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Multiplicity of positive radial solutions for an elliptic system on an annulus

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1. Introduction

Let $\Omega = \{x \in \mathbb{R}^n\colon R_1 < |x| < R_2,\ R_1, R_2 > 0\}$ be an annulus with boundary $\partial\Omega$. In this paper we seek the existence of positive radial solutions of the elliptic system

$$\begin{aligned} &\Delta u + \lambda k_1(|x|) f(u,v) = 0 \quad \text{in } \Omega,\\ &\Delta v + \lambda k_2(|x|) g(u,v) = 0 \quad \text{in } \Omega,\\ &\alpha_1 u + \beta_1 \frac{\partial u}{\partial n} = 0, \quad \alpha_2 v + \beta_2 \frac{\partial v}{\partial n} = 0 \quad \text{on } |x| = R_1,\\ &\gamma_1 u + \delta_1 \frac{\partial u}{\partial n} = 0, \quad \gamma_2 v + \delta_2 \frac{\partial v}{\partial n} = 0 \quad \text{on } |x| = R_2, \end{aligned} \tag{1}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \geq 0$ and $\rho_i \equiv \gamma_i\beta_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0$ for $i = 1, 2$. By a positive solution of (1) is meant a solution $(u,v) \in C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ with $u \geq 0, v \geq 0$ in Ω. By virtue of the strong maximum principle and conditions (A_1)–(A_3) below, it follows that $u > 0, v > 0$ in Ω.

Motivated by some recent results in [2,5] for scalar boundary value problems, we prove the following:

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Theorem 1.1. *Assume*

(A_1) : λ *is a positive parameter.*

(A_2) : $k_1, k_2 : [R_1, R_2] \to [0, \infty)$ *are continuous and do not vanish identically on any subinterval of* $[R_1, R_2]$.

(A_3) : $f, g : [0, \infty) \times [0, \infty) \to (0, \infty)$ *are continuous.*

(A_4) : $f(u_1, v_1) \le f(u_2, v_2), g(u_1, v_1) \le g(u_2, v_2)$ *for* $0 \le u_1 \le u_2, 0 \le v_1 \le v_2$.

(A_5) :

$$f_\infty \equiv \lim_{(u,v)\to\infty} \frac{f(u,v)}{u+v} = \infty, \quad g_\infty \equiv \lim_{(u,v)\to\infty} \frac{g(u,v)}{u+v} = \infty.$$

Then there exists a positive number λ^* *such that* (1) *has at least two positive solutions for* $0 < \lambda < \lambda^*$, *at least one positive solution for* $\lambda = \lambda^*$ *and no positive solution for* $\lambda > \lambda^*$.

Remark. With some obvious modifications in the proofs that follow, Theorem 1.1 can be seen to hold for the weakly coupled elliptic system:

$$\begin{aligned} &\Delta u + \lambda k_1(|x|) f(v) = 0 \quad \text{in } R_1 < |x| < R_2, \\ &\Delta v + \lambda k_2(|x|) g(u) = 0 \quad \text{in } R_1 < |x| < R_2, \end{aligned} \tag{2}$$

where

$$f_\infty = \lim_{v\to\infty} \frac{f(v)}{v} = \infty,$$

$$g_\infty = \lim_{u\to\infty} \frac{g(u)}{u} = \infty.$$

Related results for system (1) can be found in [3,6], and for system (2) in [1] and the references therein.

In proving Theorem 1.1 we shall employ upper and lower solution methods together with the following fixed-point index results [4]:

Lemma 1.2. *Let* X *be a Banach space and* K *a cone in* X. *For* $r > 0$, *define* $K_r = \{x \in K: \|x\| < r\}$. *Assume that* $T : \overline{K_r} \to K$ *is a compact map such that* $Tx \ne x$ *for* $x \in \partial K_r$.

(i) If $\|x\| \le \|Tx\|$ for $x \in \partial K_r$, *then*

$$i(T, K_r, K) = 0.$$

(ii) If $\|x\| \ge \|Tx\|$ *for* $x \in \partial K_r$, *then*

$$i(T, K_r, K) = 1.$$

Lemma 1.3. *Let* X *be a Banach space*, K *a cone in* X *and* Ω *a bounded open set in* X. *Let* $0 \in \Omega$ *and* $T : K \cap \bar{\Omega} \to K$ *be condensing. Suppose that* $Tx \ne \mu x$, *for all* $x \in K \cap \partial\Omega$ *and all* $\mu \ge 1$. *Then*

$$i(T, K \cap \Omega, K) = 1.$$

The paper is organized as follows. In Section 2 we prove that (1) has a positive solution for λ sufficiently small, and no positive solution for λ sufficiently large. The proof of Theorem 1.1 is then given in Section 3.

2. Existence and nonexistence

For radial solutions $u = u(r), v = v(r)$, (1) is equivalent to

$$\begin{aligned}
&u''(r) + \frac{n-1}{r}u'(r) + \lambda k_1(r) f(u(r), v(r)) = 0 \quad \text{in } R_1 < r < R_2,\\
&v''(r) + \frac{n-1}{r}v'(r) + \lambda k_2(r) g(u(r), v(r)) = 0 \quad \text{in } R_1 < r < R_2,\\
&\alpha_1 u(R_1) - \beta_1 u'(R_1) = 0, \quad \alpha_2 v(R_1) - \beta_2 v'(R_1) = 0,\\
&\gamma_1 u(R_2) + \delta_1 u'(R_2) = 0, \quad \gamma_2 v(R_2) + \delta_2 v'(R_2) = 0.
\end{aligned} \tag{3}$$

By applying the change of variables $s = -\int_r^{R_2}(1/t^{n-1})\,\mathrm{d}t$, followed by the change of variables $t = (m-s)/m$, where $m = -\int_{R_1}^{R_2}(1/t^{n-1})\,\mathrm{d}t$, (3) can be brought into the form

$$\begin{aligned}
&u''(t) + \lambda h_1(t) f(u(t),\ v(t)) = 0, \quad 0 < t < 1,\\
&v''(t) + \lambda h_2(t) g(u(t),\ v(t)) = 0, \quad 0 < t < 1,\\
&\alpha_1 u(0) - \beta_1 u'(0) = 0, \quad \alpha_2 v(0) - \beta_2 v'(0) = 0,\\
&\gamma_1 u(1) + \delta_1 u'(1) = 0, \quad \gamma_2 v(1) + \delta_2 v'(1) = 0,
\end{aligned} \tag{4}$$

where

$$h_1(t) = m^2 r^{2(n-1)}(m(1-t))k_1(r(m(1-t))),$$

$$h_2(t) = m^2 r^{2(n-1)}(m(1-t))k_2(r(m(1-t))),$$

and β_i, δ_i are relabels of $-\beta_i R_1^{1-n}/m, -\delta_i R_2^{1-n}/m$, respectively. It is easy to check that h_1, h_2 satisfy (A_2) on $[0, 1]$.

On the other hand, (4) is equivalent to the system of integral equations

$$\begin{aligned}
&u(t) = \lambda \int_0^1 k_1(t,s) h_1(s) f(u(s), v(s))\,\mathrm{d}s,\\
&v(t) = \lambda \int_0^1 k_2(t,s) h_2(s) f(u(s), v(s))\,\mathrm{d}s,
\end{aligned} \tag{5}$$

where $k_i(t,s), i = 1, 2$ is the Green's function

$$k_i(t,s) = \frac{1}{\rho_i}\begin{cases} (\gamma_i + \delta_i - \gamma_i t)(\beta_i + \alpha_i s), & s \le t,\\ (\beta_i + \alpha_i t)(\gamma_i + \delta_i - \gamma_i s), & t \le s. \end{cases}$$

It is easy to check that for $i = 1,2$

$$k_i(t,s) > 0 \quad \text{for all } (t,s) \in [\tfrac{1}{4}, \tfrac{3}{4}] \times [\tfrac{1}{4}, \tfrac{3}{4}],$$

$$k_i(t,s) \geq \tfrac{1}{16} k_i(z,s) \quad \text{for all } t \in [\tfrac{1}{4}, \tfrac{3}{4}],\ s \in [0,1],\ z \in [0,1].$$

From now on we concentrate on (5). Indeed, any positive solution of (5) is a positive radial solution of (1).

Let

$$A(u,v)(t) = \lambda \int_0^1 k_1(t,s) h_1(s) f(u(s), v(s))\, \mathrm{d}s,$$

$$B(u,v)(t) = \lambda \int_0^1 k_2(t,s) h_2(s) g(u(s), v(s))\, \mathrm{d}s,$$

$$T(u,v)(t) = (A(u,v)(t), B(u,v)(t)).$$

Then (5) is equivalent to the fixed-point equation

$$T(u,v) = (u,v)$$

in the usual Banach space $X = C[0,1] \times C[0,1]$ with $\|(u,v)\| = \|u\| + \|v\|$, where $\|u\| = \sup_{t\in[0,1]} |u(t)|$.

Let K be the cone defined by

$$K = \left\{ (u,v) \in X\colon u,v \geq 0,\ \min_{1/4\leq t\leq 3/4} (u(t) + v(t)) \geq \frac{1}{16}(\|u\| + \|v\|) \right\}$$

and let C be the cone defined by

$$C = \{(u,v) \in X\colon u,v \geq 0\}.$$

Lemma 2.1. *$T: X \to X$ is completely continuous and $T(C) \subset K$.*

Proof. To prove $T(C) \subset K$, choose $(u,v) \in C$. Then for $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$A(u,v)(t) \geq \frac{\lambda}{16} \int_0^1 k_1(z,s) h_1(s) f(u(s), v(s))\, \mathrm{d}s = \frac{1}{16} A(u,v)(z)$$

for all $z \in [0,1]$, and so

$$\min_{t\in[1/4,3/4]} A(u,v)(t) \geq \tfrac{1}{16} \|A(u,v)\|.$$

Similarly,

$$\min_{t\in[1/4,3/4]} B(u,v)(t) \geq \tfrac{1}{16} \|B(u,v)\|,$$

i.e., $T(u,v) \in K$; hence, $T(C) \subset K$. The complete continuity of T is obvious. □

In the following we set:

$$M = \max\left\{ \max_{(t,s)\in[0,1]\times[0,1]} k_1(t,s), \max_{(t,s)\in[0,1]\times[0,1]} k_2(t,s) \right\} > 0,$$

$$m = \min\left\{ \min_{(t,s)\in[1/4,3/4]\times[1/4,3/4]} k_1(t,s), \min_{(t,s)\in[1/4,3/4]\times[1/4,3/4]} k_2(t,s) \right\} > 0.$$

The main result of this section is the following:

Theorem 2.2. *Suppose that either $f_\infty = \infty$ or $g_\infty = \infty$. Then for λ sufficiently small,* (5) *has at least one positive solution, whereas for λ sufficiently large,* (5) *has no positive solution.*

Proof. If $q > 0$, then it follows from (A_2) and (A_3) that

$$\beta_1(q) \equiv M \max_{(u,v)\in K, \|(u,v)\|=q} \left(\int_0^1 h_1(s) f(u(s), v(s))\,\mathrm{d}s \right) > 0,$$

$$\beta_2(q) \equiv M \max_{(u,v)\in K, \|(u,v)\|=q} \left(\int_0^1 h_2(s) g(u(s), v(s))\,\mathrm{d}s \right) > 0.$$

Let $\beta(q) \equiv \max(\beta_1(q), \beta_2(q))$. For any number $0 < r_1$, let $\sigma_1 = r_1/2\beta(r_1)$ and set

$$K_{r_1} = \{(u, v) \in K\colon \|(u, v)\| < r_1\}.$$

Then for $\lambda < \sigma_1$ and $(u, v) \in \partial K_{r_1}$, we have

$$A(u, v)(t) < \sigma_1 M \int_0^1 h_1(s) f(u(s), v(s))\,\mathrm{d}s \le \sigma_1 \beta(r_1) = r_1/2.$$

In a similar way, $B(u, v)(t) < r_1/2$, which implies

$$\|T(u, v)\| < r_1 = \|(u, v)\|,$$

for $(u, v) \in \partial K_{r_1}$. By Lemma 1.2,

$$i(T, K_{r_1}, K) = 1.$$

Now assume $f_\infty = \infty$. (The proof is similar if $g_\infty = \infty$.) Then there is an $H > 0$ such that $f(u, v) \ge \eta(u + v)$ for $u + v \ge H$, where η is chosen so that

$$\frac{\lambda m \eta}{16} \int_{1/4}^{3/4} h_1(s)\,\mathrm{d}s > 1.$$

Let $r_2 \ge 16H$, and set

$$K_{r_2} = \{(u, v) \in K\colon \|(u, v)\| < r_2\}.$$

If $(u, v) \in \partial K_{r_2}$, then

$$\min_{1/4 \le t \le 3/4} (u(t) + v(t)) \ge \tfrac{1}{16} \|(u, v)\| \ge H.$$

Hence, for any $t \in [1/4, 3/4]$

$$\begin{aligned} A(u,v)(t) &\geq \lambda m \int_{1/4}^{3/4} h_1(s) f(u(s), v(s))\,\mathrm{d}s \\ &\geq \lambda m \eta \int_{1/4}^{3/4} h_1(s)(u(s)+v(s))\,\mathrm{d}s \\ &\geq \frac{\lambda m \eta}{16} \int_{1/4}^{3/4} h_1(s) \|(u,v)\|\,\mathrm{d}s \\ &> \|(u,v)\| \end{aligned}$$

and therefore,

$$\|T(u,v)\| \geq A(u,v)(t) > \|(u,v)\|$$

for $(u,v) \in \partial K_{r_2}$. Another application of Lemma 1.2 gives

$$i(T, K_{r_2}, K) = 0.$$

Since we can adjust r_1, r_2 so that $r_1 < r_2$, it follows from the additivity of the fixed point index that $i(T, K_{r_2} \backslash \overline{K}_{r_1}, K) = -1$. Thus, T has a fixed point in $K_{r_2} \backslash \overline{K}_{r_1}$ which is the desired positive solution of (5).

To prove the nonexistence part, we note that $f_\infty = \infty$ implies the existence of a constant $c > 0$ such that $f(u,v) \geq c(u+v)$ for $u, v \geq 0$. Let $(u,v) \in X$ be a positive solution of (5). By Lemma 2.1, $(u,v) \in K$, and thus

$$\min_{1/4 \leq t \leq 3/4} (u(t)+v(t)) \geq \tfrac{1}{16}(\|u\| + \|v\|).$$

For $t \in [1/4, 3/4]$, we have

$$\begin{aligned} u(t) &\geq \lambda m c \int_{1/4}^{3/4} h_1(s)(u(s)+v(s))\,\mathrm{d}s \\ &\geq \frac{\lambda m c}{16} \int_{1/4}^{3/4} h_1(y)(\|u\| + \|v\|)\,\mathrm{d}s \\ &> \|(u,v)\|, \end{aligned}$$

for λ large enough, which is a contradiction. □

3. Multiplicity

We first need the following a priori estimate.

Lemma 3.1. *If either $f_\infty = \infty$ or $g_\infty = \infty$, then there is a constant $b_I > 0$ such that $\|(u,v)\| \leq b_I$ for all positive solutions (u,v) of* (5), *where λ belongs to a compact subset I of* $(0,\infty)$.

Proof. Suppose there is a sequence $\{(u_n, v_n)\}$ of positive solutions of (5), with corresponding λ_n belonging to a compact subset of $(0,\infty)$, such that $\lim_{n\to\infty}\|(u_n, v_n)\| = \infty$. By Lemma 2.1, $(u_n, v_n) \in K$, and so

$$\min_{1/4\le t\le 3/4}(u_n(t) + v_n(t)) \ge \tfrac{1}{16}(\|u_n\| + \|v_n\|).$$

Without loss of generality assume $f_\infty = \infty$. As before, there is an $H > 0$ such that $f(u, v) \ge \eta(u + v)$ for all $u + v \ge H$, where η is chosen so that

$$\frac{\lambda_n m\eta}{16}\int_{1/4}^{3/4} h_1(s)\,\mathrm{d}s > 1.$$

Choosing n large enough so that $\frac{1}{16}(\|u_n\| + \|v_n\|) \ge H$, and taking $t \in [\frac14, \frac34]$, we have

$$\begin{aligned}\|u_n\| &\ge u_n(t),\\ &\ge \lambda_n m\eta\int_{1/4}^{3/4} h_1(s)(u_n(s) + v_n(s))\,\mathrm{d}s\\ &\ge \frac{\lambda_n m\eta}{16}\int_{1/4}^{3/4} h_1(s)(\|u_n\| + \|v_n\|)\,\mathrm{d}s\\ &> \|u_n\| + \|v_n\|\end{aligned}$$

which is a contradiction. □

In the following, we shall write $(u_1, v_1) \le (u_2, v_2)$ if $u_1(t) \le u_2(t),\ v_1(t) \le v_2(t)$ holds for all $t \in [0, 1]$.

We say that the pair $(\bar u, \bar v) \in C[0, 1] \times C[0, 1]$ is an upper solution of the system (5) if $(\bar u, \bar v) \ge (0, 0)$ and

$$\begin{aligned}\bar u(t) &\ge \lambda\int_0^1 k_1(t,s)h_1(s)f(\bar u(s), \bar v(s))\,\mathrm{d}s,\\ \bar v(t) &\ge \lambda\int_0^1 k_2(t,s)h_2(s)f(\bar u(s), \bar v(s))\,\mathrm{d}s.\end{aligned} \tag{6}$$

A lower solution $(\underline{u}, \underline{v}) \ge (0, 0)$ is defined similarly by reversing the inequalities in (6).

Lemma 3.2. *If there exists an upper solution* $(\bar u, \bar v)$ *of* (5), *then there is a positive solution* (u, v) *of* (5) *with*

$$(0, 0) \le (u, v) \le (\bar u, \bar v).$$

Proof. By taking into account the monotonicity conditions (A_4), and noting that $(0, 0)$ is a lower solution of (5), it follows that the usual monotone iteration scheme applies. □

Now let Γ denote the set of $\lambda > 0$ such that a positive solution of (5) exists, and set $\lambda^* = \sup\Gamma$. By Theorem 2.2, Γ is nonempty and bounded, and thus $0 < \lambda^* < \infty$. We claim that $\lambda^* \in \Gamma$. To see this, let $\lambda_n \to \lambda^*$, where $\lambda_n \in \Gamma$. Since the λ_n are bounded, Lemma 3.1 implies that the corresponding solutions (u_n, v_n) are bounded. By the compactness of the integral operators A and B, it easily follows that $\lambda^* \in \Gamma$. Let (u^*, v^*) be a positive solution of (5) corresponding to λ^*.

Lemma 3.3. *Let* $0 < \lambda < \lambda^*$. *Then there exists* $\varepsilon^* > 0$ *such that* $(u^* + \varepsilon, v^* + \varepsilon)$, $0 \le \varepsilon \le \varepsilon^*$ *is an upper solution of* (5).

Proof. Since $(u^*, v^*) \geq (0, 0)$, there is a constant c such that $f(u^*(t), v^*(t)) \geq c > 0$, $g(u^*(t), v^*(t)) \geq c > 0$ for all $t \in [0, 1]$. By uniform continuity, there is an ε^* such that

$$|f(u^*(t) + \varepsilon, v^*(t) + \varepsilon) - f(u^*(t), v^*(t))| < c(\lambda^* - \lambda)/\lambda,$$
$$|g(u^*(t) + \varepsilon, v^*(t) + \varepsilon) - g(u^*(t), v^*(t))| < c(\lambda^* - \lambda)/\lambda,$$

for all $t \in [0, 1]$, $0 \le \varepsilon \le \varepsilon^*$. Now

$$\begin{aligned} u^*(t) + \varepsilon \geq \lambda^* \int_0^1 & k_1(t,s)h_1(s)f(u^*(s), v^*(s))\,\mathrm{d}s \\ &= \lambda \int_0^1 k_1(t,s)h_1(s)f(u^*(s) + \varepsilon, v^*(s) + \varepsilon)\,\mathrm{d}s \\ &\quad - \lambda \int_0^1 k_1(t,s)h_1(s)[f(u^*(s) + \varepsilon, v^*(s) + \varepsilon) - f(u^*(s), v^*(s))]\,\mathrm{d}s \\ &\quad + (\lambda^* - \lambda) \int_0^1 k_1(t,s)h_1(s)f(u^*(s), v^*(s))\,\mathrm{d}s \\ &\geq \lambda \int_0^1 k_1(t,s)h_1(s)f(u^*(s) + \varepsilon, v^*(s) + \varepsilon)\,\mathrm{d}s, \end{aligned}$$

where the first inequality is strict if $\varepsilon > 0$. A similar computation holds for $v^* + \varepsilon$, thus establishing the result. □

Proof of Theorem 1.1. Let $0 < \lambda < \lambda^*$. Since (u^*, v^*) is an upper solution of (5), Lemma 3.2 implies the existence of a positive solution (u_λ, v_λ) of (5) with $(0, 0) \le (u_\lambda, v_\lambda) \le (u^*, v^*)$. Thus for $0 < \lambda \le \lambda^*$ a positive solution exists, whereas for $\lambda > \lambda^*$ a positive solution does not exist.

We next establish the existence of a second positive solution of (5) for $0 < \lambda < \lambda^*$. Consider

$$\Omega = \{(u, v) \in X:\ -\varepsilon < u(t) < u^*(t) + \varepsilon, -\varepsilon < v(t) < v^*(t) + \varepsilon,\ t \in [0, 1]\},$$

where $\varepsilon > 0$ is given in Lemma 3.3. Then Ω is bounded and open in X, $(0, 0) \in \Omega$, and $T : K \cap \bar{\Omega} \to K$ is condensing since it is compact. Moreover, $(u_\lambda, v_\lambda) \in \Omega$ for $0 < \lambda \leqslant \lambda^*$.

Let $(u, v) \in K \cap \partial\Omega$. Then there is a t_0 such that either $u(t_0) = u^*(t_0) + \varepsilon$ or $v(t_0) = v^*(t_0) + \varepsilon$. Assuming the first case holds, and taking into account (A_4), we have

$$\begin{aligned} A(u, v)(t_0) &= \lambda \int_0^1 k_1(t_0,s)h_1(s)f(u(s),v(s))\,\mathrm{d}s \\ &\leqslant \lambda \int_0^1 k_1(t_0,s)h_1(s)f(u^*(s)+\varepsilon,v^*(s)+\varepsilon)\,\mathrm{d}s \\ &< u^*(t_0)+\varepsilon \\ &= u(t_0)\leqslant \mu u(t_0) \end{aligned}$$

for all $\mu \geq 1$. Similarly, in the second case, $B(u, v)(t_0) < \mu v(t_0)$. Thus $T(u, v) \neq \mu(u, v)$ for all $(u, v) \in K \cap \partial\Omega$ and all $\mu \geq 1$. By Lemma 1.3,

$$i(T, K \cap \Omega, K) = 1.$$

Next, let $r = \max\{b_I + 1, r_2, \|(u^* + \varepsilon, v^* + \varepsilon)\|\}$, where b_I is given in Lemma 3.1, and r_2 is given in the proof of Theorem 2.2. Set

$$K_r = \{(u, v) \in K\colon \|(u, v)\| < r\}.$$

Lemma 3.1 implies that $T(u, v) \neq (u, v)$ for $(u, v) \in \partial K_r$. Furthermore, if $(u, v) \in \partial K_r$, then, as in the proof of Theorem 2.2, we see that $\|T(u, v)\| \geq \|(u, v)\|$. Consequently, Lemma 1.2 implies $i(T, K_r, K) = 0$, and by the additivity of the fixed point index we get

$$i(T, K_r \backslash \overline{K \cap \Omega}, K) = -1.$$

Thus, T has a fixed point in $K_r \backslash \overline{K \cap \Omega}$, which establishes the existence of a second positive solution.

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