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On the number of positive solutions of elliptic systems

Donal O'Regan[*1] and **Haiyan Wang**[**2]

[1] Department of Mathematics, National University of Ireland, Galway, Ireland
[2] Department of Mathematical Sciences & Applied Computing, Arizona State University, Phoenix, AZ 85069-7100, USA

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The paper deals with the existence, multiplicity and nonexistence of positive radial solutions for the elliptic system $\text{div}(|\nabla u_i|^{p-2}\nabla u_i)+\lambda k_i(|x|)f^i(u_1,\ldots,u_n)=0,\ p>1,\ R_1<|x|<R_2,\ u_i(x)=0$, on $|x|=R_1$ and R_2, $i=1,\ldots,n$, $x\in\mathbb{R}^N$, where k_i and f^i, $i=1,\ldots,n$, are continuous and nonnegative functions. Let $\mathbf{u}=(u_1,\ldots,u_n)$, $\varphi(t)=|t|^{p-2}t$, $f_0^i=\lim_{\|\mathbf{u}\|\to 0}\frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}$, $f_\infty^i=\lim_{\|\mathbf{u}\|\to\infty}\frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}$, $i=1,\ldots,n$, $\mathbf{f}=(f^1,\ldots,f^n)$, $\mathbf{f}_0=\sum_{i=1}^n f_0^i$ and $\mathbf{f}_\infty=\sum_{i=1}^n f_\infty^i$. We prove that either $\mathbf{f}_0=0$ and $\mathbf{f}_\infty=\infty$ (superlinear), or $\mathbf{f}_0=\infty$ and $\mathbf{f}_\infty=0$ (sublinear), guarantee existence for all $\lambda>0$. In addition, if $f^i(\mathbf{u})>0$ for $\|\mathbf{u}\|>0$, $i=1,\ldots,n$, then either $\mathbf{f}_0=\mathbf{f}_\infty=0$, or $\mathbf{f}_0=\mathbf{f}_\infty=\infty$, guarantee multiplicity for sufficiently large, or small λ, respectively. On the other hand, either f_0 and $f_\infty>0$, or f_0 and $f_\infty<\infty$ imply nonexistence for sufficiently large, or small λ, respectively. Furthermore, all the results are valid for Dirichlet/Neumann boundary conditions. We shall use fixed point theorems in a cone.

1 Introduction

In this paper we consider the existence, multiplicity and nonexistence of positive radial solutions for the quasilinear elliptic system

$$\begin{cases} \text{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right)+\lambda k_1(|x|)f^1(u_1,\ldots,u_n)=0, \\ \ldots \\ \text{div}\left(|\nabla u_n|^{p-2}\nabla u_n\right)+\lambda k_n(|x|)f^n(u_1,\ldots,u_n)=0, \end{cases} \tag{1.1}$$

in the domain $0<R_1<|x|<R_2<\infty,\ x\in\mathbb{R}^N,\ N\geq 2$, with one of the following three sets of the boundary conditions,

$$u_i=0 \ \text{ on } \ |x|=R_1 \ \text{ and } \ |x|=R_2,\ \ i=1,\ldots,n, \tag{1.2a}$$

$$\partial u_i/\partial r=0 \ \text{ on } \ |x|=R_1 \ \text{ and } \ u_i=0 \ \text{ on } \ |x|=R_2,\ \ i=1,\ldots,n, \tag{1.2b}$$

$$u_i=0 \ \text{ on } \ |x|=R_1 \ \text{ and } \ \partial u_i/\partial r=0 \ \text{ on } \ |x|=R_2,\ \ i=1,\ldots,n, \tag{1.2c}$$

where $p>1$, $r=|x|$ and $\partial/\partial r$ denotes differentiation in the radial direction.

When $n=1$, (1.1) becomes the scalar quasilinear equation

$$\text{div}\left(|\nabla u|^{p-2}\nabla u\right)+\lambda k(|x|)f(u)=0,\ \text{ in } R_1<|x|<R_2,\quad x\in\mathbb{R}^N,\quad N\geq 2. \tag{1.3}$$

When $p=2$, (1.3) further reduces to the classical semilinear elliptic equation

$$\Delta u+\lambda k(|x|)f(u)=0,\ \text{ in } R_1<|x|<R_2,\quad x\in\mathbb{R}^N,\quad N\geq 2. \tag{1.4}$$

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* e-mail: donal.oregan@nuigalway.ie, Phone: +353 91 493091, Fax: +353 91 750542
** Corresponding author: e-mail: wangh@asu.edu, Phone: +1 602 543 5635, Fax: +1 602 543 3260

It was proved in [3, 12, 17] that (1.4) with boundary conditions has a positive radial solution under the assumption that f is superlinear, i.e., $f_0 = \lim_{u\to 0} \frac{f(u)}{u} = 0$ and $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u} = \infty$, or f is sublinear, i.e., $f_0 = \lim_{u\to 0} \frac{f(u)}{u} = \infty$ and $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u} = 0$. Related multiplicity and nonexistence results can be found in [1, 5, 7, 8, 14, 15, 18] and others.

In this paper, we shall show appropriate combinations of superlinearity and sublinearity of f at zero and infinity for (1.1) guarantee the existence, multiplicity and nonexistence of positive radial solutions of (1.1). For this purpose, we introduce the notation $\mathbf{f}_0$ and $\mathbf{f}_\infty$, to characterize superlinearity and sublinearity for (1.1). They are natural extensions of f_0 and f_∞ defined above for the scalar equation.

Ambrosetti, Brezis and Cerami [2], Brezis and Nirenberg [4] studied the combined effects of concave and convex nonlinearities on the number of positive solutions of the Dirchlet boundary value problem for $\Delta u + \lambda u^{p_1} + u^{p_2} = 0$ with $0 < p_1 < 1 < p_2$ in general domains. Note that $\Delta u + \lambda u^{p_1} + u^{p_2} = 0$ is equivalent to (1.4) by a simple change of variables, $u = \lambda^{\frac{1}{p_2 - p_1}} v$ and $\mu = \lambda^{\frac{p_2 - 1}{p_2 - p_1}}$, which transforms $\Delta u + \lambda u^{p_1} + u^{p_2} = 0$ into $\Delta v + \mu(v^{p_1} + v^{p_2}) = 0$. It was shown in [2] that, among others, there are two positive solutions to the Dirchlet problem for $\lambda \in (0, \Lambda)$, $\Lambda > 0$. It is important to observe that $f_0 = f_\infty = \infty$ if $f = u^{p_1} + u^{p_2}$ for $0 < p_1 < 1 < p_2 < \infty$. Thus Theorem 1.2 (d) is consistent with this result in [2]. Our results can be viewed as a generalization of the related results in [2, 4] to the quasilinear linear elliptic system (1.1). In addition, we consider the number of positive solutions of (1.1) under all possible appropriate combinations of $\mathbf{f}_0$ and $\mathbf{f}_\infty$. Our results also show that convexity and monotonicity conditions are not necessary for the existence and multiplicity results at least for annular domains.

Our arguments are based on the fixed point index. Many authors have used the fixed point index for the existence of positive solutions of differential equations, see e.g. [6, 9, 17, 18, 20]. Variational methods have been frequently used for Hamiltonian systems and gradient systems. However, there is apparently no possibility of using variational methods for the n-dimensional quasilinear elliptic system (1.1), and one has to use topological methods.

We now turn to the general assumptions for this paper. Let $\varphi(t) = |t|^{p-2}t$, $p > 1$, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^n = \underbrace{\mathbb{R}_+ \times \ldots \times \mathbb{R}_+}_{n}$. Also, for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$, let $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$. We make the assumptions:

(H1). $f^i : \mathbb{R}_+^n \to \mathbb{R}_+$ is continuous, $i = 1, \ldots, n$.

(H2). $k_i : [R_1, R_2] \to [0, \infty)$ is continuous and $k_i \not\equiv 0$ on any subinterval of $[R_1, R_2]$, $i = 1, \ldots, n$.

(H3). $f^i(u_1, \ldots, u_n) > 0$ for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| > 0$, $i = 1, \ldots, n$.

In order to state our results, let

$$\mathbf{f}(\mathbf{u}) = \big(f^1(\mathbf{u}), \ldots, f^n(\mathbf{u})\big) = \big(f^1(u_1, \ldots, u_n), \ldots, f^n(u_1, \ldots, u_n)\big).$$

Then we introduce the notation

$$f_0^i = \lim_{\|\mathbf{u}\|\to 0} \frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}, \quad f_\infty^i = \lim_{\|\mathbf{u}\|\to \infty} \frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}, \quad i = 1, \ldots, n,$$

where $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$,

$$\mathbf{f}_0 = \sum_{i=1}^n f_0^i, \quad \mathbf{f}_\infty = \sum_{i=1}^n f_\infty^i. \tag{1.5}$$

Our main results are:

Theorem 1.1 *Assume* (H1) *and* (H2) *hold.*
(a). *If* $\mathbf{f}_0 = 0$ *and* $\mathbf{f}_\infty = \infty$, *then for all* $\lambda > 0$ (1.1)–(1.2) *has a positive radial solution.*
(b). *If* $\mathbf{f}_0 = \infty$ *and* $\mathbf{f}_\infty = 0$, *then for all* $\lambda > 0$ (1.1)–(1.2) *has a positive radial solution.*

Theorem 1.2 *Assume* (H1)–(H3) *hold.*
(a). *If* $\mathbf{f}_0 = 0$ *or* $\mathbf{f}_\infty = 0$, *then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (1.1)–(1.2) *has a positive radial solution.*

(b). *If* $\mathbf{f}_0 = \infty$ *or* $\mathbf{f}_\infty = \infty$*, then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (1.1)–(1.2) *has a positive radial solution.*

(c). *If* $\mathbf{f}_0 = \mathbf{f}_\infty = 0$*, then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (1.1)–(1.2) *has two positive radial solutions.*

(d). *If* $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$*, then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (1.1)–(1.2) *has two positive radial solutions.*

(e). *If* $\mathbf{f}_0 < \infty$ *and* $\mathbf{f}_\infty < \infty$*, then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (1.1)–(1.2) *has no positive radial solution.*

(f). *If* $\mathbf{f}_0 > 0$ *and* $\mathbf{f}_\infty > 0$*, then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (1.1)–(1.2) *has no positive radial solution.*

When $\lambda = 1$, Theorem 1.1 was proved in [19]. For $N = 1$, (1.1) becomes a system of ordinary differential equations. Theorems 1.1 and 1.2 for systems of ordinary differential equations were obtained in [20].

2 Preliminaries

A radial solution of (1.1)–(1.2) can be considered as a solution of the system

$$\begin{cases} \left(r^{N-1}\varphi(u_1'(r))\right)' + \lambda r^{N-1}k_1(r)f^1(u_1,\dots,u_n) = 0, \\ \dots \\ \left(r^{N-1}\varphi(u_n'(r))\right)' + \lambda r^{N-1}k_n(r)f^n(u_1,\dots,u_n) = 0, \end{cases} \tag{2.1}$$

$0 < R_1 < r < R_2 < \infty$, with one of the following three sets of the boundary conditions,

$$u_i(R_1) = u_i(R_2) = 0, \quad i = 1,\dots,n, \tag{2.2a}$$

$$u_i'(R_1) = u_i(R_2) = 0, \quad i = 1,\dots,n, \tag{2.2b}$$

$$u_i(R_1) = u_i'(R_2) = 0, \quad i = 1,\dots,n. \tag{2.2c}$$

We shall treat classical solutions of (2.1)–(2.2), namely vector-valued functions $\mathbf{u} = (u_1(r),\dots,u_n(r)) \in C^1([R_1,R_2],\mathbb{R}^n)$ with $\varphi(u_i') \in C^1(R_1,R_2)$, $i = 1,\dots,n$, which satisfies (2.1) for $r \in (R_1,R_2)$ and one of (2.2). A solution $\mathbf{u}(r) = (u_1(r),\dots,u_n(r))$ is positive if $u_i(r) \ge 0$, $i = 1,\dots,n$, for all $r \in (R_1,R_2)$ and there is at least one nontrivial component of $\mathbf{u}$. In fact, we shall show that such a nontrivial component of $\mathbf{u}$ is positive on (R_1,R_2).

Applying the change of variables, $r = (R_2 - R_1)t + R_1$, we can transform (2.1)–(2.2) into the form

$$\begin{cases} (q(t)\varphi(\zeta u_1'))' + \lambda h_1(t)f^1(\mathbf{u}) = 0, \\ \dots \\ (q(t)\varphi(\zeta u_n'))' + \lambda h_n(t)f^n(\mathbf{u}) = 0, \end{cases} \tag{2.3}$$

$0 < t < 1$, with one of the following three sets of the boundary conditions,

$$\mathbf{u}(0) = \mathbf{u}(1) = 0, \tag{2.4a}$$

$$\mathbf{u}'(0) = \mathbf{u}(1) = 0, \tag{2.4b}$$

$$\mathbf{u}(0) = \mathbf{u}'(1) = 0, \tag{2.4c}$$

where

$$\mathbf{u}(t) = (u_1(t),\dots,u_n(t)), \quad q(t) = ((R_2 - R_1)t + R_1)^{N-1}, \quad \zeta = \frac{1}{R_2 - R_1}$$

and

$$h_i(t) = (R_2 - R_1)((R_2 - R_1)t + R_1)^{N-1}k_i((R_2 - R_1)t + R_1), \quad i = 1,\dots,n.$$

It is clear that $q(t) > 0 \in C[0,1]$ and is nondecreasing for $t \in [0,1]$, and $h_i : [0,1] \to [0,\infty)$ is continuous and $h_i \not\equiv 0$ on any subinterval of $[0,1]$, $i = 1,\dots,n$.

For (2.3)–(2.4) we shall prove Theorems 2.1 and 2.2, which immediately imply that Theorems 1.1 and 1.2 are true.

Theorem 2.1 *Assume* (H1)–(H2) *hold.*

(a). *If* $\mathbf{f}_0 = 0$ *and* $\mathbf{f}_\infty = \infty$*, then for all* $\lambda > 0$ (2.3)–(2.4) *has a positive solution.*

(b). *If* $\mathbf{f}_0 = \infty$ *and* $\mathbf{f}_\infty = 0$*, then for all* $\lambda > 0$ (2.3)–(2.4) *has a positive solution.*

Theorem 2.2 *Assume* (H1)–(H3) *hold.*

(a). *If* $\mathbf{f}_0 = 0$ *or* $\mathbf{f}_\infty = 0$*, then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (2.3)–(2.4) *has a positive solution.*

(b). *If* $\mathbf{f}_0 = \mathbf{f}_\infty = 0$*, then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (2.3)–(2.4) *has two positive solutions.*

(c). *If* $\mathbf{f}_0 = \infty$ *or* $\mathbf{f}_\infty = \infty$*, then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (2.3)–(2.4) *has a positive solution.*

(d). *If* $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$*, then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (2.3)–(2.4) *has two positive solutions.*

(e). *If* $\mathbf{f}_0 < \infty$ *and* $\mathbf{f}_\infty < \infty$*, then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (2.3)–(2.4) *has no positive solution.*

(f). *If* $\mathbf{f}_0 > 0$ *and* $\mathbf{f}_\infty > 0$*, then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (2.3)–(2.4) *has no positive solution.*

We recall some concepts and conclusions on the fixed point index in a cone in [9, 10]. Let E be a Banach space and K be a closed, nonempty subset of E. K is said to be a cone if (i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geq 0$ and (ii) $u, -u \in K$ imply $u = 0$. Assume Ω is a bounded open subset in E with the boundary $\partial\Omega$, and let $T : K \cap \overline{\Omega} \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial\Omega \cap K$, then the fixed point index $i(T, K \cap \Omega, K)$ is defined. If $i(T, K \cap \Omega, K) \neq 0$, then T has a fixed point in $K \cap \Omega$. The following well-known result of the fixed point index is crucial in our arguments.

Lemma 2.3 ([9, 10]). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|x\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|x\| = r\}$.*

(i) *If* $\|Tx\| \geq \|x\|$ *for* $x \in \partial K_r$*, then*

$$i(T, K_r, K) = 0.$$

(ii) *If* $\|Tx\| \leq \|x\|$ *for* $x \in \partial K_r$*, then*

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.3 to (2.3)–(2.4), let X be the Banach space $\underbrace{C[0,1] \times \ldots \times C[0,1]}_{n}$ and, for $\mathbf{u} = (u_1, \ldots, u_n) \in X$,

$$\|\mathbf{u}\| = \sum_{i=1}^{n} \sup_{t \in [0,1]} |u_i(t)|.$$

For $\mathbf{u} \in X$ or $\mathbb{R}^n_+$, $\|\mathbf{u}\|$ denotes the norm of $\mathbf{u}$ in X or $\mathbb{R}^n_+$, respectively.

Define K a cone in X by

$$K = \left\{ \mathbf{u} = (u_1, \ldots, u_n) \in X \ : \ u_i(t) \geq 0, \ t \in [0,1], \ i = 1, \ldots, n, \text{ and } \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \sum_{i=1}^{n} u_i(t) \geq \frac{1}{4}\|\mathbf{u}\| \right\}.$$

Also, define, for r a positive number, Ω_r by

$$\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| < r\}.$$

Note that $\partial\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| = r\}$.

Let $\mathbf{T}_\lambda : K \to X$ be a map with components $\left(T^1_\lambda, \ldots, T^n_\lambda\right)$. We define T^i_λ, $i = 1, \ldots, n$, by

$$T^i_\lambda \mathbf{u}(t) = \begin{cases} \displaystyle\int_0^t \frac{1}{\zeta} \varphi^{-1}\left(\frac{1}{q(s)} \lambda \int_s^{\sigma_i} h_i(\tau) f^i(\mathbf{u}(\tau))\, d\tau\right) ds, & 0 \leq t \leq \sigma_i, \\ \displaystyle\int_t^1 \frac{1}{\zeta} \varphi^{-1}\left(\frac{1}{q(s)} \lambda \int_{\sigma_i}^{s} h_i(\tau) f^i(\mathbf{u}(\tau))\, d\tau\right) ds, & \sigma_i \leq t \leq 1, \end{cases} \tag{2.5}$$

where $\sigma_i = 0$ for (2.3)–(2.4b) and $\sigma_i = 1$ for (2.3)–(2.4c). For (2.3)–(2.4a) $\sigma_i \in (0,1)$ is a solution of the equation

$$\Theta^i \mathbf{u}(t) = 0, \quad 0 \le t \le 1, \tag{2.6}$$

where the map $\Theta^i : K \to C[0,1]$ is defined by

$$\begin{aligned}\Theta^i \mathbf{u}(t) = &\int_0^t \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^t \lambda h_i(\tau) f^i(\mathbf{u}(\tau))\, d\tau\right) ds \\ &- \int_t^1 \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_t^s \lambda h_i(\tau) f^i(\mathbf{u}(\tau))\, d\tau\right) ds, \quad 0 \le t \le 1.\end{aligned} \tag{2.7}$$

By virtue of Lemma 2.4, the operator $\mathbf{T}_\lambda$ is well-defined.

Lemma 2.4 ([18, 20]) *Assume* (H1)–(H2) *hold. Then, for any* $\mathbf{u} \in K$ *and* $i = 1, \ldots, n$, $\Theta^i \mathbf{u}(t) = 0$ *has at least one solution in* $(0,1)$. *In addition, if* $\sigma_i^1 < \sigma_i^2 \in (0,1)$, $i = 1, \ldots, n$, *are two solutions of* $\Theta^i \mathbf{u}(t) = 0$, *then* $h_i(t) f^i(\mathbf{u}(t)) \equiv 0$ *for* $t \in \left[\sigma_i^1, \sigma_i^2\right]$ *and any* $\sigma_i \in \left[\sigma_i^1, \sigma_i^2\right]$ *is also a solution of* $\Theta^i \mathbf{u}(t) = 0$. *Furthermore,* $\mathbf{T}_\lambda^i \mathbf{u}(t)$, $i = 1, \ldots, n$, *is independent of the choice of* $\sigma_i \in \left[\sigma_i^1, \sigma_i^2\right]$.

The following lemma is a standard result due to the concavity of $u(t)$ on $[0,1]$.

Lemma 2.5 *Assume* $u \in C^1[0,1]$ *with* $u(t) \ge 0$ *for* $t \in [0,1]$, *and* $q > 0$ *is nondecreasing for* $t \in [0,1]$. *If* $q(t)\varphi(\zeta u')$ *is nonincreasing on* $[0,1]$, *then* $u(t) \ge \min\{t, 1-t\} \sup_{t\in[0,1]}|u(t)|$, $t \in [0,1]$. *In particular,*

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) \ge \frac{1}{4} \sup_{t\in[0,1]} |u(t)|.$$

P r o o f. Note that φ^{-1} is increasing and $q(t)$ is nondecreasing. Because $q(t)\varphi(\zeta u')$ is nonincreasing on $[0,1]$, we find that $u'(t)$ is nonincreasing. Hence, for $0 \le t_0 < t < t_1 \le 1$,

$$u(t) - u(t_0) = \int_{t_0}^t u'(s)\, ds \ge (t - t_0) u'(t)$$

and

$$u(t_1) - u(t) = \int_t^{t_1} u'(s)\, ds \le (t_1 - t) u'(t),$$

from which, we have

$$u(t) \ge \frac{(t-t_0)u(t_1) + (t_1 - t)u(t_0)}{t_1 - t_0}.$$

Considering the above inequality on $[0,\sigma]$ and $[\sigma,1]$, we obtain

$$\begin{aligned} &u(t) \ge t\,||u|| && \text{for} \quad t \in [0,\sigma], \\ &u(t) \ge (1-t)\,||u|| && \text{for} \quad t \in [\sigma,1], \end{aligned}$$

where $\sigma \in [0,1]$ such that $u(\sigma) = ||u||$. Hence, we have $u(t) \ge \min\{t, 1-t\}||u||$ for $t \in [0,1]$. □

We remark that, according to Lemma 2.5, any nontrivial component of nonnegative solutions of (2.3)–(2.4) is positive on $(0,1)$.

Lemma 2.6 *Assume* (H1)–(H2) *hold. Then* $\mathbf{T}_\lambda(K) \subset K$ *and* $\mathbf{T}_\lambda : K \to K$ *is compact and continuous.*

P r o o f. Lemma 2.5 implies that $\mathbf{T}_\lambda(K) \subset K$. It is not hard to see that $\mathbf{T}_\lambda$ is compact and continuous (see [20] for a proof). □

Now it is not difficult to show that $\mathbf{u} \in K$ is a solution of (2.3)–(2.4) if and only if $\mathbf{u}$ is a fixed point equation

$$\mathbf{T}_\lambda \mathbf{u} = \mathbf{u} \quad \text{in} \quad K.$$

Note that for $t > 0$, $\varphi(t) = t^{p-1}$, $p > 1$ and $\varphi^{-1}(t) = t^{\frac{1}{p-1}}$. It is easy to verify the following lemma.

Lemma 2.7 *For all* $\sigma, x \in (0,\infty)$*, we have* $\varphi^{-1}(\sigma\varphi(x)) = \varphi^{-1}(\sigma)x$.

For $i = 1,\dots,n$, let

$$\gamma_i(t) = \frac{1}{8}\left[\int_{\frac{1}{4}}^{t} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^t h_i(\tau)\,d\tau\right) ds + \int_t^{\frac{3}{4}} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_t^s h_i(\tau)\,d\tau\right) ds\right],$$

where $t \in \left[\frac{1}{4},\frac{3}{4}\right]$. It follows from (H2)that

$$\Gamma = \min\left\{\gamma_i(t) : \tfrac{1}{4} \le t \le \tfrac{3}{4},\ i = 1,\dots,n\right\} > 0.$$

Lemma 2.8 *Assume* (H1)–(H2) *hold. Let* $\mathbf{u} = (u_1(t),\dots,u_n(t)) \in K$ *and* $\eta > 0$*. If there exists a component* f^i *of* $\mathbf{f}$ *such that*

$$f^i(\mathbf{u}(t)) \ge \varphi\left(\eta\sum_{i=1}^{n} u_i(t)\right) \quad \textit{for} \quad t \in \left[\tfrac{1}{4},\tfrac{3}{4}\right]$$

then

$$\|\mathbf{T}_\lambda\mathbf{u}\| \ge \varphi^{-1}(\lambda)\Gamma\eta\,\|\mathbf{u}\|.$$

P r o o f. Note, from the definition of $\mathbf{T}_\lambda\mathbf{u}$, that $T_\lambda^i\mathbf{u}(\sigma_i)$ is the maximum value of $T_\lambda^i\mathbf{u}$ on [0,1]. If $\sigma_i \in \left[\frac{1}{4},\frac{3}{4}\right]$, we consider

$$\begin{aligned}
\|\mathbf{T}_\lambda\mathbf{u}\| &\ge \sup_{t\in[0,1]} |T_\lambda^i\mathbf{u}(t)| \\
&\ge \frac{1}{2}\left[\int_{\frac{1}{4}}^{\sigma_i} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))\,d\tau\right) ds\right. \\
&\quad \left. + \int_{\sigma_i}^{\frac{3}{4}} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))\,d\tau\right) ds\right] \\
&\ge \frac{1}{2}\left[\int_{\frac{1}{4}}^{\sigma_i} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i} \lambda h_i(\tau)\varphi\left(\eta\sum_{i=1}^{n} u_i(\tau)\right) d\tau\right) ds\right. \\
&\quad \left. + \int_{\sigma_i}^{\frac{3}{4}} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s} \lambda h_i(\tau)\varphi\left(\eta\sum_{i=1}^{n} u_i(\tau)\right) d\tau\right) ds\right],
\end{aligned}$$

and so,

$$\begin{aligned}
\|\mathbf{T}_\lambda\mathbf{u}\| &\ge \frac{1}{2}\left[\int_{\frac{1}{4}}^{\sigma_i} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i} \varphi\big(\varphi^{-1}(\lambda)\big) h_i(\tau)\varphi\left(\eta\,\frac{1}{4}\,\|\mathbf{u}\|\right) d\tau\right) ds\right. \\
&\quad \left. + \int_{\sigma_i}^{\frac{3}{4}} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s} \varphi\big(\varphi^{-1}(\lambda)\big) h_i(\tau)\varphi\left(\eta\,\frac{1}{4}\,\|\mathbf{u}\|\right) d\tau\right) ds\right] \\
&= \frac{1}{2}\left[\int_{\frac{1}{4}}^{\sigma_i} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i} h_i(\tau)\,d\tau\,\varphi\left(\varphi^{-1}(\lambda)\eta\,\frac{1}{4}\,\|\mathbf{u}\|\right)\right) ds\right. \\
&\quad \left. + \int_{\sigma_i}^{\frac{3}{4}} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s} h_i(\tau)\,d\tau\,\varphi\left(\varphi^{-1}(\lambda)\eta\,\frac{1}{4}\,\|\mathbf{u}\|\right)\right) ds\right].
\end{aligned}$$

Now, because of Lemma 2.7, we have

$$\begin{aligned}
\|\mathbf{T}_\lambda\mathbf{u}\| &\ge \frac{\varphi^{-1}(\lambda)\eta\,\|\mathbf{u}\|\frac{1}{4}}{2} \\
&\quad \times \left[\int_{\frac{1}{4}}^{\sigma_i} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i} h_i(\tau)\,d\tau\right) ds + \int_{\sigma_i}^{\frac{3}{4}} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s} h_i(\tau)\,d\tau\right) ds\right] \\
&\ge \varphi^{-1}(\lambda)\Gamma\eta\,\|\mathbf{u}\|.
\end{aligned}$$

For $\sigma_i > \frac{3}{4}$, it is easy to see

$$\|T_\lambda^i \mathbf{u}\| \geq \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\frac{3}{4}} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))\,d\tau\right) ds.$$

On the other hand, we have

$$\|T_\lambda^i \mathbf{u}\| \geq \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{\zeta}\varphi^{-1}\left(\frac{1}{q(s)}\int_{\frac{1}{4}}^{s} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))\,d\tau\right) ds \quad \text{if} \quad \sigma_i < \frac{1}{4}.$$

Also, similar arguments show that $\|\mathbf{T}_\lambda \mathbf{u}\| \geq \varphi^{-1}(\lambda)\Gamma\eta\,\|\mathbf{u}\|$ if $\sigma_i > \frac{3}{4}$ or $\sigma_i < \frac{1}{4}$. □

For each $i = 1, \ldots, n$, define a new function $\hat{f}^i(t) : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\hat{f}^i(t) = \max\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n \text{ and } \|\mathbf{u}\| \leq t\}.$$

Note that $\hat{f}_0^i = \lim_{t\to 0} \frac{\hat{f}^i(t)}{\varphi(t)}$ and $\hat{f}_\infty^i = \lim_{t\to\infty} \frac{\hat{f}^i(t)}{\varphi(t)}$.

Lemma 2.9 ([20]) *Assume* (H1) *holds. Then* $\hat{f}_0^i = f_0^i$ *and* $\hat{f}_\infty^i = f_\infty^i$, $i = 1, \ldots, n$.

Lemma 2.10 *Assume* (H1)–(H2) *hold and let* $r > 0$. *If there exits an* $\varepsilon > 0$ *such that*

$$\hat{f}^i(r) \leq \varphi(\varepsilon)\varphi(r), \quad i = 1, \ldots, n,$$

then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \varphi^{-1}(\lambda)\varepsilon \widehat{C}\,\|\mathbf{u}\| \quad \textit{for} \quad \mathbf{u} \in \partial\Omega_r,$$

where the constant $\widehat{C} = \frac{1}{\zeta}\sum_{i=1}^n \varphi^{-1}\big(\frac{1}{q(0)}\int_0^1 h_i(\tau)\,d\tau\big)$.

P r o o f. From the definition of $\mathbf{T}_\lambda$, for $\mathbf{u} \in \partial\Omega_r$, we have

$$\begin{aligned}
\|\mathbf{T}_\lambda \mathbf{u}\| &= \sum_{i=1}^n \sup_{t\in[0,1]} |T_\lambda^i \mathbf{u}(t)| \\
&\leq \frac{1}{\zeta}\sum_{i=1}^n \varphi^{-1}\left[\frac{1}{q(0)}\int_0^1 \lambda h_i(\tau) f^i(\mathbf{u}(\tau))\,d\tau\right] \\
&\leq \frac{1}{\zeta}\sum_{i=1}^n \varphi^{-1}\left[\frac{1}{q(0)}\int_0^1 h_i(\tau)\,d\tau\,\lambda \hat{f}^i(r)\right] \\
&\leq \frac{1}{\zeta}\sum_{i=1}^n \varphi^{-1}\left[\frac{1}{q(0)}\int_0^1 h_i(\tau)\,d\tau\,\lambda\varphi(\varepsilon)\varphi(r)\right].
\end{aligned}$$

Note that $\lambda = \varphi\big(\varphi^{-1}(\lambda)\big)$. Then Lemma 2.7 implies that

$$\begin{aligned}
\|\mathbf{T}_\lambda \mathbf{u}\| &\leq \frac{1}{\zeta}\sum_{i=1}^n \varphi^{-1}\left(\frac{1}{q(0)}\int_0^1 h_i(\tau)\,d\tau\,\varphi\big(\varphi^{-1}(\lambda)\varepsilon r\big)\right) \\
&= \varphi^{-1}(\lambda)\varepsilon r \frac{1}{\zeta}\sum_{i=1}^n \varphi^{-1}\left(\frac{1}{q(0)}\int_0^1 h_i(\tau)\,d\tau\right) \\
&= \varphi^{-1}(\lambda)\varepsilon\widehat{C}\,\|\mathbf{u}\|.
\end{aligned}$$
□

The following two lemmas are weak forms of Lemmas 2.8 and 2.10.

Lemma 2.11 *Assume* (H1)–(H3) *hold. If* $\mathbf{u} \in \partial\Omega_r$, $r > 0$, *then*

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq 4\varphi^{-1}\big(\lambda \widehat{m}_r\big)\Gamma$$

where $\widehat{m}_r = \min\big\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n$ *and* $\frac{1}{4}r \leq \|\mathbf{u}\| \leq r,\ i = 1, \ldots, n.\big\} > 0$.

P r o o f. Since $\lambda f^i(\mathbf{u}(t)) \geq \lambda \widehat{m}_r = \varphi(\varphi^{-1}(\lambda \widehat{m}_r))$ for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$, $i = 1, \ldots, n$, it is easy to see that this lemma can be shown in a similar manner as in Lemma 2.8. □

Lemma 2.12 *Assume* (H1)–(H3) *hold. If* $\mathbf{u} \in \partial\Omega_r$, $r > 0$, *then*

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \varphi^{-1}(\lambda)\varphi^{-1}\big(\widehat{M_r}\big)\widehat{C},$$

where $\widehat{M_r} = \max\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}^n_+$ *and* $\|\mathbf{u}\| \leq r,\ i = 1, \ldots, n\} > 0$ *and* $\widehat{C}$ *is the positive constant defined in Lemma* 2.10.

P r o o f. Since $f^i(\mathbf{u}(t)) \leq \widehat{M_r} = \varphi\big(\varphi^{-1}\big(\widehat{M_r}\big)\big)$ for $t \in [0, 1]$, $i = 1, \ldots, n$, it is easy to see that this lemma can be shown in a similar manner as in Lemma 2.10. □

3 Proof of Theorem 2.1

Part (a). $\mathbf{f}_0 = 0$ implies that $f_0^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.9 that $\hat{f}_0^i = 0$, $i = 1, \ldots, n$. Therefore, we can choose $r_1 > 0$ so that $\hat{f}^i(r_1) \leq \varphi(\varepsilon)\varphi(r_1)$, $i = 1, \ldots, n$, where the constant $\varepsilon > 0$ satisfies

$$\varphi^{-1}(\lambda)\varepsilon\widehat{C} < 1,$$

and $\widehat{C}$ is the positive constant defined in Lemma 2.10. We have by Lemma 2.10 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \varphi^{-1}(\lambda)\varepsilon\widehat{C}\,\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_1}.$$

Now, since $\mathbf{f}_\infty = \infty$, there exists a component f^i of $\mathbf{f}$ such that $f^i_\infty = \infty$. Therefore, there is an $\widehat{H} > 0$ such that

$$f^i(\mathbf{u}) \geq \varphi(\eta)\varphi(\|\mathbf{u}\|)$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n_+$ and $\|\mathbf{u}\| \geq \widehat{H}$, where $\eta > 0$ is chosen so that

$$\varphi^{-1}(\lambda)\Gamma\eta > 1.$$

Let $r_2 = \max\big\{2r_1, 4\widehat{H}\big\}$. If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_2}$, then

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \sum_{i=1}^{n} u_i(t) \geq \frac{1}{4}\|\mathbf{u}\| = \frac{1}{4}r_2 \geq \widehat{H},$$

which implies that

$$f^i(\mathbf{u}(t)) \geq \varphi(\eta)\varphi\left(\sum_{i=1}^{n} u_i(t)\right) = \varphi\left(\eta\sum_{i=1}^{n} u_i(t)\right) \quad \text{for} \quad t \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right].$$

It follows from Lemma 2.8 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \varphi^{-1}(\lambda)\Gamma\eta\,\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

By Lemma 2.3,

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0.$$

It follows from the additivity of the fixed point index that

$$i\big(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \overline{\Omega}_{r_1}, K\big) = -1.$$

Thus, $i\big(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \overline{\Omega}_{r_1}, K\big) \neq 0$, which implies $\mathbf{T}_\lambda$ has a fixed point $\mathbf{u} \in \Omega_{r_2} \setminus \overline{\Omega}_{r_1}$ by the existence property of the fixed point index. The fixed point $\mathbf{u} \in \Omega_{r_2} \setminus \overline{\Omega}_{r_1}$ is the desired positive solution of (2.3)–(2.4).

Part (b). If $\mathbf{f}_0 = \infty$, there exists a component f^i such that $f_0^i = \infty$. Therefore, there is an $r_1 > 0$ such that

$$f^i(\mathbf{u}) \geq \varphi(\eta)\varphi(\|\mathbf{u}\|)$$

for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \leq r_1$, where $\eta > 0$ is chosen so that

$$\varphi^{-1}(\lambda)\Gamma\eta > 1.$$

If $\mathbf{u} = (u_1, \dots, u_n) \in \partial\Omega_{r_1}$, then

$$f^i(\mathbf{u}(t)) \geq \varphi(\eta)\varphi\left(\sum_{i=1}^n u_i(t)\right) = \varphi\left(\eta \sum_{i=1}^n u_i(t)\right), \quad \text{for} \quad t \in [0,1].$$

Lemma 2.8 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \varphi^{-1}(\lambda)\Gamma\eta\,\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_1}.$$

We now determine Ω_{r_2}. Notice that $\mathbf{f}_\infty = 0$ implies that $f_\infty^i = 0$, $i = 1, \dots, n$. It follows from Lemma 2.9 that $\hat{f}_\infty^i = 0$, $i = 1, \dots, n$. Therefore there is an $r_2 > 2r_1$ such that

$$\hat{f}^i(r_2) \leq \varphi(\varepsilon)\varphi(r_2), \quad i = 1, \dots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$\varphi^{-1}(\lambda)\varepsilon\widehat{C} < 1,$$

and $\widehat{C}$ is the positive constant defined in Lemma 2.10. Thus, we have by Lemma 2.10 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \varphi^{-1}(\lambda)\varepsilon\widehat{C}\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

By Lemma 2.3,

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1.$$

It follows from the additivity of the fixed point index that $i\big(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \overline{\Omega}_{r_1}, K\big) = 1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \overline{\Omega}_{r_1}$, which is the desired positive solution of (2.3)–(2.4). □

4 Proof of Theorem 2.2

Part (a). Fix a number $r_1 > 0$. Lemma 2.11 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\|, \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_1}, \quad \lambda > \lambda_0.$$

If $\mathbf{f}_0 = 0$, then $f_0^i = 0$, $i = 1, \dots, n$. It follows from Lemma 2.9 that

$$\hat{f}_0^i = 0, \quad i = 1, \dots, n.$$

Therefore, we can choose $0 < r_2 < r_1$ so that

$$\hat{f}^i(r_2) \leq \varphi(\varepsilon)\varphi(r_2), \quad i = 1, \dots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$\varphi^{-1}(\lambda)\varepsilon\widehat{C} < 1,$$

and $\widehat{C}$ is the positive constant defined in Lemma 2.10. We have by Lemma 2.10 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leq \varphi^{-1}(\lambda)\varepsilon\widehat{C}\,\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

If $\mathbf{f}_\infty = 0$, then $f^i_\infty = 0, i = 1, \dots, n$. It follows from Lemma 2.9 that $\hat{f}^i_\infty = 0, i = 1, \dots, n$. Therefore there is an $r_3 > 2r_1$ such that

$$\hat{f}^i(r_3) \le \varphi(\varepsilon)\varphi(r_3), \quad i = 1, \dots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$\varphi^{-1}(\lambda)\varepsilon\widehat{C} < 1,$$

and $\widehat{C}$ is the positive constant defined in Lemma 2.10. Thus, we have by Lemma 2.10 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \le \varphi^{-1}(\lambda)\varepsilon\widehat{C}\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.3 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0, \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 1.$$

Thus $i\big(\mathbf{T}_\lambda, \Omega_{r_1} \setminus \overline{\Omega}_{r_2}, K\big) = -1$ and $i\big(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K\big) = 1$. Hence, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ according to $\mathbf{f}_0 = 0$ or $\mathbf{f}_\infty = 0$, respectively. Consequently, (2.3)–(2.4) has a positive solution for $\lambda > \lambda_0$.

Part (b). Fix a number $r_1 > 0$. Lemma 2.12 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_1}, \quad 0 < \lambda < \lambda_0.$$

If $\mathbf{f}_0 = \infty$, there exists a component f^i of $\mathbf{f}$ such that $f^i_0 = \infty$. Therefore, there is a positive number $r_2 < r_1$ such that

$$f^i(\mathbf{u}) \ge \varphi(\eta)\varphi(\|\mathbf{u}\|)$$

for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n_+$ and $\|\mathbf{u}\| \le r_2$, where $\eta > 0$ is chosen so that

$$\varphi^{-1}(\lambda)\Gamma\eta > 1.$$

Then

$$f^i(\mathbf{u}(t)) \ge \varphi(\eta)\varphi\left(\sum_{i=1}^n u_i(t)\right) = \varphi\left(\eta\sum_{i=1}^n u_i(t)\right),$$

for $\mathbf{u} = (u_1, \dots, u_n) \in \partial\Omega_{r_2}$, $t \in [0, 1]$. Lemma 2.8 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \ge \varphi^{-1}(\lambda)\Gamma\eta\,\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_2}.$$

If $\mathbf{f}_\infty = \infty$, there exists a component f^i of $\mathbf{f}$ such that $f^i_\infty = \infty$. Therefore, there is an $\widehat{H} > 0$ such that

$$f^i(\mathbf{u}) \ge \varphi(\eta)\varphi(\|\mathbf{u}\|)$$

for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n_+$ and $\|\mathbf{u}\| \ge \widehat{H}$, where $\eta > 0$ is chosen so that

$$\varphi^{-1}(\lambda)\Gamma\eta > 1.$$

Let $r_3 = \max\big\{2r_1, 4\widehat{H}\big\}$. If $\mathbf{u} = (u_1, \dots, u_n) \in \partial\Omega_{r_3}$, then

$$\min_{\frac14 \le t \le \frac34} \sum_{i=1}^n u_i(t) \ge \frac{1}{4}\,\|\mathbf{u}\| = \frac{1}{4}\,r_3 \ge \widehat{H},$$

which implies that

$$f^i(\mathbf{u}(t)) \ge \varphi(\eta)\varphi\left(\sum_{i=1}^n u_i(t)\right) = \varphi\left(\eta\sum_{i=1}^n u_i(t)\right) \quad \text{for} \quad t \in \left[\tfrac14, \tfrac34\right].$$

It follows from Lemma 2.8 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \varphi^{-1}(\lambda)\Gamma\eta\,\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.3 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1, \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 0,$$

and hence, $i\big(\mathbf{T}_\lambda, \Omega_{r_1} \setminus \overline{\Omega}_{r_2}, K\big) = 1$ and $i\big(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K\big) = -1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ according to $f_0 = \infty$ or $f_\infty = \infty$, respectively. Consequently, (2.3)–(2.4) has a positive solution for $0 < \lambda < \lambda_0$.

Part (c). Fix two numbers $0 < r_3 < r_4$. Lemma 2.11 implies that there exists a $\lambda_0 > 0$ such that we have, for $\lambda > \lambda_0$,

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\|, \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_i} \quad (i = 3, 4).$$

Since $\mathbf{f}_0 = 0$ and $\mathbf{f}_\infty = 0$, it follows from the proof of Theorem 2.2 (a) that we can choose $0 < r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_i} \quad (i = 1, 2).$$

It follows from Lemma 2.3 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1, \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1,$$

and

$$i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 0, \quad i(\mathbf{T}_\lambda, \Omega_{r_4}, K) = 0$$

and hence, $i\big(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K\big) = -1$ and $i\big(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \overline{\Omega}_{r_4}, K\big) = 1$. Thus, $\mathbf{T}_\lambda$ has two fixed points $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ such that $\mathbf{u}_1(t) \in \Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ and $\mathbf{u}_2(t) \in \Omega_{r_2} \setminus \overline{\Omega}_{r_4}$, which are the desired distinct positive solutions of (2.3)–(2.4) for $\lambda > \lambda_0$ satisfying

$$r_1 < \|\mathbf{u}_1\| < r_3 < r_4 < \|\mathbf{u}_2\| < r_2.$$

Part (d). Fix two numbers $0 < r_3 < r_4$. Lemma 2.12 implies that there exists a $\lambda_0 > 0$ such that we have, for $0 < \lambda < \lambda_0$,

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\|, \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_i} \quad (i = 3, 4).$$

Since $\mathbf{f}_0 = \infty$ and $\mathbf{f}_\infty = \infty$, it follows from the proof of Theorem 2.2 (b) that we can choose $0 < r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\|, \quad \text{for} \quad \mathbf{u} \in \partial\Omega_{r_i} \quad (i = 1, 2).$$

It follows from Lemma 2.3 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0, \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0,$$

and

$$i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 1, \quad i(\mathbf{T}_\lambda, \Omega_{r_4}, K) = 1$$

and hence, $i\big(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K\big) = 1$ and $i\big(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \overline{\Omega}_{r_4}, K\big) = -1$. Thus, $\mathbf{T}_\lambda$ has two fixed points $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ such that $\mathbf{u}_1(t) \in \Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ and $\mathbf{u}_2(t) \in \Omega_{r_2} \setminus \overline{\Omega}_{r_4}$, which are the desired distinct positive solutions of (2.3)–(2.4) for $\lambda < \lambda_0$ satisfying

$$r_1 < \|\mathbf{u}_1\| < r_3 < r_4 < \|\mathbf{u}_2\| < r_2.$$

Part (e). Since $\mathbf{f}_0 < \infty$ and $\mathbf{f}_\infty < \infty$, then $f_0^i < \infty$ and $f_\infty^i < \infty$, $i = 1, \dots, n$. Therefore, for each $i = 1, \dots, n$, there exist positive numbers ε_1^i, ε_2^i, r_1^i and r_2^i such that $r_1^i < r_2^i$,

$$f^i(\mathbf{u}) \le \varepsilon_1^i \varphi(\|\mathbf{u}\|) \quad \text{for} \quad \mathbf{u} \in \mathbb{R}_+^n, \quad \|\mathbf{u}\| \le r_1^i,$$

and

$$f^i(\mathbf{u}) \le \varepsilon_2^i \varphi(\|\mathbf{u}\|) \quad \text{for} \quad \mathbf{u} \in \mathbb{R}_+^n, \quad \|\mathbf{u}\| \ge r_2^i.$$

Let

$$\varepsilon^i = \max\left\{\varepsilon_1^i, \varepsilon_2^i, \max\left\{\frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)} : \mathbf{u} \in \mathbb{R}_+^n, \quad r_1^i \le \|\mathbf{u}\| \le r_2^i\right\}\right\} > 0 \quad \text{and} \quad \varepsilon = \max_{i=1,\dots,n}\{\varepsilon^i\} > 0.$$

Thus, we have

$$f^i(\mathbf{u}) \le \varepsilon \varphi(\|\mathbf{u}\|) \quad \text{for} \quad \mathbf{u} \in \mathbb{R}_+^n, \quad i = 1, \dots, n.$$

Assume $\mathbf{v}(t)$ is a positive solution of (2.3)–(2.4). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0$, where

$$\lambda_0 = \varphi\left(\frac{1}{\frac{1}{\zeta}\sum_{i=1}^n \varphi^{-1}\left(\frac{\varepsilon}{q(0)}\int_0^1 h_i(\tau)\,d\tau\right)}\right).$$

In fact, for $0 < \lambda < \lambda_0$, since $\mathbf{T}_\lambda \mathbf{v}(t) = \mathbf{v}(t)$ for $t \in [0, 1]$, we find

$$\begin{aligned}
\|\mathbf{v}\| &= \|\mathbf{T}_\lambda \mathbf{v}\| \\
&\le \frac{1}{\zeta}\sum_{i=1}^n \varphi^{-1}\left(\frac{1}{q(0)}\int_0^1 h_i(\tau)\varepsilon\,d\tau\,\lambda\varphi(\|\mathbf{v}\|)\right) \\
&= \frac{1}{\zeta}\sum_{i=1}^n \varphi^{-1}\left(\frac{1}{q(0)}\int_0^1 h_i(\tau)\varepsilon\,d\tau\,\varphi\big(\varphi^{-1}(\lambda)\|\mathbf{v}\|\big)\right) \\
&= \varphi^{-1}(\lambda)\frac{1}{\zeta}\sum_{i=1}^n \varphi^{-1}\left(\frac{\varepsilon}{q(0)}\int_0^1 h_i(\tau)\,d\tau\right)\|\mathbf{v}\| \\
&< \|\mathbf{v}\|,
\end{aligned}$$

which is a contradiction.

Part (f). Since $\mathbf{f}_0 > 0$ and $\mathbf{f}_\infty > 0$, there exist two components f^i and f^j of $\mathbf{f}$ such that $f_0^i > 0$ and $f_\infty^j > 0$. Therefore, there exist positive numbers η_1, η_2, r_1 and r_2 such that $r_1 < r_2$,

$$f^i(\mathbf{u}) \ge \eta_1 \varphi(\|\mathbf{u}\|) \quad \text{for} \quad \mathbf{u} \in \mathbb{R}_+^n, \quad \|\mathbf{u}\| \le r_1,$$

and

$$f^j(\mathbf{u}) \ge \eta_2 \varphi(\|\mathbf{u}\|) \quad \text{for} \quad \mathbf{u} \in \mathbb{R}_+^n, \quad \|\mathbf{u}\| \ge r_2.$$

Let

$$\eta_3 = \min\left\{\eta_1, \eta_2, \min\left\{\frac{f^j(\mathbf{u})}{\varphi(\|\mathbf{u}\|)} : \mathbf{u} \in \mathbb{R}_+^n, \ \frac{1}{4}r_1 \le \|\mathbf{u}\| \le r_2\right\}\right\} > 0.$$

Thus, we have

$$f^i(\mathbf{u}) \ge \eta_3 \varphi(\|\mathbf{u}\|) \quad \text{for} \quad \mathbf{u} \in \mathbb{R}_+^n, \quad \|\mathbf{u}\| \le r_1,$$

and

$$f^j(\mathbf{u}) \ge \eta_3 \varphi(\|\mathbf{u}\|) \quad \text{for} \quad \mathbf{u} \in \mathbb{R}_+^n, \quad \|\mathbf{u}\| \ge \frac{1}{4}r_1.$$

Since $\eta_3\varphi(\|\mathbf{u}\|) = \varphi\big(\varphi^{-1}(\eta_3)\big)\varphi(\|\mathbf{u}\|)$, it follows that

$$f^i(\mathbf{u}) \geq \varphi\big(\varphi^{-1}(\eta_3)\|\mathbf{u}\|\big) \quad \text{for} \quad \mathbf{u} \in \mathbb{R}^n_+, \quad \|\mathbf{u}\| \leq r_1, \tag{4.1}$$

and

$$f^j(\mathbf{u}) \geq \varphi\big(\varphi^{-1}(\eta_3)\|\mathbf{u}\|\big) \quad \text{for} \quad \mathbf{u} \in \mathbb{R}^n_+, \quad \|\mathbf{u}\| \geq \frac{1}{4}r_1. \tag{4.2}$$

Assume $\mathbf{v}(t) = (v_1, \ldots, v_n)$ is a positive solution of (2.3)–(2.4). We will show that this leads to a contradiction for $\lambda > \lambda_0 = \varphi\big(\frac{1}{\Gamma\varphi^{-1}(\eta_3)}\big)$. In fact, if $\|\mathbf{v}\| \leq r_1$, (4.1) implies that

$$f^i(\mathbf{v}(t)) \geq \varphi\left(\varphi^{-1}(\eta_3)\sum_{i=1}^{n} v_i(t)\right), \quad \text{for} \quad t \in [0,1].$$

On the other hand, if $\|\mathbf{v}\| > r_1$, then

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \sum_{i=1}^{n} v_i(t) \geq \frac{1}{4}\|\mathbf{v}\| > \frac{1}{4}r_1,$$

which, together with (4.2), implies that

$$f^j(\mathbf{v}(t)) \geq \varphi\left(\varphi^{-1}(\eta_3)\sum_{i=1}^{n} v_i(t)\right), \quad \text{for} \quad t \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right].$$

Since $\mathbf{T}_\lambda\mathbf{v}(t) = \mathbf{v}(t)$ for $t \in [0,1]$, it follows from Lemma 2.8 that, for $\lambda > \lambda_0$,

$$\|\mathbf{v}\| = \|\mathbf{T}_\lambda\mathbf{v}\| \geq \varphi^{-1}(\lambda)\Gamma\varphi^{-1}(\eta_3)\,\|\mathbf{v}\| > \|\mathbf{v}\|,$$

which is a contradiction. □

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