Multiplicity of positive radial solutions for an elliptic system on an annulus

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1. Introduction

Let $\Omega = \{ x \in \mathbb{R}^n : R_1 < |x| < R_2, \ R_1, R_2 > 0 \}$ be an annulus with boundary $\partial \Omega$. In this paper we seek the existence of positive radial solutions of the elliptic system

$$
\begin{align*}
\Delta u + \lambda k_1(|x|)f(u,v) &= 0 \quad \text{in} \ \Omega, \\
\Delta v + \lambda k_2(|x|)g(u,v) &= 0 \quad \text{in} \ \Omega, \\
\gamma_1 u + \delta_1 \frac{\partial u}{\partial n} &= 0, \quad \gamma_2 v + \delta_2 \frac{\partial v}{\partial n} = 0 \quad \text{on} \ |x| = R_1, \\
1 u + 1 \frac{\partial u}{\partial n} &= 0 \quad \text{on} \ |x| = R_2,
\end{align*}
$$

(1)

where $\gamma_i, \delta_i \geq 0$ and $\rho_i \equiv \gamma_i \delta_i + z_i \gamma_i + z_i \delta_i > 0$ for $i = 1, 2$. By a positive solution of (1) is meant a solution $(u,v) \in C^2(\Omega) \times C^2(\Omega)$ with $u \geq 0, v \geq 0$ in $\Omega$. By virtue of the strong maximum principle and conditions $(A_1)$--$(A_3)$ below, it follows that $u > 0, v > 0$ in $\Omega$.

Motivated by some recent results in [2,5] for scalar boundary value problems, we prove the following:

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Theorem 1.1. Assume

\( (A_1) \): \( \lambda \) is a positive parameter.

\( (A_2) \): \( k_1, k_2 : [R_1, R_2] \to [0, \infty) \) are continuous and do not vanish identically on any subinterval of \([R_1, R_2]\).

\( (A_3) \): \( f, g : [0, \infty) \times [0, \infty) \to (0, \infty) \) are continuous.

\( (A_4) \): \( f(u_1, v_1) \leq f(u_2, v_2), g(u_1, v_1) \leq g(u_2, v_2) \) for \( 0 \leq u_1 \leq u_2, 0 \leq v_1 \leq v_2 \).

\( (A_5) \):

\[
\lim_{(u, v) \to \infty} \frac{f(u, v)}{u + v} = 1, \quad \lim_{(u, v) \to \infty} \frac{g(u, v)}{u + v} = 1.
\]

Then there exists a positive number \( \lambda^* \) such that (1) has at least two positive solutions for \( 0 < \lambda < \lambda^* \), at least one positive solution for \( \lambda = \lambda^* \) and no positive solution for \( \lambda > \lambda^* \).

Remark. With some obvious modifications in the proofs that follow, Theorem 1.1 can be seen to hold for the weakly coupled elliptic system:

\[
\Delta u + ik_1(|x|)f(v) = 0 \quad \text{in} \quad R_1 < |x| < R_2,
\]
\[
\Delta v + ik_2(|x|)g(u) = 0 \quad \text{in} \quad R_1 < |x| < R_2,
\]

where
\[
f_{\infty} = \lim_{t \to \infty} \frac{f(t)}{t} = \infty, \quad g_{\infty} = \lim_{t \to \infty} \frac{g(t)}{t} = \infty.
\]

Related results for system (1) can be found in [3,6], and for system (2) in [1] and the references therein.

In proving Theorem 1.1 we shall employ upper and lower solution methods together with the following fixed-point index results [4]:

Lemma 1.2. Let \( X \) be a Banach space and \( K \) a cone in \( X \). For \( r > 0 \), define \( K_r = \{ x \in K : ||x|| < r \} \). Assume that \( T : K_r \to K \) is a compact map such that \( Tx \neq x \) for \( x \in \partial K_r \).

(i) If \( ||x|| \leq ||Tx|| \) for \( x \in \partial K_r \), then

\( i(T, K_r, K) = 0 \).

(ii) If \( ||x|| \geq ||Tx|| \) for \( x \in \partial K_r \), then

\( i(T, K_r, K) = 1 \).

Lemma 1.3. Let \( X \) be a Banach space, \( K \) a cone in \( X \) and \( \Omega \) a bounded open set in \( X \). Let \( 0 \in \Omega \) and \( T : K \cap \bar{\Omega} \to K \) be condensing. Suppose that \( Tx \neq \mu x \) for all \( x \in K \cap \partial \Omega \) and all \( \mu \geq 1 \). Then

\( i(T, K \cap \Omega, K) = 1 \).
The paper is organized as follows. In Section 2 we prove that (1) has a positive solution for \( \lambda \) sufficiently small, and no positive solution for \( \lambda \) sufficiently large. The proof of Theorem 1.1 is then given in Section 3.

2. Existence and nonexistence

For radial solutions \( u = u(r), v = v(r) \), (1) is equivalent to

\[
\begin{align*}
&u''(r) + \frac{n-1}{r} u'(r) + \lambda k_1(r) f(u(r), v(r)) = 0 \quad \text{in } R_1 < r < R_2, \\
v''(r) + \frac{n-1}{r} v'(r) + \lambda k_2(r) g(u(r), v(r)) = 0 \quad \text{in } R_1 < r < R_2, \\
&\gamma_1 u(R_2) + \delta_1 u'(R_2) = 0, \quad \gamma_2 v(R_2) + \delta_2 v'(R_2) = 0.
\end{align*}
\]

(3)

By applying the change of variables \( s = -\int_{R_1}^{r} (1/n-1) \, dt \), followed by the change of variables \( t = (m - s)/m \), where \( m = -\int_{R_1}^{R_2} (1/n-1) \, dt \), (3) can be brought into the form

\[
\begin{align*}
&u''(t) + \lambda h_1(t) f(u(t), v(t)) = 0, \quad 0 < t < 1, \\
v''(t) + \lambda h_2(t) g(u(t), v(t)) = 0, \quad 0 < t < 1, \\
&\gamma_1 u(0) + \delta_1 u'(0) = 0, \quad \gamma_2 v(0) + \delta_2 v'(0) = 0, \\
&\gamma_1 u(1) + \delta_1 u'(1) = 0, \quad \gamma_2 v(1) + \delta_2 v'(1) = 0,
\end{align*}
\]

(4)

where

\[
\begin{align*}
h_1(t) &= m^2 r^{2(n-1)}(m(1-t)) k_1(r(m(1-t))), \\
h_2(t) &= m^2 r^{2(n-1)}(m(1-t)) k_2(r(m(1-t))),
\end{align*}
\]

and \( \beta_i, \delta_i \) are relabels of \(-\beta_i R_1^{1-n}/m, -\delta_i R_2^{1-n}/m\), respectively. It is easy to check that \( h_1, h_2 \) satisfy \( (A_2) \) on \([0, 1]\).

On the other hand, (4) is equivalent to the system of integral equations

\[
\begin{align*}
u(t) &= \lambda \int_0^1 k_1(t,s) h_1(s) f(u(s), v(s)) \, ds, \\
v(t) &= \lambda \int_0^1 k_2(t,s) h_2(s) g(u(s), v(s)) \, ds,
\end{align*}
\]

(5)

where \( k_i(t,s), i = 1, 2 \) is the Green’s function

\[
k_i(t,s) = \frac{1}{\rho_i} \begin{cases} 
(\gamma_i + \delta_i - \gamma_i t)(\beta_i + \alpha_i s), & s \leq t, \\
(\beta_i + \alpha_i t)(\gamma_i + \delta_i - \gamma_i s), & t \leq s.
\end{cases}
\]
It is easy to check that for \( i = 1, 2 \)
\[
k_i(t, s) > 0 \quad \text{for all} \quad (t, s) \in \left[ \frac{1}{4}, \frac{3}{4} \right] \times \left[ \frac{1}{4}, \frac{3}{4} \right],
\]
\[
k_i(t, s) \geq \frac{1}{16} k_i(z, s) \quad \text{for all} \quad t \in \left[ \frac{1}{4}, \frac{3}{4} \right], \quad s \in [0, 1], \quad z \in [0, 1].
\]

From now on we concentrate on (5). Indeed, any positive solution of (5) is a positive radial solution of (1).

Let
\[
A(u, v)(t) = \lambda \int_0^1 k_1(t, s) h_1(s) f(u(s), v(s)) \, ds,
\]
\[
B(u, v)(t) = \lambda \int_0^1 k_2(t, s) h_2(s) g(u(s), v(s)) \, ds,
\]
\[
T(u, v)(t) = (A(u, v)(t), B(u, v)(t)).
\]

Then (5) is equivalent to the fixed-point equation
\[
T(u, v) = (u, v)
\]
in the usual Banach space \( X = C[0, 1] \times C[0, 1] \) with \( \|(u, v)\| = \|u\| + \|v\| \), where
\[
\|u\| = \sup_{t \in [0, 1]} |u(t)|.
\]

Let \( K \) be the cone defined by
\[
K = \{ (u, v) \in X : u, v \geq 0, \quad \min_{1/4 \leq t \leq 3/4} (u(t) + v(t)) \geq \frac{1}{16} (\|u\| + \|v\|) \}
\]
and let \( C \) be the cone defined by
\[
C = \{ (u, v) \in X : u, v \geq 0 \}.
\]

**Lemma 2.1.** \( T : X \to X \) is completely continuous and \( T(C) \subseteq K \).

**Proof.** To prove \( T(C) \subseteq K \), choose \( (u, v) \in C \). Then for \( t \in \left[ \frac{1}{4}, \frac{3}{4} \right] \),
\[
A(u, v)(t) = \lambda \int_0^1 k_1(t, s) h_1(s) f(u(s), v(s)) \, ds \geq \frac{1}{16} A(u, v)(z)
\]
for all \( z \in [0, 1] \), and so
\[
\min_{t \in [1/4, 3/4]} A(u, v)(t) \geq \frac{1}{16} \|A(u, v)\|.
\]

Similarly,
\[
\min_{t \in [1/4, 3/4]} B(u, v)(t) \geq \frac{1}{16} \|B(u, v)\|,
\]
i.e., \( T(u, v) \in K \); hence, \( T(C) \subseteq K \). The complete continuity of \( T \) is obvious. \( \square \)

In the following we set:
\[
M = \max \left\{ \max_{(t, s) \in [0, 1] \times [0, 1]} k_1(t, s), \max_{(t, s) \in [0, 1] \times [0, 1]} k_2(t, s) \right\} > 0,
\]
The main result of this section is the following:

**Theorem 2.2.** Suppose that either \( f_\infty = \infty \) or \( g_\infty = \infty \). Then for \( \lambda \) sufficiently small, (5) has at least one positive solution, whereas for \( \lambda \) sufficiently large, (5) has no positive solution.

**Proof.** If \( q > 0 \), then it follows from (A2) and (A3) that

\[
\beta_1(q) \equiv M \max_{(u,v) \in K,\|u,v\| = q} \left( \int_0^1 h_1(s)f(u(s),v(s))\,ds \right) > 0,
\]

\[
\beta_2(q) \equiv M \max_{(u,v) \in K,\|u,v\| = q} \left( \int_0^1 h_2(s)g(u(s),v(s))\,ds \right) > 0.
\]

Let \( \beta(q) \equiv \max(\beta_1(q),\beta_2(q)) \). For any number \( 0 < r_1 \), let \( \sigma_1 = r_1/2\beta(r_1) \) and set

\[ K_{r_1} = \{(u,v) \in K : \|(u,v)\| < r_1\}. \]

Then for \( \lambda < \sigma_1 \) and \((u,v) \in \partial K_{r_1} \), we have

\[
A(u,v)(t) < \sigma_1 M \int_0^1 h_1(s)f(u(s),v(s))\,ds \leq \sigma_1 \beta(r_1) = r_1/2.
\]

In a similar way, \( B(u,v)(t) < r_1/2 \), which implies

\[
\|T(u,v)\| < r_1 = \|(u,v)\|,
\]

for \((u,v) \in \partial K_{r_1} \). By Lemma 1.2,

\[
i(T,K_{r_1},K) = 1.
\]

Now assume \( f_\infty = \infty \). (The proof is similar if \( g_\infty = \infty \).) Then there is an \( H > 0 \) such that \( f(u,v) \geq \eta(u + v) \) for \( u + v \geq H \), where \( \eta \) is chosen so that

\[
\frac{\lambda m n}{16} \int_{1/4}^{3/4} h_1(s)\,ds > 1.
\]

Let \( r_2 \geq 16H \), and set

\[ K_{r_2} = \{(u,v) \in K : \|(u,v)\| < r_2\}. \]

If \((u,v) \in \partial K_{r_2} \), then

\[
\min_{1/4 \leq t \leq 3/4} (u(t) + v(t)) \geq \frac{1}{16}\|u,v\| \geq H.
\]
Hence, for any $t \in [1/4, 3/4]$
\[
A(u, v)(t) \geq \lambda m \int_{1/4}^{3/4} h_1(s) f(u(s), v(s)) \, ds \\
\geq \lambda mn \int_{1/4}^{3/4} h_1(s)(u(s) + v(s)) \, ds \\
\geq \frac{\lambda m}{16} \int_{1/4}^{3/4} h_1(s) \|u\| \|v\| \, ds \\
> \|(u, v)\|
\]
and therefore,
\[
\|T(u, v)\| \geq A(u, v)(t) > \|(u, v)\|
\]
for $(u, v) \in \partial K_{r_1}$. Another application of Lemma 1.2 gives
\[
i(T, K_{r_1}, K) = 0.
\]
Since we can adjust $r_1$, $r_2$ so that $r_1 < r_2$, it follows from the additivity of the fixed point index that $i(T, K_{r_2} \setminus K_{r_1}, K) = -1$. Thus, $T$ has a fixed point in $K_{r_2} \setminus K_{r_1}$ which is the desired positive solution of (5).

To prove the nonexistence part, we note that $f_\infty = \infty$ implies the existence of a constant $c > 0$ such that $f(u, v) \geq c(u + v)$ for $u, v \geq 0$. Let $(u, v) \in X$ be a positive solution of (5). By Lemma 2.1, $(u, v) \in K$, and thus
\[
\min_{1/4 \leq t \leq 3/4} (u(t) + v(t)) \geq \frac{1}{16}(\|u\| + \|v\|).
\]
For $t \in [1/4, 3/4]$, we have
\[
u(t) \geq \lambda mc \int_{1/4}^{3/4} h_1(s)(u(s) + v(s)) \, ds \\
\geq \frac{\lambda mc}{16} \int_{1/4}^{3/4} h_1(y)(\|u\| + \|v\|) \, ds \\
> \|(u, v)\|,
\]
for $\lambda$ large enough, which is a contradiction. \(\square\)

3. Multiplicity

We first need the following a priori estimate.

**Lemma 3.1.** If either $f_\infty = \infty$ or $g_\infty = \infty$, then there is a constant $b_1 > 0$ such that $\|(u, v)\| \leq b_1$ for all positive solutions $(u, v)$ of (5), where $\lambda$ belongs to a compact subset $I$ of $(0, \infty)$. 
Proof. Suppose there is a sequence \( \{ (u_n, v_n) \} \) of positive solutions of (5), with corresponding \( \lambda_n \) belonging to a compact subset of \((0, \infty)\), such that \( \lim_{n \to \infty} \|(u_n, v_n)\| = \infty \).

By Lemma 2.1, \((u_n, v_n) \in K\), and so
\[
\min_{1/4 \leq t \leq 3/4} (u_n(t) + v_n(t)) \geq \frac{1}{16}(\|u_n\| + \|v_n\|).
\]

Without loss of generality assume \( f_{\infty} = \infty \). As before, there is an \( H > 0 \) such that \( f(u, v) \geq \eta(u + v) \) for all \( u + v \geq H \), where \( \eta \) is chosen so that
\[
\frac{\lambda_n \eta}{16} \int_{1/4}^{3/4} h_1(s) \, ds > 1.
\]

Choosing \( n \) large enough so that \( \frac{\lambda_n \eta}{16} \int_{1/4}^{3/4} h_1(s) \, ds \geq H \), and taking \( t \in [1/4, 3/4] \), we have
\[
\|u_n\| \geq u_n(t),
\]
\[
\geq \lambda_n \eta \int_{1/4}^{3/4} h_1(s)(u_n(s) + v_n(s)) \, ds
\]
\[
\geq \frac{\lambda_n \eta}{16} \int_{1/4}^{3/4} h_1(s)(\|u_n\| + \|v_n\|) \, ds
\]
\[
> \|u_n\| + \|v_n\|
\]
which is a contradiction. \( \square \)

In the following, we shall write \((u_1, v_1) \leq (u_2, v_2)\) if \( u_1(t) \leq u_2(t)\), \( v_1(t) \leq v_2(t)\) holds for all \( t \in [0, 1]\).

We say that the pair \((\bar{u}, \bar{v})\) \( \in C[0, 1] \times C[0, 1]\) is an upper solution of the system (5) if \((\bar{u}, \bar{v}) \geq (0, 0)\) and
\[
\bar{u}(t) \geq \lambda \int_0^1 k_1(t, s) h_1(s) f(\bar{u}(s), \bar{v}(s)) \, ds,
\]
\[
\bar{v}(t) \geq \lambda \int_0^1 k_2(t, s) h_2(s) f(\bar{u}(s), \bar{v}(s)) \, ds.
\]

A lower solution \((\underline{u}, \underline{v}) \geq (0, 0)\) is defined similarly by reversing the inequalities in (6).

Lemma 3.2. If there exists an upper solution \((\bar{u}, \bar{v})\) of (5), then there is a positive solution \((u, v)\) of (5) with
\[
(0, 0) \leq (u, v) \leq (\bar{u}, \bar{v}).
\]

Proof. By taking into account the monotonicity conditions (A_4), and noting that \((0, 0)\) is a lower solution of (5), it follows that the usual monotone iteration scheme applies. \( \square \)

Now let \( \Gamma \) denote the set of \( \lambda > 0 \) such that a positive solution of (5) exists, and set \( \lambda^* = \sup \Gamma \). By Theorem 2.2, \( \Gamma \) is nonempty and bounded, and thus \( 0 < \lambda^* < \infty \). We
claim that $\lambda^* \in \Gamma$. To see this, let $\lambda_n \to \lambda^*$, where $\lambda_n \in \Gamma$. Since the $\lambda_n$ are bounded, Lemma 3.1 implies that the corresponding solutions $(u_n, v_n)$ are bounded. By the compactness of the integral operators $A$ and $B$, it easily follows that $\lambda^* \in \Gamma$. Let $(u^*, v^*)$ be a positive solution of (5) corresponding to $\lambda^*$.

Lemma 3.3. Let $0 < \lambda < \lambda^*$. Then there exists $\varepsilon^* > 0$ such that $(u^* + \varepsilon, v^* + \varepsilon)$, $0 \leq \varepsilon \leq \varepsilon^*$, is an upper solution of (5).

Proof. Since $(u^*, v^*) \geq (0, 0)$, there is a constant $c$ such that $f(u^*(t), v^*(t)) \geq c > 0$, $g(u^*(t), v^*(t)) \geq c > 0$ for all $t \in [0, 1]$. By uniform continuity, there is an $\varepsilon^*$ such that

\[
|f(u^*(t) + \varepsilon, v^*(t) + \varepsilon) - f(u^*(t), v^*(t))| < c(\lambda^* - \lambda)/\lambda,
\]

\[
|g(u^*(t) + \varepsilon, v^*(t) + \varepsilon) - g(u^*(t), v^*(t))| < c(\lambda^* - \lambda)/\lambda,
\]

for all $t \in [0, 1]$, $0 \leq \varepsilon \leq \varepsilon^*$. Now

\[
u^*(t) + \varepsilon \geq \lambda^* \int_0^1 k_1(t,s)h_1(s)f(u^*(s), v^*(s)) \, ds
\]

\[= \lambda \int_0^1 k_1(t,s)h_1(s)f(u^*(s) + \varepsilon, v^*(s) + \varepsilon) \, ds
\]

\[-\lambda \int_0^1 k_1(t,s)h_1(s)[f(u^*(s) + \varepsilon, v^*(s) + \varepsilon) - f(u^*(s), v^*(s))] \, ds
\]

\[+ (\lambda^* - \lambda) \int_0^1 k_1(t,s)h_1(s)f(u^*(s), v^*(s)) \, ds
\]

\[\geq \lambda \int_0^1 k_1(t,s)h_1(s)f(u^*(s) + \varepsilon, v^*(s) + \varepsilon) \, ds,
\]

where the first inequality is strict if $\varepsilon > 0$. A similar computation holds for $v^* + \varepsilon$, thus establishing the result. \qed

Proof of Theorem 1.1. Let $0 < \lambda < \lambda^*$. Since $(u^*, v^*)$ is an upper solution of (5), Lemma 3.2 implies the existence of a positive solution $(u_\varepsilon, v_\varepsilon)$ of (5) with $(0, 0) \leq (u_\varepsilon, v_\varepsilon) \leq (u^*, v^*)$. Thus for $0 < \lambda \leq \lambda^*$ a positive solution exists, whereas for $\lambda > \lambda^*$ a positive solution does not exist.

We next establish the existence of a second positive solution of (5) for $0 < \lambda < \lambda^*$. Consider

\[\Omega = \{(u, v) \in X; -\varepsilon < u(t) < u^*(t) + \varepsilon, -\varepsilon < v(t) < v^*(t) + \varepsilon, t \in [0, 1]\},\]

where $\varepsilon > 0$ is given in Lemma 3.3. Then $\Omega$ is bounded and open in $X$, $(0, 0) \in \Omega$, and $T : K \cap \Omega \to K$ is condensing since it is compact. Moreover, $(u_\varepsilon, v_\varepsilon) \in \Omega$ for $0 < \lambda \leq \lambda^*$.

Let $(u, v) \in K \cap \Omega$. Then there is a $t_0$ such that either $u(t_0) = u^*(t_0) + \varepsilon$ or $v(t_0) = v^*(t_0) + \varepsilon$. Assuming the first case holds, and taking into account (A4),
we have
\[ A(u, v)(t_0) = \lambda \int_0^1 k_1(t_0, s)h_1(s)f(u(s), v(s)) \, ds \]
\[ \leq \lambda \int_0^1 k_1(t_0, s)h_1(s)f(u^*(s) + \varepsilon, v^*(s) + \varepsilon) \, ds \]
\[ < u^*(t_0) + \varepsilon \]
\[ = u(t_0) \leq \mu u(t_0) \]
for all \( \mu \geq 1 \). Similarly, in the second case, \( B(u, v)(t_0) < \mu v(t_0) \). Thus \( T(u, v) \neq \mu (u, v) \)
for all \((u, v) \in K \cap \partial \Omega\) and all \( \mu \geq 1 \). By Lemma 1.3,
\[ i(T, K \cap \Omega, K) = 1. \]

Next, let \( r = \max\{b_I + 1, r_2, \|(u^* + \varepsilon, v^* + \varepsilon)\|\} \), where \( b_I \) is given in Lemma 3.1, and \( r_2 \) is given in the proof of Theorem 2.2. Set
\[ K_r = \{(u, v) \in K : \|(u, v)\| < r\}. \]

Lemma 3.1 implies that \( T(u, v) \neq (u, v) \) for \((u, v) \in \partial K_r\). Furthermore, if \((u, v) \in \partial K_r\),
then, as in the proof of Theorem 2.2, we see that \( \|T(u, v)\| \geq \|(u, v)\| \). Consequently,
Lemma 1.2 implies \( i(T, K_r, K) = 0 \), and by the additivity of the fixed point index we get
\[ i(T, K_r \setminus K \cap \Omega, K) = -1. \]
Thus, \( T \) has a fixed point in \( K_r \setminus K \cap \Omega \), which establishes the existence of a second positive solution.

References