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ON THE EXISTENCE OF POSITIVE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the existence of positive solutions of the equation $u'' + a(t)f(u) = 0$ with linear boundary conditions. We show the existence of at least one positive solution if f is either superlinear or sublinear by a simple application of a Fixed Point Theorem in cones.

1. INTRODUCTION

In this paper we shall consider the second-order boundary value problem (BVP)

$$u'' + a(t)f(u) = 0, \qquad 0 < t < 1; \tag{1.1}$$

$$\begin{aligned} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{aligned} \tag{1.2}$$

The following conditions will be assumed throughout:

(A.1) $f \in C([0, \infty), [0, \infty))$,
(A.2) $a \in C([0, 1], [0, \infty))$ and $a(t) \not\equiv 0$ on any subinterval of $[0, 1]$.
(A.3) $\alpha, \beta, \gamma, \delta \geq 0$ and $\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

The BVP (1.1), (1.2) arises in many different areas of applied mathematics and physics; see [1–3, 6, 12, 13] for some references along this line. Additional existence results may be found in [4, 7, 8, 10, 11]. Our purpose here is to give an existence result for positive solutions to the BVP (1.1), (1.2), assuming that f is either superlinear or sublinear. We do not require any monotonicity assumptions on f. To be precise, we introduce the notation

$$f_0 := \lim_{u\to 0} \frac{f(u)}{u}, \qquad f_\infty := \lim_{u\to\infty} \frac{f(u)}{u}.$$

Thus, $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. By a positive solution of (1.1), (1.2) we understand a solution $u(t)$ which is positive on $0 < t < 1$ and satisfies

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the differential equation (1.1) for $0 < t < 1$ and the boundary conditions (1.2). By a change of variable, the existence of a positive solution of (1.1), (1.2) may be shown to be equivalent to the existence of a positive radial solution of the semilinear elliptic equation $\Delta u + g(|x|)f(u) = 0$ in the annulus $R_1 < |x| < R_2$ subject to certain boundary conditions for $|x| = R_1$ and $|x| = R_2$. (Here $|x|$ denotes the Euclidean norm.) We refer to [11] for some additional details.

2. Existence results

The main result of this paper is

Theorem 1. *Assume* (A.1)–(A.3) *hold. Then the BVP* (1.1), (1.2) *has at least one positive solution in the case*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ (*superlinear*), *or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$ (*sublinear*).

It will be seen in the proof that Theorem 1 is also valid for the more general equation

$$u'' + f(t, u) = 0 \tag{1.1*}$$

with the same boundary conditions (1.2), provided we assume a certain uniformity with respect to the t variable. We state this more general result as

Corollary 1. *Assume f is continuous, $f(t, u) \geq 0$ for $t \in [0, 1]$, and $u \geq 0$ with $f(t, u) \not\equiv 0$ on any subinterval of* $[0, 1]$ *for $u > 0$; and let condition* (A.3) *hold. Then the BVP* $(1.1)^*$, (1.2) *has at least one positive solution in the case*

(i)* $\lim_{u\to 0+} \max_{t\in[0,1]} \frac{f(t,u)}{u} = 0$ *and* $\lim_{u\to\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} = \infty$, *or*
(ii)* $\lim_{u\to 0+} \min_{t\in[0,1]} \frac{f(t,u)}{u} = \infty$ *and* $\lim_{u\to\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} = 0$.

The proof of Theorem 1 will be based on an application of the following Fixed Point Theorem due to Krasnoselskii [9]. The proof of Corollary 1 follows from the proof of Theorem 1 with obvious slight modifications which we shall omit.

Theorem 2 [4, 9]. *Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A\colon K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

We will apply the first and second parts of the above Fixed Point Theorem to the superlinear and sublinear cases, respectively.

Proof of Theorem 1. *Superlinear case.* Suppose then that $f_0 = 0$ and $f_\infty = \infty$. We wish to show the existence of a positive solution of (1.1), (1.2). Now (1.1), (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = \int_0^1 k(t, s)a(s)f(u(s))\,ds := Au(t), \qquad u \in C[0, 1].$$

Here $k(t, s)$ denotes the Green's function for the BVP

$$u'' = 0; \tag{2.1}$$

$$\begin{aligned} \alpha u(0) - \beta u'(0) &= 0, \\ \gamma u(1) + \delta u'(1) &= 0 \end{aligned} \tag{2.2}$$

and is explicitly given by

$$k(t, s) = \begin{cases} \frac{1}{\rho}(\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \le s \le t \le 1, \\ \frac{1}{\rho}(\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \le t \le s \le 1. \end{cases}$$

We let K be the cone in $C[0, 1]$ given by

$$K = \left\{ u \in C[0, 1]: u(t) \ge 0, \min_{1/4 \le t \le 3/4} u(t) \ge M\|u\| \right\} \tag{2.3}$$

where $\|u\| = \sup_{[0,1]} |u(t)|$ and

$$M = \min\left\{ \frac{\gamma + 4\delta}{4(\gamma + \delta)}, \frac{\alpha + 4\beta}{4(\alpha + \beta)} \right\}. \tag{2.4}$$

We define

$$\varphi(t) := (\gamma + \delta - \gamma t), \quad \psi(t) := \beta + \alpha t, \quad 0 \le t \le 1, \tag{2.5}$$

so that

$$k(t, s) = \begin{cases} \frac{1}{\rho}\varphi(t)\psi(s), & 0 \le s \le t \le 1, \\ \frac{1}{\rho}\varphi(s)\psi(t), & 0 \le t \le s \le 1. \end{cases} \tag{2.6}$$

Observe that $k(t, s) \le \frac{1}{\rho}\varphi(s)\psi(s) = k(s, s)$, $0 \le t, s \le 1$, so that, if $u \in K$, then

$$Au(t) = \int_0^1 k(t, s)a(s)f(u(s))\,ds \le \int_0^1 k(s, s)a(s)f(u(s))\,ds \tag{2.7}$$

and hence

$$\|Au\| \le \int_0^1 k(s, s)a(s)f(u(s))\,ds. \tag{2.8}$$

Furthermore, for $\frac{1}{4} \le t \le \frac{3}{4}$

$$\frac{k(t, s)}{k(s, s)} = \begin{cases} \dfrac{\varphi(t)}{\varphi(s)}, & s \le t, \\ \dfrac{\psi(t)}{\psi(s)}, & t \le s; \end{cases} \ge \begin{cases} \dfrac{\gamma + 4\delta}{4(\gamma + \delta)}, & s \le t, \\ \dfrac{\alpha + 4\beta}{4(\alpha + \beta)}, & t \le s, \end{cases}$$

so

$$\frac{k(t, s)}{k(s, s)} \ge M, \qquad \frac{1}{4} \le t \le \frac{3}{4}.$$

Hence, if $u \in K$,

$$\begin{aligned} \min_{1/4 \le t \le 3/4} Au(t) &= \min_{1/4 \le t \le 3/4} \int_0^1 k(t, s)a(s)f(u(s))\,ds \\ &\ge M \int_0^1 k(s, s)a(s)f(u(s))\,ds \ge M\|Au\|. \end{aligned}$$

Therefore, $AK \subset K$. Moreover, it is easy to see that $A: K \to K$ is completely continuous.

Now, since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \le \eta u$, for $0 < u \le H_1$, where $\eta > 0$ satisfies

$$\eta \int_0^1 k(s, s)a(s)\, ds \le 1. \tag{2.9}$$

Thus, if $u \in K$ and $\|u\| = H_1$, then from (2.7) and (2.9)

$$Au(t) \le \int_0^1 k(s, s)a(s)f(u(s)) \le \|u\|, \qquad 0 \le t \le 1. \tag{2.10}$$

Now if we let

$$\Omega_1 := \{u \in E: \|u\| < H_1\} \tag{2.11}$$

then (2.10) shows that

$$\|Au\| \le \|u\|, \qquad u \in K \cap \partial\Omega_1. \tag{2.12}$$

Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(u) \ge \mu u$, $u \ge \widehat{H}_2$, where $\mu > 0$ is chosen so that

$$M\mu \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)a(s)\, ds \ge 1. \tag{2.13}$$

Let $H_2 := \max\{2H_1, \widehat{H}_2/M\}$ and $\Omega_2 := \{u \in E: \|u\| < H_2\}$. Then $u \in K$ and $\|u\| = H_2$ implies

$$\min_{1/4 \le t \le 3/4} u(t) \ge M\|u\| \ge \widehat{H}_2$$

and so

$$\begin{aligned} Au(\tfrac{1}{2}) &= \int_0^1 k(\tfrac{1}{2}, s)a(s)f(u(s))\, ds \ge \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)a(s)f(u(s))\, ds \\ &\ge \mu \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)a(s)u(s)\, ds \ge \mu M\|u\| \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)a(s)\, ds \ge \|u\|. \end{aligned}$$

Hence, $\|Au\| \ge \|u\|$ for $u \in K \cap \partial\Omega_2$.

Therefore, by the first part of the Fixed Point Theorem, it follows that A has a fixed point in $K \cap \overline{\Omega}_2 \backslash \Omega_1$ such that $H_1 \le \|u\| \le H_2$. Further, since $k(t, s) > 0$, it follows that $u(t) > 0$ for $0 < t < 1$. This completes the superlinear part of the theorem.

Sublinear case. Suppose next that $f_0 = \infty$ and $f_\infty = 0$. We first choose $H_1 > 0$ such that $f(u) \ge \hat{\eta} u$ for $0 < u \le H_1$, where

$$\hat{\eta} M \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)a(s)\, ds \ge 1 \tag{2.14}$$

(M is as in the first part of the proof). Then for $u \in K$ and $\|u\| = H_1$ we have

$$\begin{aligned} Au(\tfrac{1}{2}) &= \int_0^1 k(\tfrac{1}{2}, s)a(s)f(u(s))\, ds \\ &\ge \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)a(s)f(u(s))\, ds \ge \hat{\eta} \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)a(s)u(s)\, ds \\ &\ge \hat{\eta} M\|u\| \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)a(s)\, ds \ge \|u\| \quad \text{[by (2.14)]}. \end{aligned}$$

Thus, we may let $\Omega_1 := \{u \in E: \|u\| < H_1\}$ so that

$$\|Au\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_1.$$

Now, since $f_\infty = 0$, there exists $\widehat{H}_2 > 0$ so that $f(u) \leq \lambda u$ for $u \geq \widehat{H}_2$ where $\lambda > 0$ satisfies

$$\lambda \int_0^1 k(s, s)a(s)\, ds \leq 1. \tag{2.15}$$

We consider two cases:

Case (i). Suppose f is bounded, say $f(u) \leq N$ for all $u \in (0, \infty)$. In this case choose $H_2 := \max\{2H_1, N\int_0^1 k(s, s)a(s)\, ds\}$ so that for $u \in K$ with $\|u\| = H_2$ we have

$$Au(t) = \int_0^1 k(t, s)a(s)f(u(s))\, ds \leq N \int_0^1 k(s, s)a(s)\, ds \leq H_2$$

and therefore $\|Au\| \leq \|u\|$.

Case (ii). If f is unbounded, then let $H_2 > \max\{2H_1, \widehat{H}_2\}$ and such that

$$f(u) \leq f(H_2) \quad \text{for } 0 < u \leq H_2.$$

(We are able to do this since f is unbounded.)

Then for $u \in K$ and $\|u\| = H_2$ we have

$$\begin{aligned} Au(t) &= \int_0^1 k(t, s)a(s)f(u(s))\, ds \leq \int_0^1 k(s, s)a(s)f(u(s))\, ds \\ &\leq \int_0^1 k(s, s)a(s)f(H_2)\, ds \leq \lambda H_2 \int_0^1 k(s, s)a(s)\, ds \leq H_2 = \|u\|. \end{aligned}$$

Therefore, in either case we may put

$$\Omega_2 := \{u \in E: \|u\| < H_2\},$$

and for $u \in K \cap \partial\Omega_2$ we have $\|Au\| \leq \|u\|$. By the second part of the Fixed Point Theorem it follows that BVP (1.1), (1.2) has a positive solution, and this completes the proof of the theorem.

References

1. C. Bandle, C. V. Coffman, and M. Marcus, *Nonlinear elliptic problems in annular domains*, J. Differential Equations **69** (1987), 322–345.
2. C. Bandle and M. K. Kwong, *Semilinear elliptic problems in annular domains*, J. Appl. Math. Phys. **40** (1989), 245–257.
3. C. V. Coffman and M. Marcus, *Existence and uniqueness results for semilinear Dirichlet problems in Annuli*, Arch. Rational Mech. Anal. **108** (1989), 293–307.
4. K. Deimling, *Nonlinear functional analysis*, Springer, New York, 1985.
5. L. H. Erbe, *Boundary value problems for ordinary differential equations*, Rocky Mountain J. Math. **1** (1970), 709–729.
6. X. Garaizar, *Existence of positive radial solutions for semilinear elliptic problems in the annulus*, J. Differential Equations **70** (1987), 69–72.
7. G. B. Gustafson and K. Schmitt, *Nonzero solutions of boundary value problems for second order ordinary and delay-differential equations*, J. Differential Equations **12** (1972), 127–147.

8. Georges Iffland, *Positive solutions of a problem of Emden-Fowler type with a free boundary*, SIAM J. Math. Anal. **18** (1987), 283–292.

9. M. A. Krasnoselskii, *Positive solutions of operator equations*, Noordhoff, Groningen, 1964.

10. R. A. Moore and Z. Nehari, *Nonoscillation theorems for a class of nonlinear differential equations*, Trans. Amer. Math. Soc. **93** (1959), 30–52.

11. J. Santanilla, *Nonnegative solutions to boundary value problems for nonlinear first and second order ordinary differential equations*, J. Math. Anal. Appl. **126** (1987), 397–408.

12. Haiyan Wang, *On the existence of positive solutions for semilinear elliptic equations in the annulus*, J. Differential Equations (to appear).

13. J. S. W. Wong, *On the generalized Emden-Fowler equation*, SIAM Rev. **17** (1975), 339–360.

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