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A note on positive periodic solutions of delayed differential equations

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ABSTRACT

We consider the existence of positive ω-periodic solutions for the periodic equation $x'(t) = a(t)e^{x(t)}x(t) - \lambda b(t)f(x(t-\tau(t)))$, where $a, b \in C(\mathbb{R}, [0,\infty))$ are ω-periodic, $\int_0^\omega a(t)dt > 0$, $\int_0^\omega b(t)dt > 0$, $f \in C([0,\infty),[0,\infty))$, and $f(u) > 0$ for $u > 0$, $\tau(t)$ is a continuous ω-periodic function.

1. Introduction

In recent years, there has been considerable interest in the existence of positive periodic solutions of the following equation:

$$x'(t) = a(t)g(x(t))x(t) - \lambda b(t)f(x(t-\tau(t))). \tag{1.1}$$

See, for example, [1–6]. (1.1) has been proposed as a model for a variety of biological processes. See, for example, the above references and [7,8].

The existence results in the literature are largely based on the assumption that $g(x(t))$ is constant or bounded. It is interesting to know whether there is a positive solution to (1.1) when $g(x(t))$ is not necessarily bounded. In this short note, we take $g(x) = e^x$ and consider the existence of an positive ω-periodic solution of the equation

$$x'(t) = a(t)e^{x(t)}x(t) - \lambda b(t)f(x(t-\tau(t))), \tag{1.2}$$

where $\lambda > 0$ is a positive parameter. We shall show that (1.2) has a positive ω-periodic solution when λ is sufficiently large. Apparently, our results can be extended to more general $g(x)$. Our arguments are based on a well-known fixed point theorem (Lemma 2.1).

Let $\mathbb{R} = (-\infty, \infty)$. We make the following assumptions:

(H1) $a, b \in C(\mathbb{R}, [0,\infty))$ are ω-periodic functions, $\int_0^\omega a(t)dt > 0$, $\int_0^\omega b(t)dt > 0$. $\tau \in C(\mathbb{R},\mathbb{R})$ is an ω-periodic function.
(H2) $f : [0,\infty) \to [0,\infty)$ is continuous. $f(u) > 0$ for $u > 0$.

Also, let $\sigma = e^{-\int_0^\omega a(t)dt} < 1$, $m(L) = \min\{f(u) : \frac{\sigma^{e^L}(1-\sigma)}{1-\sigma^{e^L}} L \le u \le L\} > 0,\ L > 0$.

Our main result is:

Theorem 1.1. *Assume* (H1)–(H2) *hold and* $\lim_{u\to 0^+} \frac{f(u)}{u} = 0$. *For each* $L > 0$, *there exists a* $\lambda_0 = L\frac{1-\sigma^{e^L}}{m(L)\sigma^{e^L}\int_0^\omega b(s)ds} > 0$ *such that* (1.2) *has a positive* ω*-periodic solution* u *with* $\sup_{t\in[0,\omega]} u(t) \le L$ *for* $\lambda > \lambda_0$.

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2. Preliminaries

We consider the modified equation

$$x'(t) = a(t)g_L(x(t))x(t) - \lambda b(t)f(x(t-\tau(t))), \tag{2.3}$$

where $g_L(u) = e^u$ for $0 \le u \le L$ and $g_L(u) = e^L$ for $u \ge L$. For each $L > 0$, if we can find a positive ω-periodic solution u for (2.3) and $\sup_{t\in[0,\omega]} u(t) \le L$, then u is also a positive ω-periodic solution to (1.2).

The following well-known result of the fixed point index is crucial in our arguments.

Lemma 2.1 ([9–11]). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous and such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|x\| = r\}$.*

(i) *If $\|Tx\| \ge \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

(ii) *If $\|Tx\| \le \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.1 to (2.3), let X be the Banach space $\{u(t) : u(t) \in C(\mathbb{R}, \mathbb{R}), u(t+\omega) = u(t)\}$ with $\|u\| = \sup_{t\in[0,\omega]} |u(t)|,\ u \in X$.

Let K be a cone in X defined by

$$K = \left\{ u \in X : u(t) \ge \frac{\sigma^{e^L}(1-\sigma)}{1-\sigma^{e^L}}\|u\|, t \in [0, \omega] \right\},$$

and denote Ω_r by

$$\Omega_r = \{u \in K : \|u\| < r\}.$$

Note that $\partial\Omega_r = \{u \in K : \|u\| = r\}$.

Following Wang [5], let $T_\lambda : K \to X$ be a map defined by

$$T_\lambda u(t) = \lambda \int_t^{t+\omega} G_L(t, s)b(s)f(u(s-\tau(s)))\mathrm{d}s, \tag{2.4}$$

where

$$G_L(t, s) = \frac{e^{-\int_t^s a(\theta)g_L(u(\theta))\mathrm{d}\theta}}{1 - e^{-\int_0^\omega a(\theta)g_L(u(\theta))\mathrm{d}\theta}}.$$

Note that

$$1 \le g_L(u) \le e^L$$

and

$$\frac{\sigma^{e^L}}{1-\sigma^{e^L}} \le G_L(t, s) \le \frac{1}{1-\sigma}, \quad t \le s \le t+\omega.$$

Lemma 2.2. *Assume* (H1)–(H2) *hold. Then $T_\lambda(K) \subset K$ and $T_\lambda : K \to K$ is compact and continuous.*

Proof. In view of the definition of K, for $u \in K$, we have

$$\begin{aligned}
(T_\lambda u)(t+\omega) &= \lambda \int_{t+\omega}^{t+2\omega} G_L(t+\omega, s)b(s)f(u(s-\tau(s)))\mathrm{d}s \\
&= \lambda \int_t^{t+\omega} G_L(t+\omega, \theta+\omega)b(\theta+\omega)f(u(\theta+\omega-\tau(\theta+\omega)))\mathrm{d}\theta \\
&= \lambda \int_t^{t+\omega} G_L(t, s)b(s)f(u(s-\tau(s)))\mathrm{d}s \\
&= (T_\lambda u)(t).
\end{aligned}$$

It is easy to see that $\int_t^{t+\omega} b(s)f(u(s-\tau(s)))\mathrm{d}s$ is a constant because of the periodicity of $b(t)f(u(t-\tau(t)))$. One can show that, for $u \in K$ and $t \in [0,\omega]$,

$$\begin{aligned} T_\lambda u(t) &\geq \frac{\sigma^{\mathrm{e}^L}}{1-\sigma^{\mathrm{e}^L}}\lambda\int_t^{t+\omega} b(s)f(u(s-\tau(s)))\mathrm{d}s \\ &= \frac{\sigma^{\mathrm{e}^L}}{1-\sigma^{\mathrm{e}^L}}\lambda\int_0^{\omega} b(s)f(u(s-\tau(s)))\mathrm{d}s \\ &= \frac{\sigma^{\mathrm{e}^L}(1-\sigma)}{1-\sigma^{\mathrm{e}^L}}\frac{1}{1-\sigma}\lambda\int_0^{\omega} b(s)f(u(s-\tau(s)))\mathrm{d}s \\ &\geq \frac{\sigma^{\mathrm{e}^L}(1-\sigma)}{1-\sigma^{\mathrm{e}^L}}\|T_\lambda u\|. \end{aligned}$$

Thus $T_\lambda(K) \subset K$ and it is easy to show that $T_\lambda : K \to K$ is compact and continuous. □

Lemma 2.3. *Assume* (H1)–(H2) *hold. Then a positive ω-periodic solution of* (2.3) *is equivalent to a fixed point of T_λ in K.*

Proof. If $u \in K$ and $T_\lambda u = u$, then

$$\begin{aligned} u'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda\int_t^{t+\omega} G_L(t,s)b(s)f(u(s-\tau(s)))\mathrm{d}s\right) \\ &= \lambda G_L(t,t+\omega)b(t+\omega)f(u(t+\omega-\tau(t+\omega))) - \lambda G_L(t,t)b(t)f(u(t-\tau(t))) + a(t)g_L(u(t))T_\lambda u(t) \\ &= \lambda[G_L(t,t+\omega)-G_L(t,t)]b(t)f(u(t-\tau(t))) + a(t)g_L(u(t))T_\lambda u(t) \\ &= a(t)g_L(u(t))u(t) - \lambda b(t)f(u(t-\tau(t))). \end{aligned}$$

Thus u is a positive ω-periodic solution of (2.3). On the other hand, if u is a positive ω-periodic solution of (2.3), then $\lambda b(t)f(u(t-\tau(t))) = a(t)g_L(u(t))u(t) - u'(t)$ and

$$\begin{aligned} T_\lambda u(t) &= \lambda\int_t^{t+\omega} G_L(t,s)b(s)f(u(s-\tau(s)))\mathrm{d}s \\ &= \int_t^{t+\omega} G_L(t,s)\big(a(s)g_L(u(s))u(s) - u'(s)\big)\mathrm{d}s \\ &= \int_t^{t+\omega} G_L(t,s)a(s)g_L(u(s))u(s)\mathrm{d}s - \int_t^{t+\omega} G_L(t,s)u'(s)\mathrm{d}s \\ &= \int_t^{t+\omega} G_L(t,s)a(s)g_L(u(s))u(s)\mathrm{d}s - G_L(t,s)u(s)\,|_t^{t+\omega} - \int_t^{t+\omega} G_L(t,s)a(s)g_L(u(s))u(s)\mathrm{d}s \\ &= u(t). \end{aligned}$$

Furthermore, in view of the proof of Lemma 2.2, we also have $u(t) \geq \frac{\sigma^{\mathrm{e}^L}(1-\sigma)}{1-\sigma^{\mathrm{e}^L}}\|u\|$ for $t \in [0,\omega]$. Thus u is a fixed point of T_λ in K. □

Lemma 2.4. *Assume* (H1)–(H2) *hold. If $u \in \partial\Omega_L$, then*

$$\|T_\lambda u\| \geq \lambda\frac{\sigma^{\mathrm{e}^L}\int_0^\omega b(s)\mathrm{d}s}{1-\sigma^{\mathrm{e}^L}}m(L).$$

Proof. Since $f(u(t)) \geq m(L)$ for $t \in \mathbb{R}$, it is easy to see that

$$\begin{aligned} (T_\lambda u)(t) &\geq \frac{\sigma^{\mathrm{e}^L}}{1-\sigma^{\mathrm{e}^L}}\lambda\int_0^{\omega} b(s)f(u(s-\tau(s)))\mathrm{d}s \\ &\geq \frac{\sigma^{\mathrm{e}^L}\int_0^\omega b(s)\mathrm{d}s}{1-\sigma^{\mathrm{e}^L}}\lambda m(L). \quad \square \end{aligned}$$

Lemma 2.5. *Assume* (H1)–(H2) *hold and let $r > 0$. If $u \in \partial\Omega_r$ and there exists an $\varepsilon > 0$ such that $f(u(t)) \leq \varepsilon u(t)$ for $t \in [0,\omega]$, then*

$$\|T_\lambda u\| \leq \lambda\varepsilon\|u\|\frac{\int_0^\omega b(s)\mathrm{d}s}{1-\sigma}.$$

Proof. From the definition of T, for $u \in \partial\Omega_r$, we have

$$\begin{aligned}\|T_\lambda u\| &\le \frac{1}{1-\sigma}\lambda\int_0^\omega b(s)f(u(s-\tau(s)))\mathrm{d}s\\ &\le \frac{1}{1-\sigma}\lambda\int_0^\omega b(s)\varepsilon u(s-\tau(s))\mathrm{d}s\\ &\le \frac{1}{1-\sigma}\lambda\int_0^\omega b(s)\mathrm{d}s\varepsilon\|u\|\\ &= \frac{\int_0^\omega b(s)\mathrm{d}s}{1-\sigma}\lambda\varepsilon\|u\|. \quad\square\end{aligned}$$

3. Proof of Theorem 1.1

Proof. Let $r_1 = L$. By Lemma 2.4 we infer that there exists a $\lambda_0 = L\frac{1-\sigma^{e^L}}{m(L)\sigma^{e^L}\int_0^\omega b(s)\mathrm{d}s} > 0$ such that, for $u \in \partial\Omega_{r_1}, \lambda > \lambda_0$,

$$\begin{aligned}\|T_\lambda u\| &\ge \lambda\frac{\sigma^{e^L}\int_0^\omega b(s)\mathrm{d}s}{1-\sigma^{e^L}}m(L)\\ &> L\frac{1-\sigma^{e^L}}{m(L)\sigma^{e^L}\int_0^\omega b(s)\mathrm{d}s}\,\frac{\sigma^{e^L}\int_0^\omega b(s)\mathrm{d}s}{1-\sigma^{e^L}}m(L)\\ &= L = \|u\|.\end{aligned}$$

Since $\lim_{u\to 0^+}\frac{f(u)}{u} = 0$, we can choose $0 < r_2 < r_1$ such that $f(u) \le \varepsilon u$ for $0 \le u \le r_2$, where the constant $\varepsilon > 0$ satisfies

$$\lambda\varepsilon\frac{\int_0^\omega b(s)\mathrm{d}s}{1-\sigma} < 1.$$

Thus $f(u(t)) \le \varepsilon u(t)$ for $u \in \partial\Omega_{r_2}$ and $t \in [0,\omega]$. We have by Lemma 2.5 that

$$\|T_\lambda u\| \le \lambda\varepsilon\frac{\int_0^\omega b(s)\mathrm{d}s}{1-\sigma}\|u\| < \|u\| \quad \text{for } u \in \partial\Omega_{r_2}.$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0, \qquad i(T_\lambda, \Omega_{r_2}, K) = 1.$$

Thus $i(T_\lambda, \Omega_{r_1}\setminus\bar{\Omega}_{r_2}, K) = -1$ and T_λ has a fixed point u in $\Omega_{r_1}\setminus\bar{\Omega}_{r_2}$, which is a positive ω-periodic solution of (2.3) for $\lambda > \lambda_0$. Note that $\|u\| \le L$; it is also a positive ω-periodic solution to (1.2). $\square$

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