A note on positive periodic solutions of delayed differential equations

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\textbf{A B S T R A C T}

We consider the existence of positive $\omega$-periodic solutions for the periodic equation $x'(t) = a(t)e^{\int_0^t g(t) dt}x(t) - \lambda b(t)f(x(t - \tau(t)))$, where $a, b \in C(\mathbb{R}, [0, \infty))$ are $\omega$-periodic, $\int_0^\omega a(t)dt > 0$, $\int_0^\omega b(t)dt > 0$, $f \in C([0, \infty), [0, \infty))$, and $f(u) > 0$ for $u > 0$, $\tau(t)$ is a continuous $\omega$-periodic function.

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1. Introduction

In recent years, there has been considerable interest in the existence of positive periodic solutions of the following equation:

\begin{equation}
x'(t) = a(t)g(x(t))x(t) - \lambda b(t)f(x(t - \tau(t))).
\end{equation}

See, for example, [1–6]. (1.1) has been proposed as a model for a variety of biological processes. See, for example, the above references and [7,8].

The existence results in the literature are largely based on the assumption that $g(x(t))$ is constant or bounded. It is interesting to know whether there is a positive solution to (1.1) when $g(x(t))$ is not necessarily bounded. In this short note, we take $g(x) = e^x$ and consider the existence of an positive $\omega$-periodic solution of the equation

\begin{equation}
x'(t) = a(t)e^{\int_0^t g(t) dt}x(t) - \lambda b(t)f(x(t - \tau(t))),
\end{equation}

where $\lambda > 0$ is a positive parameter. We shall show that (1.2) has a positive $\omega$-periodic solution when $\lambda$ is sufficiently large. Apparently, our results can be extended to more general $g(x)$. Our arguments are based on a well-known fixed point theorem (Lemma 2.1).

Let $\mathbb{R} = (-\infty, \infty)$. We make the following assumptions:

(H1) $a, b \in C(\mathbb{R}, [0, \infty))$ are $\omega$-periodic functions, $\int_0^\omega a(t)dt > 0$, $\int_0^\omega b(t)dt > 0$, $\tau \in C(\mathbb{R}, \mathbb{R})$ is an $\omega$-periodic function.
(H2) $f : [0, \infty) \rightarrow [0, \infty)$ is continuous, $f(u) > 0$ for $u > 0$.

Also, let $\sigma = e^{-\int_0^\tau g(t)dt} < 1$, $m(\lambda) = \min\{f(u) : \sigma^{1-\sigma}(1-\sigma)L \leq u \leq L\} > 0$, $L > 0$.

Our main result is:

\textbf{Theorem 1.1.} Assume (H1)–(H2) hold and $\lim_{u \rightarrow 0^+} f(u) = 0$. For each $L > 0$, there exists a $\lambda_0 = L \frac{1-\sigma^L}{m(\lambda)\sigma^{\frac{1-\sigma}{1-\sigma}}} > 0$ such that (1.2) has a positive $\omega$-periodic solution $u$ with $\sup_{t \in [0, \omega]} u(t) \leq L$ for $\lambda > \lambda_0$.

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2. Preliminaries

We consider the modified equation
\[
x'(t) = a(t)g_t(u(t))x(t) - \lambda b(t)f(x(t - \tau(t))),
\]  
where \(g_t(u) = e^u\) for \(0 \leq u \leq L\) and \(g_t(u) = e^L\) for \(u \geq L\). For each \(L > 0\), if we can find a positive \(\omega\)-periodic solution \(u\) for (2.3) and \(\sup_{t \in [0, \omega)} u(t) \leq L\), then \(u\) is also a positive \(\omega\)-periodic solution to (1.2).

The following well-known result of the fixed point index is crucial in our arguments.

**Lemma 2.1** ([9–11]). Let \(E\) be a Banach space and \(K\) a cone in \(E\). For \(r > 0\), define \(K_r = \{ u \in K : \|u\| < r \}\). Assume that \(T : K_r \rightarrow K\) is completely continuous and such that \(Tx \neq x\) for \(x \in \partial K_r = \{ u \in K : \|u\| = r \}\).

(i) If \(\|Tx\| \geq \|x\|\) for \(x \in \partial K_r\), then
\[i(T, K_r, K) = 0.\]

(ii) If \(\|Tx\| \leq \|x\|\) for \(x \in \partial K_r\), then
\[i(T, K_r, K) = 1.\]

In order to apply Lemma 2.1 to (2.3), let \(X\) be the Banach space \(\{u(t) : u(t) \in C(\mathbb{R}, \mathbb{R}), u(t + \omega) = u(t)\}\) with \(\|u\| = \sup_{t \in [0, \omega]} |u(t)|, u \in X\).

Let \(K\) be a cone in \(X\) defined by
\[K = \left\{ u \in X : u(t) \geq \frac{\sigma^t (1 - \sigma)}{1 - \sigma^t} \|u\|, t \in [0, \omega] \right\},\]
and denote \(\Omega_r\) by
\[\Omega_r = \{ u \in K : \|u\| < r \}.\]

Note that \(\Omega_r = \{ u \in K : \|u\| = r \}\).

Following Wang [5], let \(T_\lambda : K \rightarrow X\) be a map defined by
\[
T_\lambda u(t) = \lambda \int_t^{t+\omega} G_t(t, s)b(s)f(u(s - \tau(s)))ds,
\]
where
\[G_t(t, s) = \frac{e^{-\int_t^s a(\theta)g_t(u(\theta))d\theta}}{1 - e^{-\int_0^\omega a(\theta)g_t(u(\theta))d\theta}}.
\]

Note that
\[1 \leq g_t(u) \leq e^L\]
and
\[\frac{\sigma^t}{1 - \sigma^t} \leq G_t(t, s) \leq \frac{1}{1 - \sigma}, t \leq s \leq t + \omega.
\]

**Lemma 2.2.** Assume (H1)–(H2) hold. Then \(T_\lambda(K) \subset K\) and \(T_\lambda : K \rightarrow K\) is compact and continuous.

**Proof.** In view of the definition of \(K\), for \(u \in K\), we have
\[
(T_\lambda u)(t + \omega) = \lambda \int_t^{t+2\omega} G_t(t + \omega, s)b(s)f(u(s - \tau(s)))ds
\]
\[
= \lambda \int_t^{t+\omega} G_t(t + \omega, \theta + \omega)b(\theta + \omega)f(u(\theta + \omega - \tau(\theta + \omega)))d\theta
\]
\[
= \lambda \int_t^{t+\omega} G_t(t, s)b(s)f(u(s - \tau(s)))ds
\]
\[
= (T_\lambda u)(t).
\]
It is easy to see that \( \int_{t}^{t+\omega} b(s)f(u(s - \tau(s)))ds \) is a constant because of the periodicity of \( b(t)f(u(t - \tau(t))) \). One can show that, for \( u \in K \) and \( t \in [0, \omega] \),
\[
T_\lambda u(t) = \frac{\sigma^e}{1 - \sigma^e \lambda} \int_{t}^{t+\omega} b(s)f(u(s - \tau(s)))ds
\]

Furthermore, \( T_\lambda \) and \( T_\lambda u = u \) then,
\[
\lambda G_\lambda(t, t + \omega)b(t + \omega)f(u(t + \omega - \tau(t + \omega))) - \lambda G_\lambda(t, t)b(t)f(u(t - \tau(t))) + a(t)g_\lambda(u(t))T_\lambda u(t)
\]

Thus \( u \) is a positive \( \omega \)-periodic solution of (2.3). On the other hand, if \( u \) is a positive \( \omega \)-periodic solution of (2.3), then
\[
\lambda b(t)f(u(t - \tau(t))) = a(t)g_\lambda(u(t))u(t) - u'(t)
\]

Furthermore, in view of the proof of Lemma 2.2, we also have \( u(t) \geq \frac{\sigma^e(1 - \sigma)}{1 - \sigma^e} \| u \| \) for \( t \in [0, \omega] \). Thus \( u \) is a fixed point of \( T_\lambda \) in \( K \).

**Lemma 2.3.** Assume (H1)–(H2) hold. Then a positive \( \omega \)-periodic solution of (2.3) is equivalent to a fixed point of \( T_\lambda \) in \( K \).

**Proof.** If \( u \in K \) and \( T_\lambda u = u \), then
\[
u'(t) = \frac{d}{dt} \left( \int_{t}^{t+\omega} G_\lambda(t, s)b(s)f(u(s - \tau(s)))ds \right)
\]

Thus \( u \) is a positive \( \omega \)-periodic solution of (2.3). On the other hand, if \( u \) is a positive \( \omega \)-periodic solution of (2.3), then
\[
\lambda b(t)f(u(t - \tau(t))) = a(t)g_\lambda(u(t))u(t) - u'(t)
\]

Furthermore, in view of the proof of Lemma 2.2, we also have \( u(t) \geq \frac{\sigma^e(1 - \sigma)}{1 - \sigma^e} \| u \| \) for \( t \in [0, \omega] \). Thus \( u \) is a fixed point of \( T_\lambda \) in \( K \).

**Lemma 2.4.** Assume (H1)–(H2) hold. If \( u \in \partial \Omega_t \), then
\[
\| T_\lambda u \| \geq \frac{\lambda \sigma^e}{1 - \sigma^e \lambda} \int_{0}^{\omega} b(s)ds
\]

**Proof.** Since \( f(u(t)) \geq m(L) \) for \( t \in \mathbb{R} \), it is easy to see that
\[
(T_\lambda u)(t) \geq \frac{\sigma^e}{1 - \sigma^e \lambda} \int_{0}^{\omega} b(s)f(u(s - \tau(s)))ds
\]

Furthermore, in view of the proof of Lemma 2.2, we also have \( u(t) \geq \frac{\sigma^e(1 - \sigma)}{1 - \sigma^e} \| u \| \) for \( t \in [0, \omega] \). Thus \( u \) is a fixed point of \( T_\lambda \) in \( K \).

**Lemma 2.5.** Assume (H1)–(H2) hold and let \( r > 0 \). If \( u \in \partial \Omega_r \) and there exists an \( \varepsilon > 0 \) such that \( f(u(t)) \leq \varepsilon u(t) \) for \( t \in [0, \omega] \), then
\[
\| T_\lambda u \| \leq \frac{\lambda \varepsilon \| u \|}{1 - \varepsilon}
\]
Let $r_1 = L$. By Lemma 2.4 we infer that there exists a $\lambda_0 = L \frac{1 - \sigma^e}{m(L) \sigma^e \int_0^\sigma b(s)ds} > 0$ such that, for $u \in \partial \Omega_1$, $\lambda > \lambda_0$,

$$
\|T_{\lambda}u\| \geq \lambda \frac{\sigma^e \int_0^\sigma b(s)ds}{1 - \sigma^e} m(L)
$$

$$
= L \frac{1 - \sigma^e}{m(L) \sigma^e \int_0^\sigma b(s)ds} \frac{\sigma^e \int_0^\sigma b(s)ds}{1 - \sigma^e} m(L)
$$

$$
= L = \|u\|.
$$

Since $\lim_{u \to 0^+} \frac{f(u)}{u} = 0$, we can choose $0 < r_2 < r_1$ such that $f(u) \leq \varepsilon u$ for $0 \leq u \leq r_2$, where the constant $\varepsilon > 0$ satisfies

$$
\lambda \varepsilon \frac{\int_0^\sigma b(s)ds}{1 - \sigma} < 1.
$$

Thus $f(u(t)) \leq \varepsilon u(t)$ for $u \in \partial \Omega_2$ and $t \in [0, \omega]$. We have by Lemma 2.5 that

$$
\|T_{\lambda}u\| \leq \lambda \varepsilon \frac{\int_0^\sigma b(s)ds}{1 - \sigma} \|u\| < \|u\| \quad \text{for} \quad u \in \partial \Omega_2.
$$

It follows from Lemma 2.1 that

$$
i(T_{\lambda}, \Omega_1, K) = 0, \quad i(T_{\lambda}, \Omega_2, K) = 1.
$$

Thus $i(T_{\lambda}, \Omega_1 \setminus \Omega_2, K) = -1$ and $T_{\lambda}$ has a fixed point $u$ in $\Omega_1 \setminus \Omega_2$, which is a positive $\omega$-periodic solution of (2.3) for $\lambda > \lambda_0$. Note that $\|u\| \leq L$; it is also a positive $\omega$-periodic solution to (1.2). \hfill \Box

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References


