Qinhuangdao Bird Reserve and Banding Station, this real data is collected from six different locations at different times (from 2001 to 2014). Table 1 shows the discrete density distribution of boyciana $\tilde{x}_2(t, z_i)(i = 1, \ldots, 6)$ at six bird-watching locations in normalization format. The findings are that the density value at Location 1 is higher than the other five locations. It is worth noticing that this observatory is located in the forest where few people enter and is far away from the coastline. This confirms that boyciana depends on and prefers a region without interference or less interference for building their nest. Now, since

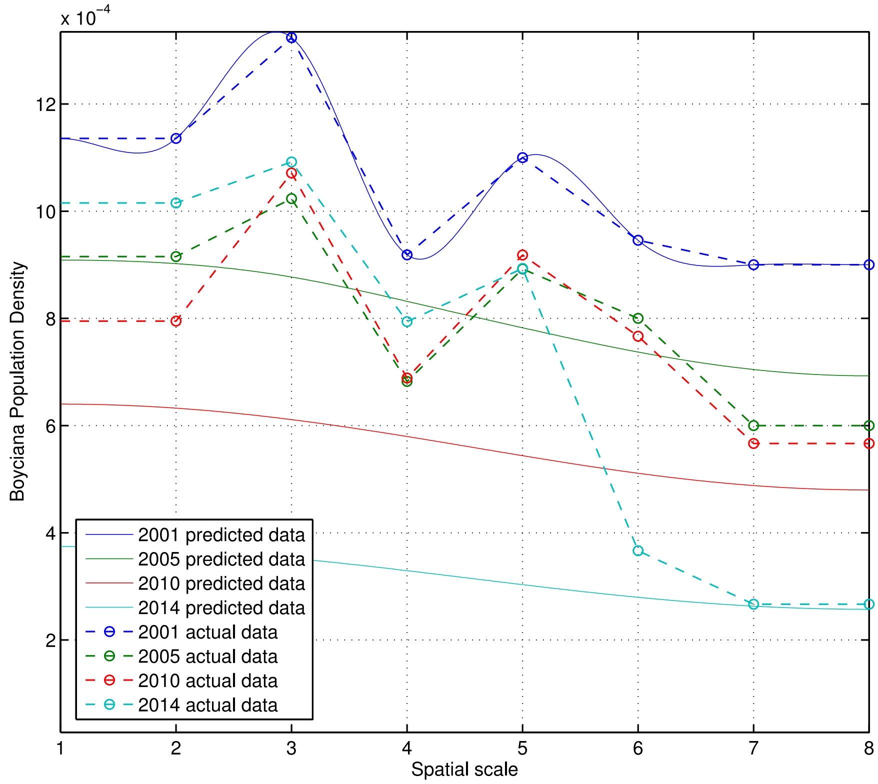


Fig. 6. Predicted vs. Actual data of boyciana density. The x-axis represents the spatial scale. The y-axis is density. From top to bottom, the first four predicted curves are in years of 2001, 2005, 2010, 2014, respectively.

Table 1
Boyciana density distribution $\tilde{x}_2(t, z)$ in six observatories.

Time (year)	2001	2002	$\cdots$	2014
z_1 (Loc.1)	0.001136	0.001047	$\cdots$	0.000717
z_2 (Loc.2)	0.001324	0.001150	$\cdots$	0.000981
z_3 (Loc.3)	0.000919	0.000802	$\cdots$	0.000742
z_4 (Loc.4)	0.000946	0.000909	$\cdots$	0.000766
z_5 (Loc.5)	0.001100	0.000966	$\cdots$	0.000396
z_6 (Loc.6)	0.000900	0.000866	$\cdots$	0.000266

Table 2
Normalization fish density distribution $\tilde{x}_1(t, z)$.

Time (year)	2001	2002	$\cdots$	2014
z_1	0.1	0.09	$\cdots$	0.03
z_2	0.12	0.12	$\cdots$	0.07
z_3	0.14	0.14	$\cdots$	0.07
z_4	0.16	0.15	$\cdots$	0.10
z_5	0.18	0.18	$\cdots$	0.12
z_6	0.2	0.19	$\cdots$	0.14

these locations have different linear distances from the Beidaihe coastline, accordingly we discrete the spatial axis z as six points $z_1, \dots, z_6$ which correspond to the six bird observatories locations.

For the fish density distribution $\tilde{x}_1(t, z)$, it should be noticed that the "fish density" represents the fish food available quantity. Considering the range of boyciana foraging activities is at about 0.5–5 km, assume that the distance from the boyciana's habitat to the coastline represents degree of different fish food catching. That is the quantity of available fish food linearly negative dependent on the distance to the coastline. Thus, in Table 2, the density distribution at z_6 position $\tilde{x}_1(t, z_6)$ is normalized created by the real quantity of fish production of Qinhuangdao in the past 14 years (from 2001 to 2014). And the other positions $\tilde{x}_1(t, z_i)(i = 1, \dots, 6)$ is linearly decreasing with respect to the coastline distance z.

4.2. Spatial assumption

The reasonable assumptions are made to reduce the spatial domain of the original model (2.1). From Section 4.1, it is clear that the populations densities of boyciana and fish are assumed as single space variables. Mathematically, the state variables vector satisfies

$$x(t, z_1, z_2) = x(t, z_1).$$

Then the spatial domain Ω is longitudinal compression into a 'pipe' line

$$\Omega = (0, l).$$

Moreover, the wetland boundary $\partial\Omega$ is changed into two points: $\mathbf{z} = 0, \mathbf{z} = l$ which are both closed without flux under Neumann BC. For the convenience of writing, we define $x(t, z) = x(t, z_1)$ and choose the spatial domain of system (2.1) as $\Omega = (0, \pi)$ to simplify calculation.

Under above assumptions, (2.1) becomes an 1D-spatial PDAEs as following:

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = d_1 \dfrac{\partial^2 x_1}{\partial z^2} + x_1 \left(1 - x_1 - \dfrac{c x_2}{x_1 + \alpha x_2} - h_1 x_3\right), z \in (0, \pi), t > 0, \\ \dfrac{\partial x_2}{\partial t} = d_2 \dfrac{\partial^2 x_2}{\partial z^2} + x_2 \left(-d + \dfrac{m x_1}{x_1 + \alpha x_2} - h_2 x_3\right), z \in (0, \pi), t > 0, \\ 0 = \dfrac{\partial^2 x_3}{\partial z^2} + r x_3 (1 - x_3), z \in (0, \pi), \\ \dfrac{\partial x_1}{\partial z} = \dfrac{\partial x_2}{\partial z} = \dfrac{\partial x_3}{\partial z} = 0, z \in \{0, \pi\}, t > 0, \\ x_1(0, z) = x_1^0(z), x_2(0, z) = x_2^0(z), z \in [0, \pi]. \end{cases} \tag{4.1}$$

In addition, the corresponding S-L problem of (4.1) has the eigenvalues and the eigenfunctions [27]:

$$\mu_n = n^2, \phi_n(z) = \sqrt{\frac{2}{\pi}}\cos(nz), \quad n \in \mathbb{N}_0. \tag{4.2}$$

Therefore, the first eigenvalue is $\mu_1([0, \pi]) = 1$.

4.3. Maximum–minimum norm optimization model fitting

In this subsection, the numerical PDAEs fitting method is investigated to predict the development of boyciana population. The following maximum–minimum norm optimization algorithm is built to optimal the parameters.

$$\begin{cases} \min \sum_{i=1}^{2}\sum_{j=1}^{14}\sum_{k=1}^{6} \dfrac{|x_i(t_j, z_k) - \tilde{x}_i(t_j, z_k)|}{|\tilde{x}_i(t_j, z_k)|} \\ s.t. \\ \dfrac{\partial x_1}{\partial t} = d_1 \Delta x_1 + x_1\left(1 - x_1 - \dfrac{cx_2}{x_1 + \alpha x_2} - h_1 x_3(z)\right), \\ \dfrac{\partial x_2}{\partial t} = d_2 \Delta x_2 + x_2\left(-d + \dfrac{mx_1}{x_1 + \alpha x_2} - h_2 x_3(z)\right), \\ \left.\dfrac{\partial x_1}{\partial z}\right|_{z=0,\pi} = \left.\dfrac{\partial x_2}{\partial z}\right|_{z=0,\pi} = 0, \\ x_1(0, z) = \tilde{x}_1(0, z), x_2(0, z) = \tilde{x}_2(0, z), \\ d_1, d_2, c, m, \alpha, d, h_1, h_2, r > 0. \end{cases} \tag{4.3}$$

Here, the optimal parameters are d_1, d_2, c, m, α, d, h_1, h_2 and r.

For the initial density value of the above optimal problem, $x_1(0, z)$ and $x_2(0, z)$ is taken from the actual data in 2001.

With the Neumann BC $\frac{\partial x_1}{\partial z} = \frac{\partial x_2}{\partial z} = 0$, we use the technique in numerical analysis

$$0 = \frac{\mathrm{d}x(z)}{\mathrm{d}z} \approx \frac{x(\xi) - x(z)}{\xi - z}.$$

Consequently, in the spatial direction, two sets of data are added to each side of the observation data. The added data values are the same as their adjacent values.

The numerical optimal program is designed with MATLAB software. The initial input parameters is

$$d_1 = d_2 = 0.001, d = 0.3, c = 1, m = 1,$$
$$\alpha = 0.5, h_1 = 0.1, h_2 = 0.3, r = 1.$$

Combining with the boundary value in Tables 1 and 2, we get the numerical optimization result:

$$d_1 = 0.1185, d_2 = 0.5773, d = 0.1343, c = 0.8996,$$

$$m = 0.5093, \alpha = 0.5535, h_1 = 1.8102, h_2 = 0.0172, r = 6.0440.$$

Fig. 6 illustrates the predicted results for boyciana. The dashed lines denote the actual observations for the density in different year, while the starred lines illustrate the density predicted by our PDAEs Model (4.3). Here, the average prediction accuracy is 95.17%. It is effective from the statistical perspective.

Conclusion

In this work, the boyciana–fish reaction-diffusion system coupled with elliptic human distribution equation is considered. It is a generalization of the classical parabolic PDEs system. Because some theoretical results cannot be directly applied in the singular derivative coefficient matrix E situation, the novel energy estimation method is provided to investigate the global stability of PDAEs model. The maximum–minimum norm optimization model is built to optimize the parameters of PDAEs (2.1). The numerical results show the effectiveness of the development model.

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