Advances in Differential Equations Volume 8, Number 1, January 2003, Pages 111–128

# ON THE STRUCTURE OF POSITIVE RADIAL SOLUTIONS FOR QUASILINEAR EQUATIONS IN ANNULAR DOMAINS

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**Abstract.** We study the existence, multiplicity and nonexistence of positive radial solutions to boundary value problems for the quasilinear equation $\operatorname{div}(A(|\nabla u|)\nabla u) + \lambda h(|x|)f(u) = 0$ in annular domains under general assumptions on the function $A(u)$. Various possible behaviors of the quotient $\frac{f(u)}{A(u)u}$ at zero and infinity are considered. We shall use fixed point theorems for operators on a Banach space.

## 1. Introduction

In this paper we consider the existence, multiplicity and nonexistence of positive radial solutions for the quasilinear equation

$$\operatorname{div}(A(|\nabla u|)\nabla u) + \lambda k(|x|)f(u) = 0, \text{ in } R_1 < |x| < R_2, \ x \in \mathbb{R}^n, \ n \geq 2 \tag{1.1}$$

with one of the following three sets of boundary conditions,

$$u = 0 \text{ on } |x| = R_1 \text{ and } |x| = R_2, \tag{1.2a}$$

$$\partial u/\partial r = 0 \text{ on } |x| = R_1 \text{ and } u = 0 \text{ on } |x| = R_2, \tag{1.2b}$$

$$u = 0 \text{ on } |x| = R_1 \text{ and } \partial u/\partial r = 0 \text{ on } |x| = R_2, \tag{1.2c}$$

where $r = |x|$ and $\partial/\partial r$ denotes differentiation in the radial direction, and $0 < R_1 < R_2 < \infty$.

Ni and Serrin [14, 15] established some existence and non-existence theorems for radial ground state solutions of quasilinear equations of the form (1.1) in $\mathbb{R}^n$. The function $A$ originates from a variety of practical applications, for instance, the degenerate m-Laplace operator, namely $A(|p|) =$

Accepted for publication: September 2002.
AMS Subject Classifications: 35J65, 35P30.

$|p|^{m-2}$, $m > 1$. When $A \equiv 1$ we recall that (1.1) reduces to the classical semilinear elliptic equation

$$\Delta u + \lambda k(|x|)f(u) = 0, \text{ in } R_1 < |x| < R_2, \; x \in \mathbb{R}^n, \; n \geq 2. \tag{1.3}$$

(1.3)-(1.2) has been studied by Bandle, Coffman and Marcus [1], Bandle and Peletier [2], Coffman and Kwong [3], Lin [12], Ni and Nussbaum [13] and many others [5, 7, 9, 11, 16, 18, 19]. In particular, when $f$ is non-negative, Bandle, Coffman and Marcus [1], Coffman and Marcus [5] and Lin [12] have established the existence of positive radial solutions of (1.3)-(1.2) under the assumption that $f$ is superlinear, i.e., $\lim_{u\to 0} \frac{f(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, which improves various earlier existence results by many authors under some additional assumptions on $f$.

On the other hand, the present author [16] has established the existence of positive radial solutions of (1.3)-(1.2) under the assumption that $f$ is sublinear, i.e., $\lim_{u\to 0} \frac{f(u)}{u} = \infty$ and $\lim_{u\to\infty} \frac{f(u)}{u} = 0$. In fact, we shall show that the existence, multiplicity and nonexistence of positive radial solutions of (1.3)-(1.2) are characterized by the asymptotic behaviors of the quotient $\frac{f(u)}{u}$ at zero and infinity.

The main purpose of this paper is to examine the quasilinear problem (1.1)-(1.2) under general assumptions on the function $A$. In this paper we introduce a new and general assumption (see H1) on the function $A$, which covers the two important cases $A \equiv 1$ and $A(|p|) = |p|^{m-2}, m > 1$, i.e., the degenerate $m$-Laplace operator.

Under such assumption, we are able to show that the structure of the positive radial solution set of (1.1)-(1.2) is exactly the same as that of the two special cases $A \equiv 1$ and $A(|p|) = |p|^{m-2}, m > 1$, in the sense that Theorems 1.1 and 1.2 hold for the general problem (1.1)-(1.2) and the two special cases. We consider not only existence, but also multiplicity and nonexistence. Our results (Theorems 1.1 and 1.2) generalize and extend the work of many authors [1, 2, 3, 5, 7, 11, 12, 13, 16, 18, 19]. Furthermore, most of our results are new even for the case $A(|p|) = |p|^{m-2}, m > 1$.

Finally, our arguments in this paper are closely related to those of [16], in which the present author uses the fixed point index for compact maps, which is based on Leray-Schauder degree theory, to study (1.1)-(1.2) for $A \equiv 1$.

Let $\varphi(t) := A(|t|)t$. We make the assumptions:

(H1) $\varphi$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and there exist two increasing homeomorphisms $\psi_1$ and $\psi_2$ from $(0,\infty)$ onto

$(0,\infty)$ such that

$$\psi_1(\sigma)\varphi(t) \le \varphi(\sigma t) \le \psi_2(\sigma)\varphi(t), \text{ for all } \sigma \text{ and } t > 0.$$

(H2) $k : [R_1, R_2] \to [0,\infty)$ is continuous and $k(t) \not\equiv 0$ on any subinterval of $[R_1, R_2]$.
(H3) $f : [0,\infty) \to [0,\infty)$ is continuous.
(H4) $f(u) > 0$ for $u > 0$.

In order to state our results we introduce the notation

$$f_0 := \lim_{u\to 0} \frac{f(u)}{\varphi(u)} \quad \text{and} \quad f_\infty := \lim_{u\to\infty} \frac{f(u)}{\varphi(u)}.$$

Our main results are:

**Theorem 1.1.** *Assume* (H1)-(H3) *hold.*

(a) *If $f_0 = 0$ and $f_\infty = \infty$, then for all $\lambda > 0$* (1.1)-(1.2) *has a positive radial solution.*
(b) *If $f_0 = \infty$ and $f_\infty = 0$, then for all $\lambda > 0$* (1.1)-(1.2) *has a positive radial solution.*

**Theorem 1.2.** *Assume* (H1)-(H4) *hold.*

(a) *If $f_0 = 0$ or $f_\infty = 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$* (1.1)-(1.2) *has a positive radial solution.*
(b) *If $f_0 = \infty$ or $f_\infty = \infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$* (1.1)-(1.2) *has a positive radial solution.*
(c) *If $f_0 = f_\infty = 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$* (1.1)-(1.2) *has two positive radial solutions.*
(d) *If $f_0 = f_\infty = \infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$* (1.1)-(1.2) *has two positive radial solutions.*
(e) *If $f_0 < \infty$ and $f_\infty < \infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$* (1.1)-(1.2) *has no positive radial solution.*
(f) *If $f_0 > 0$ and $f_\infty > 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$* (1.1)-(1.2) *has no positive radial solution.*

We should mention that, on an annulus, there are non-radially symmetric positive solutions of (1.1)-(1.2) even when $f(u) = u^p$, which was first observed by Brezis and Nirenberg [4].

## 2. Preliminaries

A radial solution of (1.1)-(1.2) can be considered as a solution of the equation

$$\left(r^{n-1}\varphi(u'(r))\right)' + \lambda r^{n-1}k(r)f(u(r)) = 0, \text{ in } R_1 < r < R_2, \tag{2.1}$$

with one of the following three sets of boundary conditions,

$$u(R_1) = u(R_2) = 0, \tag{2.2a}$$

$$u'(R_1) = u(R_2) = 0, \tag{2.2b}$$

$$u(R_1) = u'(R_2) = 0. \tag{2.2c}$$

We shall treat classical solutions of (2.1)-(2.2), namely functions $u$ of class $C^1$ on $[R_1, R_2]$ with $\varphi(u') \in C^1(R_1, R_2)$, which satisfies (2.1)-(2.2) for $r \in (R_1, R_2)$. A solution $u$ is positive if $u(r) > 0$ for all $r \in (R_1, R_2)$.

Applying change of variables, $r = (R_2 - R_1)t + R_1$, we can transform (2.1)-(2.2) into the form

$$(q(t)\varphi(p(t)u'))' + \lambda h(t) f(u) = 0, \quad 0 < t < 1 \tag{2.3}$$

with one of the following three sets of boundary conditions,

$$u(0) = u(1) = 0, \tag{2.4a}$$

$$u'(0) = u(1) = 0, \tag{2.4b}$$

$$u(0) = u'(1) = 0, \tag{2.4c}$$

where

$$q(t) := ((R_2 - R_1)t + R_1)^{n-1}, \quad p(t) := \frac{1}{R_2 - R_1}$$

and

$$h(t) := (R_2 - R_1)((R_2 - R_1)t + R_1)^{n-1} k((R_2 - R_1)t + R_1).$$

It is easy to see that (H1)-(H2) imply

$$(P) \begin{cases} p(t) \ \textit{and} \ q(t) \in C[0,1] \ \textit{with} \ p > 0 \ \textit{and} \ q > 0 \ \textit{for} \ t \in [0,1] \\ \textit{and} \ q(t) \ \textit{is nondecreasing on} \ [0,1]. \\ \varphi \text{ is an odd, increasing homeomorphism from } \mathbb{R} \text{ onto } \mathbb{R} \text{ and} \\ \text{there exist two increasing homeomorphisms } \psi_1 \text{ and } \psi_2 \text{ from} \\ (0,\infty) \text{ onto } (0,\infty) \text{ such that} \\ \qquad \psi_1(\sigma)\varphi(t) \le \varphi(\sigma t) \le \psi_2(\sigma)\varphi(t), \ \text{ for all } \sigma \text{ and } t > 0. \\ h : [0,1] \to [0,\infty) \ \text{ is continuous and does not vanish identically} \\ \text{on any subinterval of } [0,1]. \end{cases}$$

For (2.3)-(2.4) we shall prove Theorems 2.1 and 2.2, which immediately imply that Theorems 1.1 and 1.2 are true. In addition, Part (g) of Theorem 2.2 also holds for (1.1)-(1.2).

Although the functions $p$, $q$, $\varphi$ and $h$ are of the special forms defined above, we remark that Theorems 2.1 and 2.2, including Lemmas 2.4-2.11, hold even for general functions $p$, $q$, $\varphi$ and $h$ if they satisfy the property (P).

In the following proof we only use the property (P) of $p$, $q$, $\varphi$ and $h$ and do not rely on any special form that $p$, $q$, $\varphi$ and $h$ may have. Define

$$\rho := \Big[\int_0^1 \frac{1}{p(s)}ds\Big]^{-1} \min\Big\{\int_0^{\frac{1}{4}} \frac{1}{p(t)}dt, \int_{\frac{3}{4}}^1 \frac{1}{p(t)}dt\Big\} > 0,$$

and

$$\gamma(t) := \frac{\rho}{2}\Big[\int_{\frac{1}{4}}^t \frac{1}{p(s)}\psi_2^{-1}\Big(\frac{1}{q(1)}\int_s^t h(\tau)d\tau\Big)ds + \int_t^{\frac{3}{4}} \frac{1}{p(s)}\psi_2^{-1}\Big(\frac{1}{q(1)}\int_t^s h(\tau)d\tau\Big)ds\Big],$$

where $t \in [\frac{1}{4}, \frac{3}{4}]$. It follows from (H1)-(H2) that $\Gamma := \min_{\frac{1}{4}\le t\le\frac{3}{4}} \gamma(t) > 0$.

**Theorem 2.1.** *Assume* (H1)-(H3) *hold.*

(a) *If* $f_0 = 0$ *and* $f_\infty = \infty$, *then for all* $\lambda > 0$ (2.3)-(2.4) *has a positive solution.*

(b) *If* $f_0 = \infty$ *and* $f_\infty = 0$, *then for all* $\lambda > 0$ (2.3)-(2.4) *has a positive solution.*

**Theorem 2.2.** *Assume* (H1)-(H4) *hold.*

(a) *If* $f_0 = 0$ *or* $f_\infty = 0$, *then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (2.3)-(2.4) *has a positive solution.*

(b) *If* $f_0 = \infty$ *or* $f_\infty = \infty$, *then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (2.3)-(2.4) *has a positive solution.*

(c) *If* $f_0 = f_\infty = 0$, *then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (2.3)-(2.4) *has two positive solutions.*

(d) *If* $f_0 = f_\infty = \infty$, *then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (2.3)-(2.4) *has two positive solutions.*

(e) *If* $f_0 < \infty$ *and* $f_\infty < \infty$, *then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (2.3)-(2.4) *has no positive solution.*

(f) *If* $f_0 > 0$ *and* $f_\infty > 0$, *then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (2.3)-(2.4) *has no positive solution.*

(g) *If* $0 < f_0 < \infty$, $0 < f_\infty < \infty$ *and either*

$$\psi_2\Big(\frac{1}{\Gamma\psi_2^{-1}(f_0)}\Big) < \lambda < \psi_1\Big(\frac{1}{\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\psi_1^{-1}(f_\infty)}\Big)$$

*or*

$$\psi_2\Big(\frac{1}{\Gamma\psi_2^{-1}(f_\infty)}\Big) < \lambda < \psi_1\Big(\frac{1}{\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\psi_1^{-1}(f_0)}\Big),$$

*then* (2.3)-(2.4) *has a positive solution.*

The following well-known result of the fixed point index is crucial in our arguments.

**Lemma 2.3.** ([6, 8, 10]). *Let $E$ be a Banach space and $K$ a cone in $E$. For $r > 0$, define $K_r := \{u \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r := \{u \in K : \|x\| = r\}$.*
(i) *If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.*
(ii) *If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.*

In order to apply Lemma 2.3 to (2.3)-(2.4), let $X$ be the Banach space $C[0,1]$ with $\|u\| = \sup_{t\in[0,1]} |u(t)|$. Define $K$ be a cone in $X$ by

$$K := \{u \in X : u(t) \geq 0, \quad \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq \rho\|u\|\}.$$

Also, define, for $r$ a positive number, $\Omega_r$ by $\Omega_r := \{u \in K : \|u\| < r\}$. Note that $\partial\Omega_r = \{u \in K : \|u\| = r\}$.

Let the map $T_\lambda : K \to X$ be defined by

$$T_\lambda u(t) := \begin{cases} \int_0^t \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(s)}\lambda \int_s^\sigma h(\tau)f(u(\tau))d\tau)ds, & 0 \leq t \leq \sigma, \\ \int_t^1 \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(s)}\lambda \int_\sigma^s h(\tau)f(u(\tau))d\tau)ds, & \sigma \leq t \leq 1, \end{cases}$$

where $\sigma = 0$ for (2.3)-(2.4b) and $\sigma = 1$ for (2.3)-(2.4c). For (2.3)-(2.4a) $\sigma \in (0,1)$ is a solution of the equation

$$Z_1(t) = Z_2(t), \tag{2.5}$$

where

$$Z_1(t) = \int_0^t \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(s)}\lambda \int_s^t h(\tau)f(u(\tau))d\tau)ds, \quad 0 \leq t \leq 1,$$

and

$$Z_2(t) = \int_t^1 \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(s)}\lambda \int_t^s h(\tau)f(u(\tau))d\tau)ds, \quad 0 \leq t \leq 1.$$

Note that (2.5) has at least one solution in $(0,1)$. In fact, if $h(\tau)f(u(\tau)) \equiv 0$ on [0,1], we may choose any $\sigma \in (0,1)$. If there is a $\tau \in (0,1)$ such that $h(\tau)f(u(\tau)) > 0$, then $Z_1(0) - Z_2(0) < 0$ and $Z_1(1) - Z_2(1) > 0$. Since $Z_1(t) - Z_2(t)$ is nondecreasing continuous function defined on [0,1], (2.5) has at least one solution in (0,1). Moreover, if $\sigma_1$ and $\sigma_2 \in (0,1)$ are solutions of (2.5), it is not difficult to show that $h(\tau)f(u(\tau)) \equiv 0$ on $[\sigma_1, \sigma_2]$. Therefore, $T_\lambda u(t)$ is independent of the choice of $\sigma \in [\sigma_1, \sigma_2]$ and then the operator is well defined.

**Lemma 2.4.** *Assume* (H1)-(H2) *hold. Let $u$ and $v \in X$ with $u(t) \geq 0$ and $v(t) \leq 0$ for $t \in [0,1]$. If $(q(t)\varphi(p(t)u'))' = v$, then*

$$u(t) \geq [\int_0^1 \frac{1}{p(s)}ds]^{-1} \min\{\int_0^t \frac{1}{p(s)}ds, \int_t^1 \frac{1}{p(s)}ds\}||u||, \quad t \in [0,1].$$

*In particular,*

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq \rho||u||.$$

**Proof.** Since $q(t)\varphi(p(t)u'(t))$ is nonincreasing, it follows, from $\varphi^{-1}$ is increasing and $q(t)$ is nondecreasing, that $p(t)u'(t)$ is nonincreasing. Hence, for $0 \leq t_0 < t < t_1 \leq 1$,

$$u(t) - u(t_0) = \int_{t_0}^t \frac{1}{p(s)}p(s)u'(s)ds \geq \int_{t_0}^t \frac{1}{p(s)}dsp(t)u'(t)$$

and

$$u(t_1) - u(t) = \int_t^{t_1} \frac{1}{p(s)}p(s)u'(s)ds \leq \int_t^{t_1} \frac{1}{p(s)}dsp(t)u'(t),$$

from which, we have

$$u(t) \geq [\int_{t_0}^{t_1} \frac{1}{p(s)}ds]^{-1}[\int_t^{t_1} \frac{1}{p(s)}dsu(t_0) + \int_{t_0}^t \frac{1}{p(s)}dsu(t_1)].$$

Considering the above inequality on $[0,\sigma]$ and $[\sigma,1]$, we have

$$u(t) \geq [\int_0^1 \frac{1}{p(s)}ds]^{-1} \int_0^t \frac{1}{p(s)}ds||u|| \quad \text{for} \quad t \in [0,\sigma],$$

and

$$u(t) \geq [\int_0^1 \frac{1}{p(s)}ds]^{-1} \int_t^1 \frac{1}{p(s)}ds||u|| \quad \text{for} \quad t \in [\sigma,1],$$

where $\sigma \in [0,1]$ such that $u(\sigma) = ||u||$. Hence,

$$u(t) \geq [\int_0^1 \frac{1}{p(s)}ds]^{-1} \min\{\int_0^t \frac{1}{p(s)}ds, \int_t^1 \frac{1}{p(s)}ds\}||u||, \quad t \in [0,1]. \qquad \square$$

We remark that, according to Lemma 2.4, any non-negative solution of (2.3)-(2.4) is positive unless it is identical to zero.

**Lemma 2.5.** *Assume* (H1)-(H3) *hold. Then $T_\lambda(K) \subset K$ and the map $T_\lambda : K \to K$ is completely continuous.*

**Proof.** Lemma 2.4 implies that $T_\lambda(K) \subset K$. It is not difficult to verify that $T_\lambda$ is compact and continuous. $\square$

Now it is not difficult to show that (2.3)-(2.4) is equivalent to the fixed point equation

$$T_\lambda u = u \quad \text{in} \quad \text{K}.$$

**Lemma 2.6.** *Assume* (H1) *holds. Then for all* $\sigma, t \in (0, \infty)$

$$\psi_2^{-1}(\sigma)t \le \varphi^{-1}(\sigma\varphi(t)) \le \psi_1^{-1}(\sigma)t.$$

**Proof.** Since $\sigma = \psi_1(\psi_1^{-1}(\sigma)) = \psi_2(\psi_2^{-1}(\sigma))$ and $\varphi(\varphi^{-1}(\sigma\varphi(t))) = \sigma\varphi(t)$, it follows that

$$\psi_2(\psi_2^{-1}(\sigma))\varphi(t) = \varphi(\varphi^{-1}(\sigma\varphi(t))) = \psi_1(\psi_1^{-1}(\sigma))\varphi(t).$$

On the other hand, we have by (H1)

$$\psi_1(\psi_1^{-1}(\sigma))\varphi(t) \le \varphi(\psi_1^{-1}(\sigma)t) \text{ and } \psi_2(\psi_2^{-1}(\sigma))\varphi(t) \ge \varphi(\psi_2^{-1}(\sigma)t).$$

Hence,

$$\varphi(\psi_2^{-1}(\sigma)t) \le \varphi(\varphi^{-1}(\sigma\varphi(t)) \le \varphi(\psi_1^{-1}(\sigma)t).$$

Since $\varphi^{-1}$ is increasing, we obtain

$$\psi_2^{-1}(\sigma)t \le \varphi^{-1}(\sigma\varphi(t)) \le \psi_1^{-1}(\sigma)t. \qquad \square$$

**Lemma 2.7.** *Assume* (H1)-(H3) *hold and let* $\eta > 0$. *If* $u \in K$ *and* $f(u(t)) \ge \varphi(u(t)\eta)$ *for* $t \in [\frac{1}{4}, \frac{3}{4}]$, *then*

$$\|T_\lambda u\| \ge \psi_2^{-1}(\lambda)\Gamma\eta\|u\|.$$

**Proof.** Note that from the definition of $T_\lambda u$ that $T_\lambda u(\sigma)$ is the maximum value of $T_\lambda u$ on [0,1]. If $\sigma \in [\frac{1}{4}, \frac{3}{4}]$, it follows from Lemma 2.4 and Lemma 2.6 that

$$\begin{aligned}
\|T_\lambda u\| \ge \frac{1}{2}\Big[ &\int_{\frac{1}{4}}^{\sigma} \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(1)}\lambda \int_s^{\sigma} h(\tau)f(u(\tau))d\tau)ds \\
&+ \int_{\sigma}^{\frac{3}{4}} \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(1)}\lambda \int_{\sigma}^{s} h(\tau)f(u(\tau))d\tau)ds\Big] \\
\ge \frac{1}{2}\Big[ &\int_{\frac{1}{4}}^{\sigma} \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(1)} \int_s^{\sigma} h(\tau)\psi_2(\psi_2^{-1}(\lambda))\varphi(u(\tau)\eta)d\tau)ds \\
&+ \int_{\sigma}^{\frac{3}{4}} \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(1)} \int_{\sigma}^{s} h(\tau)\psi_2(\psi_2^{-1}(\lambda))\varphi(u(\tau)\eta)d\tau)ds\Big] \\
\ge \frac{1}{2}\Big[ &\int_{\frac{1}{4}}^{\sigma} \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(1)} \int_s^{\sigma} h(\tau)\varphi(\psi_2^{-1}(\lambda)\rho\eta\|u\|)d\tau)ds
\end{aligned}$$

$$+\int_{\sigma}^{\frac{3}{4}}\frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(1)}\int_{\sigma}^{s}h(\tau)\varphi(\psi_2^{-1}(\lambda)\rho\eta\|u\|)d\tau)ds\Big]$$

$$\geq\frac{1}{2}\Big[\int_{\frac{1}{4}}^{\sigma}\frac{1}{p(s)}\psi_2^{-1}(\frac{1}{q(1)}\int_{s}^{\sigma}h(\tau)d\tau)\psi_2^{-1}(\lambda)\rho\eta\|u\|ds$$

$$+\int_{\sigma}^{\frac{3}{4}}\frac{1}{p(s)}\psi_2^{-1}(\frac{1}{q(1)}\int_{\sigma}^{s}h(\tau)d\tau)\psi_2^{-1}(\lambda)\rho\eta\|u\|ds\Big]$$

$$=\frac{\psi_2^{-1}(\lambda)\eta\rho\|u\|}{2}\Big[\int_{\frac{1}{4}}^{\sigma}\frac{1}{p(s)}\psi_2^{-1}(\frac{1}{q(1)}\int_{s}^{\sigma}h(\tau)d\tau)ds$$

$$+\int_{\sigma}^{\frac{3}{4}}\frac{1}{p(s)}\psi_2^{-1}(\frac{1}{q(1)}\int_{\sigma}^{s}h(\tau)d\tau)ds\Big]\geq\psi_2^{-1}(\lambda)\Gamma\eta\|u\|.$$

For $\sigma > \frac{3}{4}$, it is easy to see

$$\|T_\lambda u\|\geq\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(1)}\lambda\int_{s}^{\frac{3}{4}}h(\tau)f(u(\tau))d\tau)ds.$$

On the other hand, we have

$$\|T_\lambda u\|\geq\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(1)}\lambda\int_{\frac{1}{4}}^{s}h(\tau)f(u(\tau))d\tau)ds \text{ for } \sigma<\frac{1}{4}.$$

Therefore, similar arguments show that

$$\|T_\lambda u\|\geq\psi_2^{-1}(\lambda)\Gamma\eta\|u\| \text{ if } \sigma>\frac{3}{4} \text{ or } \sigma<\frac{1}{4}. \qquad \square$$

Define a new function

$$f^*(u):=\max_{0\leq t\leq u}\{f(t)\}.$$

Note that $f_0^*=\lim_{u\to 0}\frac{f^*(u)}{\varphi(u)}$ and $f_\infty^*=\lim_{u\to\infty}\frac{f^*(u)}{\varphi(u)}$.

**Lemma 2.8.** *Assume* (H1)-(H3) *hold. Then* $f_0^*=f_0$ *and* $f_\infty^*=f_\infty$.

**Proof.** It is easy to see that $f_0^*=f_0$. For the second part, we consider two cases, (a) $f(u)$ is bounded and (b) $f(u)$ is unbounded. For the case (a), it follows, from $\lim_{u\to\infty}\varphi(u)=\infty$, that $f_\infty^*=0=f_\infty$. For the case (b), for any $\delta>0$, let $M:=\max_{0\leq t\leq\delta}\{f(t)\}$ and $N_\delta:=\min\{u:u\geq\delta, f(u)\geq M\}\geq\delta$, then

$$\max_{0\leq t\leq N_\delta}\{f(t)\}=f(N_\delta).$$

Thus, for any $\delta > 0$, there exists a $N_\delta \geq \delta$ such that

$$f^*(u) = \max\{\max_{0\leq t\leq N_\delta}\{f(t)\}, \max_{N_\delta\leq t\leq u}\{f(t)\}\} = \max_{N_\delta\leq t\leq u}\{f(t)\} \text{ for } u > N_\delta.$$

Hence, it follows, from the definitions of $f_\infty$ and $f^*_\infty$, that $f^*_\infty = f_\infty$. □

**Lemma 2.9.** *Assume* (H1)-(H3) *hold and let* $r > 0$. *If there exists an* $\varepsilon > 0$ *such that* $f^*(r) \leq \psi_1(\varepsilon)\varphi(r)$, *then*

$$\|T_\lambda u\| \leq \psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\varepsilon\|u\| \text{ for } u \in \partial\Omega_r.$$

**Proof.** From the definition of $T_\lambda$, Lemma 2.4 and Lemma 2.6, for $u \in \partial\Omega_r$, we have

$$\begin{aligned}
\|T_\lambda u\| &\leq \int_0^1 \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(0)}\lambda\int_0^1 h(\tau)f(u(\tau))d\tau)ds \\
&\leq \int_0^1 \frac{1}{p(s)}\varphi^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)\lambda f^*(r)d\tau)ds \\
&\leq \int_0^1 \frac{1}{p(s)}ds\varphi^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)\psi_1(\psi_1^{-1}(\lambda))\psi_1(\varepsilon)\varphi(r)d\tau) \\
&\leq \int_0^1 \frac{1}{p(s)}ds\varphi^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau\varphi(\psi_1^{-1}(\lambda)\varepsilon r)) \\
&\leq \int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\psi_1^{-1}(\lambda)\varepsilon\|u\|. \qquad \Box
\end{aligned}$$

The following two lemmas are weak forms of Lemmas 2.7 and 2.9.

**Lemma 2.10.** *Assume* (H1)-(H4) *hold. If* $u \in \partial\Omega_r$, $r > 0$, *then*

$$\|T_\lambda u\| \geq \frac{\varphi^{-1}(\lambda\hat{m}_r)\Gamma}{\rho},$$

*where* $\hat{m}_r := \min_{\rho r\leq t\leq r}\{f(t)\} > 0$.

**Proof.** Since $\lambda f(u(t)) \geq \lambda\hat{m}_r = \varphi(\varphi^{-1}(\lambda\hat{m}_r))$ for t $\in [\frac{1}{4}, \frac{3}{4}]$, it is easy to see that this lemma can be shown in a similar manner as in Lemma 2.7. □

**Lemma 2.11.** *Assume* (H1)-(H4) *hold. If* $u \in \partial\Omega_r$, $r > 0$, *then*

$$\|T_\lambda u\| \leq \varphi^{-1}(\lambda\hat{M}_r)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau),$$

*where* $\hat{M}_r := \max_{0\le t\le r}\{f(t)\} > 0$.

**Proof.** Since $\lambda f(u(t)) \le \lambda \hat{M}_r = \varphi(\varphi^{-1}(\lambda \hat{M}_r))$ for t $\in [0,1]$, it is easy to see that this lemma can be shown in a similar manner as in Lemma 2.9. □

## 3. Proof of Theorem 2.1

**Proof.** Part (a). It follows from Lemma 2.8 that $f_0^* = 0$. Therefore, we can choose $r_1 > 0$ so that $f^*(r_1) \le \psi_1(\varepsilon)\varphi(r_1)$, where the constant $\varepsilon > 0$ satisfies

$$\varepsilon\psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(s)ds) < 1.$$

We have by Lemma 2.9 that

$$\|T_\lambda u\| \le \psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\varepsilon\|u\| < \|u\| \text{ for } u \in \partial\Omega_{r_1}.$$

Now, since $f_\infty = \infty$, there is an $\hat{H} > 0$ such that $f(u) \ge \psi_2(\eta)\varphi(u)$ for $u \ge \hat{H}$ , where $\eta > 0$ is chosen so that

$$\psi_2^{-1}(\lambda)\Gamma\eta > 1.$$

Let $r_2 = \max\{2r_1, \frac{\hat{H}}{\rho}\}$. If $u \in \partial\Omega_{r_2}$, then

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) \ge \rho\|u\| = \rho r_2 \ge \hat{H},$$

which implies that

$$f(u(t)) \ge \psi_2(\eta)\varphi(u(t)) \ \ge \varphi(u(t)\eta) \quad \text{for} \quad t \in [\frac{1}{4}, \frac{3}{4}].$$

It follows from Lemma 2.7 that

$$\|T_\lambda u\| \ge \psi_2^{-1}(\lambda)\Gamma\eta\|u\| > \|u\| \quad \text{for} \quad u \in \partial\Omega_{r_2}.$$

By Lemma 2.3, $i(T_\lambda, \Omega_{r_1}, K) = 1$ and $i(T_\lambda, \Omega_{r_2}, K) = 0$. It follows from the additivity of the fixed point index that $i(T_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = -1$. Thus, $i(T_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) \ne 0$, which implies $T_\lambda$ has a fixed point $u \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ according to the existence property of the fixed point index. The fixed point $u \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ is the desired positive solution of (2.3)-(2.4).

Part (b). If $f_0 = \infty$, there is a $r_1 > 0$ such that $f(u) \ge \psi_2(\eta)\varphi(u)$ for $0 \le u \le r_1$, where $\eta > 0$ is chosen so that $\psi_2^{-1}(\lambda)\Gamma\eta > 1$. If $u \in \partial\Omega_{r_1}$, then

$$f(u(t)) \ge \psi_2(\eta)\varphi(u(t)) \ge \varphi(u(t)\eta) \text{ for } t \in [0,1].$$

Lemma 2.7 implies that

$$\|T_\lambda u\| \ge \psi_2^{-1}(\lambda)\Gamma\eta\|u\| > \|u\| \quad for \quad u \in \partial\Omega_{r_1}.$$

We now determine $\Omega_{r_2}$. Since $f^*_\infty = f_\infty = 0$, there is a $r_2 > 2r_1$ such that $f^*(r_2) \le \psi_1(\varepsilon)\varphi(r_2)$, where the constant $\varepsilon > 0$ satisfies

$$\varepsilon\psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(s)ds) < 1.$$

Thus, we have by Lemma 2.9

$$\|T_\lambda u\| \le \psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\varepsilon\|u\| < \|u\| \text{ for } u \in \partial\Omega_{r_2}.$$

By Lemma 2.3, $i(T_\lambda, \Omega_{r_1}, K) = 0$ and $i(T_\lambda, \Omega_{r_2}, K) = 1$. It follows from the additivity of the fixed point index that $i(T_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $T_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (2.3)-(2.4).

## 4. Proof of Theorem 2.2

**Proof.** Part (a). Choose a number $r_1 > 0$. By Lemma 2.10, we infer that there exists a $\lambda_0 > 0$ such that

$$\|T_\lambda u\| > \|u\|, \text{ for } \mathrm{u} \in \partial\Omega_{\mathrm{r}_1}, \lambda > \lambda_0.$$

If $f^*_0 = f_0 = 0$, we can choose $0 < r_2 < r_1$ so that $f^*(r_2) \le \psi_1(\varepsilon)\varphi(r_2)$, where the constant $\varepsilon > 0$ satisfies

$$\varepsilon\psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(s)ds) < 1.$$

Thus, we have by Lemma 2.9 that

$$\|T_\lambda u\| \le \psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\varepsilon\|u\| < \|u\| \text{ for } u \in \partial\Omega_{r_2}.$$

If $f^*_\infty = f_\infty = 0$, there is an $r_3 > 2r_1$ such that $f^*(r_3) \le \psi_1(\varepsilon)\varphi(r_3)$, where the constant $\varepsilon > 0$ satisfies

$$\varepsilon\psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(s)ds) < 1.$$

Thus, we have

$$\|T_\lambda u\| \le \psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\varepsilon\|u\| < \|u\| \text{ for } u \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.3 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0, \quad i(T_\lambda, \Omega_{r_2}, K) = 1 \text{ and } i(T_\lambda, \Omega_{r_3}, K) = 1.$$

Thus, $i(T_\lambda, \Omega_{r_1} \setminus \bar{\Omega}_{r_2}, K) = -1$ and $i(T_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = 1$. Hence, $T_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ according to $f_0 = 0$ or $f_\infty = 0$, respectively. Consequently, (2.3)-(2.4) has a positive solution for $\lambda > \lambda_0$.

Part (b). Choose a number $r_1 > 0$. By Lemma 2.11 we infer that there exists a $\lambda_0 > 0$ such that

$$\|T_\lambda u\| < \|u\|, \text{ for } \mathrm{u} \in \partial\Omega_{\mathrm{r}_1}, 0 < \lambda < \lambda_0.$$

If $f_0 = \infty$, there is a $0 < r_2 < r_1$ such that $f(u) \geq \psi_2(\eta)\varphi(u)$ for $0 \leq u \leq r_2$, where $\eta > 0$ is chosen so that

$$\psi_2^{-1}(\lambda)\Gamma\eta > 1.$$

Then

$$f(u(t)) \geq \psi_2(\eta)\varphi(u(t)) \geq \varphi(u(t)\eta) \text{ for } u \in \partial\Omega_{r_2}, \ t \in [0, 1].$$

Lemma 2.7 implies that

$$\|T_\lambda u\| \geq \psi_2^{-1}(\lambda)\Gamma\eta\|u\| > \|u\| \text{ for } u \in \partial\Omega_{r_2}.$$

If $f_\infty = \infty$, there is an $\hat{H} > 0$ such that $f(u) \geq \psi_2(\eta)\varphi(u)$ for $u \geq \hat{H}$, where $\eta > 0$ is chosen so that

$$\psi_2^{-1}(\lambda)\Gamma\eta > 1.$$

Let $r_3 = \max\{2r_1, \frac{\hat{H}}{\rho}\}$. If $u \in \partial\Omega_{r_3}$, then

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq \rho\|u\| \geq \hat{H},$$

and hence,

$$f(u(t)) \geq \psi_2(\eta)\varphi(u(t)) \ \geq \varphi(u(t)\eta) \text{ for } t \in [\frac{1}{4}, \frac{3}{4}].$$

It follows from Lemma 2.7 that

$$\|T_\lambda u\| \geq \psi_2^{-1}(\lambda)\Gamma\eta\|u\| > \|u\| \text{ for } u \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.3 that

$$i(T_\lambda, \Omega_{r_1}, K) = 1, \quad i(T_\lambda, \Omega_{r_2}, K) = 0 \text{ and } i(T_\lambda, \Omega_{r_3}, K) = 0,$$

and hence, $i(T_\lambda, \Omega_{r_1} \setminus \bar{\Omega}_{r_2}, K) = 1$ and $i(T_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = -1$. Thus, $T_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ according to $f_0 = \infty$ or $f_\infty = \infty$, respectively. Consequently, (2.3)-(2.4) has a positive solution for $0 < \lambda < \lambda_0$.

Part (c). Choose two numbers $0 < r_3 < r_4$. By Lemma 2.10 we infer that there exists a $\lambda_0 > 0$ such that

$$\|T_\lambda u\| > \|u\|, \text{ for } \mathrm{u} \in \partial\Omega_{\mathrm{r_i}}, \ \lambda > \lambda_0, (\mathrm{i} = 3, 4).$$

Since $f_0 = 0$ and $f_\infty = 0$, it follows from the proof of Theorem 2.2 (a) that we can choose $r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|T_\lambda u\| < \|u\| \text{ for } u \in \partial\Omega_{r_i}, \ (i = 1, 2).$$

It follows from Lemma 2.3 that

$$i(T_\lambda, \Omega_{r_1}, K) = 1, \quad i(T_\lambda, \Omega_{r_2}, K) = 1,$$
$$i(T_\lambda, \Omega_{r_3}, K) = 0, \quad i(T_\lambda, \Omega_{r_4}, K) = 0$$

and hence, $i(T_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = -1$ and $i(T_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_4}, K) = 1$. Thus, $T_\lambda$ has two fixed points $u_1(t)$ and $u_2(t)$ such that $u_1(t) \in \Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ and $u_2(t) \in \Omega_{r_2} \setminus \bar{\Omega}_{r_4}$, which are the desired distinct positive solutions of (2.3)-(2.4) for $\lambda > \lambda_0$ satisfying

$$r_1 < \|u_1\| < r_3 < r_4 < \|u_2\| < r_2.$$

Part (d). Choose two numbers $0 < r_3 < r_4$. By Lemma 2.11 we infer that there exists a $\lambda_0 > 0$ such that

$$\|T_\lambda u\| < \|u\|, \text{ for } \mathrm{u} \in \partial\Omega_{\mathrm{r_i}}, 0 < \lambda < \lambda_0, (\mathrm{i} = 3, 4).$$

Since $f_0 = \infty$ and $f_\infty = \infty$, it follows from the proof of Theorem 2.2 (b) that we can choose $r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|T_\lambda u\| > \|u\| \text{ for } u \in \partial\Omega_{r_i}, \quad (i = 1, 2).$$

It follows from Lemma 2.3 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0, \quad i(T_\lambda, \Omega_{r_2}, K) = 0,$$
$$i(T_\lambda, \Omega_{r_3}, K) = 1, \quad i(T_\lambda, \Omega_{r_4}, K) = 1$$

and hence, $i(T_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = 1$ and $i(T_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_4}, K) = -1$. Thus, $T_\lambda$ has two fixed points $u_1(t)$ and $u_2(t)$ such that $u_1(t) \in \Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ and $u_2(t) \in \Omega_{r_2} \setminus \bar{\Omega}_{r_4}$, which are the desired distinct positive solutions of (2.3)-(2.4) for $0 < \lambda < \lambda_0$ satisfying

$$r_1 < \|u_1\| < r_3 < r_4 < \|u_2\| < r_2.$$

Part (e) Since $f_0 < \infty$ and $f_\infty < \infty$, there exist positive numbers $\varepsilon_1$, $\varepsilon_2$, $r_1$ and $r_2$ such that $r_1 < r_2$ and

$$f(u) \le \varepsilon_1 \varphi(u) \text{ for } u \in [0, r_1],$$

$$f(u) \le \varepsilon_2 \varphi(u) \text{ for } u \in [r_2, \infty).$$

Let $\varepsilon_3 := \max\{\varepsilon_1, \varepsilon_2, \max\limits_{r_1 \le u \le r_2}\{\frac{f(u)}{\varphi\{u\}}\}\} > 0$. Thus, we have

$$f(u) \le \varepsilon_3 \varphi(u) \text{ for } u \in [0, \infty).$$

Assume $v(t)$ is a positive solution of (2.3)-(2.4). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 := \psi_1(\frac{1}{\int_0^1 \frac{1}{p(s)} ds \psi_1^{-1}(\varepsilon_3 \frac{1}{q(0)} \int_0^1 h(\tau)d\tau)})$. Since $T_\lambda v(t) = v(t)$ for $t \in [0,1]$,

$$\begin{aligned}\|v\| = \|T_\lambda v\| &\leq \int_0^1 \frac{1}{p(s)} \varphi^{-1}(\frac{1}{q(0)} \lambda \int_0^1 h(\tau) f(v(\tau)) d\tau) ds \\ &\leq \int_0^1 \frac{1}{p(s)} ds \varphi^{-1}(\frac{1}{q(0)} \int_0^1 h(\tau) \varepsilon_3 d\tau \psi_1(\psi_1^{-1}(\lambda)) \varphi(\|v\|)) \\ &\leq \int_0^1 \frac{1}{p(s)} ds \varphi^{-1}(\frac{1}{q(0)} \int_0^1 h(\tau) \varepsilon_3 d\tau \varphi(\psi_1^{-1}(\lambda) \|v\|)) \\ &\leq \int_0^1 \frac{1}{p(s)} ds \psi_1^{-1}(\frac{1}{q(0)} \varepsilon_3 \int_0^1 h(\tau) d\tau) \psi_1^{-1}(\lambda) \|v\| < \|v\|,\end{aligned}$$

which is a contradiction.

Part (f). Since $f_0 > 0$ and $f_\infty > 0$, it follows that there exist positive numbers $\eta_1$, $\eta_2$, $r_1$ and $r_2$ such that $r_1 < r_2$ and

$$\begin{aligned} f(u) &\geq \eta_1 \varphi(u) \quad \text{for} \quad u \in [0, r_1], \\ f(u) &\geq \eta_2 \varphi(u) \quad \text{for} \quad u \in [r_2, \infty). \end{aligned}$$

Let $\eta_3 := \min\{\eta_1, \eta_2, \min_{r_1 \leq u \leq r_2} \{\frac{f(u)}{\varphi\{u\}}\}\} > 0$. Thus, we have

$$f(u) \geq \eta_3 \varphi(u) \quad \text{for} \quad u \in [0, \infty).$$

Since $\eta_3 \varphi(u) = \psi_2(\psi_2^{-1}(\eta_3)) \varphi(u)$, (H1) implies that

$$f(u) \geq \eta_3 \varphi(u) \geq \varphi(\psi_2^{-1}(\eta_3) u) \quad \text{for} \quad u \in [0, \infty).$$

Assume $v(t)$ is a positive solution of (2.3)-(2.4). We will show that this leads to a contradiction for $\lambda > \lambda_0 := \psi_2(\frac{1}{\Gamma \psi_2^{-1}(\eta_3)})$. Since $T_\lambda v(t) = v(t)$ for $t \in [0,1]$, it follows from Lemma 2.7 that

$$\|v\| = \|T_\lambda v\| \geq \psi_2^{-1}(\lambda) \Gamma \psi_2^{-1}(\eta_3) \|v\| > \|v\|,$$

which is a contradiction.

Part (g). If

$$\psi_2(\frac{1}{\Gamma \psi_2^{-1}(f_0)}) < \lambda < \psi_1(\frac{1}{\int_0^1 \frac{1}{p(s)} ds \psi_1^{-1}(\frac{1}{q(0)} \int_0^1 h(\tau) d\tau) \psi_1^{-1}(f_\infty)}),$$

there exists an $0 < \varepsilon < f_0$ such that

$$\psi_2(\frac{1}{\Gamma \psi_2^{-1}(f_0 - \varepsilon)}) < \lambda < \psi_1(\frac{1}{\int_0^1 \frac{1}{p(s)} ds \psi_1^{-1}(\frac{1}{q(0)} \int_0^1 h(\tau) d\tau) \psi_1^{-1}(f_\infty + \varepsilon)}).$$

Beginning with $f_0$, there is a $r_1 > 0$ such that $f(u) \geq (f_0 - \varepsilon)\varphi(u)$ for $0 \leq u \leq r_1$. Note that $(f_0 - \varepsilon)\varphi(u) = \psi_2(\psi_2^{-1}(f_0 - \varepsilon))\varphi(u)$. If $u \in \partial\Omega_{r_1}$, then

$$f(u(t)) \geq \psi_2(\psi_2^{-1}(f_0 - \varepsilon))\varphi(u(t)) \geq \varphi(u(t)\psi_2^{-1}(f_0 - \varepsilon)) \quad \text{for} \quad t \in [0, 1].$$

Lemma 2.7 implies that

$$\|T_\lambda u\| \geq \psi_2^{-1}(\lambda)\Gamma\psi_2^{-1}(f_0 - \varepsilon)\|u\| > \|u\| \text{ for } u \in \partial\Omega_{r_1}.$$

It remains to consider $f_\infty$. It follows from Lemma 2.8 that $\lim_{u\to\infty} \frac{f^*(u)}{\varphi(u)} = f_\infty$. Therefore, there is a $r_2 > 2r_1$ such that

$$f^*(r_2) \leq (f_\infty + \varepsilon)\varphi(r_2) = \psi_1(\psi_1^{-1}(f_\infty + \varepsilon))\varphi(r_2).$$

Lemma 2.9 implies that, for $u \in \partial\Omega_{r_2}$, we have

$$\|T_\lambda u\| \leq \psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\psi_1^{-1}(f_\infty + \varepsilon)\|u\| < \|u\|.$$

It follows from Lemma 2.3 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0 \text{ and } i(T_\lambda, \Omega_{r_2}, K) = 1.$$

Hence, $i(T_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $T_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (2.3)-(2.4).

If

$$\psi_2(\frac{1}{\Gamma\psi_2^{-1}(f_\infty)}) < \lambda < \psi_1(\frac{1}{\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\psi_1^{-1}(f_0)}),$$

there exists an $0 < \varepsilon < f_\infty$ such that

$$\psi_2(\frac{1}{\Gamma\psi_2^{-1}(f_\infty - \varepsilon)}) < \lambda < \psi_1(\frac{1}{\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\psi_1^{-1}(f_0 + \varepsilon)}).$$

Since $f_0^* = f_0$, there exists a $r_3 > 0$ such that $f^*(r_3) \leq (f_0 + \varepsilon)\varphi(r_3)$. Lemma 2.9 implies that

$$\|T_\lambda u\| \leq \psi_1^{-1}(\lambda)\int_0^1 \frac{1}{p(s)}ds\psi_1^{-1}(\frac{1}{q(0)}\int_0^1 h(\tau)d\tau)\psi_1^{-1}(f_0 + \varepsilon))\|u\| < \|u\|$$

for $u \in \partial\Omega_{r_3}$. Next, considering $f_\infty$, there is an $\hat{H} > 0$ such that $f(u) \geq (f_\infty - \varepsilon)\varphi(u)$ for $u \geq \hat{H}$. Let $r_4 = \max\{2r_3, \frac{\hat{H}}{\rho}\}$. If $u \in \partial\Omega_{r_4}$, then

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq \rho\|u\| \geq \hat{H}$$

and hence,

$$f(u(t)) \geq (f_\infty - \varepsilon)\varphi(u(t)) \geq \varphi(u(t)\psi_2^{-1}(f_\infty - \varepsilon)) \quad \text{for} \quad t \in [\frac{1}{4}, \frac{3}{4}].$$

Lemma 2.7 implies that

$$\|T_\lambda u\| \geq \psi_2^{-1}(\lambda)\Gamma\psi_2^{-1}(f_\infty - \varepsilon)\|u\| > \|u\| \quad \text{for} \quad u \in \partial\Omega_{r_4}.$$

Again it follows from Lemma 2.3 that

$$i(T_\lambda, \Omega_{r_3}, K) = 1 \ \text{ and } \ i(T_\lambda, \Omega_{r_4}, K) = 0.$$

Hence, $i(T_\lambda, \Omega_{r_4} \setminus \bar{\Omega}_{r_3}, K) = -1$. Thus, $T_\lambda$ has a fixed point in $\Omega_{r_4} \setminus \bar{\Omega}_{r_3}$, which is the desired positive solution of (2.3)-(2.4). □

## References

[1] C. Bandle, C.V. Coffman, and M. Marcus, *Nonlinear elliptic problems in annular domains,* J. Differential Equations, 69 (1987), 322-345.

[2] C. Bandle and L. A. Peletier, *Nonlinear elliptic problems with critical exponent in shrinking annuli,* Math. Ann., 280 (1988), 1-19.

[3] C.V. Coffman and M. Kwong, *Semilinear elliptic problems in annular domains,* Z. Angew. Math. Phys., 40 1989, 245-257.

[4] H. Berzis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,* Comm. Pure Appl. Math., 36 (1983), 437-477.

[5] C.V. Coffman and M. Marcus, *Existence and uniqueness results for semilinear Dirichlet problems in annuli,* Arch. Rational Mech. Anal., 108 (1989), 293–307.

[6] K, Deimling, “Nonlinear Functional Analysis,” Springer, Berlin, 1985.

[7] X. Garaizar, *Existence of positive radial solutions for semilinear elliptic equations in the annulus,* J. Differential Equations, 70 (1987), 69-92.

[8] D. Guo and V. Lakshmikantham, “Nonlinear Problems in Abstract Cones,” Academic Press, Orlando, FL, 1988.

[9] M. Kwong and L. Zhang, *Uniqueness of the positive solutions of* $\Delta u + f(u) = 0$*, in an annulus,* Differential and Integral Equations, 4 (1991), 583-599.

[10] M. Krasnoselskii, “Positive Solutions of Operator Equations,” Noordhoff, Groningen, 1964.

[11] S.S. Lin and F. M. Pai, *Existence and multiplicity of positive radial solutions for semilinear elliptic equations in annular domains,* SIAM J. Math. Anal., 22 (1991), 1500-1515.

[12] S.S. Lin, *On the existence of positive radial solutions for semilinear elliptic equations in annular domains,* J. Differential Equations, 81 (1989), 221-233.

[13] W.-M. Ni and R. D. Nussbaum, *Uniqueness and nonuniqueness for positive radial solutions of* $\Delta u + f(u, r) = 0$, Comm. Pure Appl. Math., 38 (1985), 67-108.

[14] W.-M. Ni and J. Serrin, *Non-existence theorems for singular solutions of quasilinear partial differential equations,* Comm. Pure Appl. Math., 39 (1986), 379-399.

[15] W.-M. Ni and J. Serrin, *Nonexistence theorems for quasilinear partial differential equations,* Rend. Circ. Mat. Palermo Suppl., 5 (1986), 171-185.

[16] H. Wang, *On the existence of positive solutions for semilinear elliptic equations in the annulus,* J. Differential Equations, 109 (1994), 1-7.

[17] J. Wang, *The existence of positive solutions for the one-dimensional p-Laplacian,* Proc. Amer. Math. Soc., 125 (1997), 2275-2283.

[18] S. Yadava, *Uniqueness of positive radial solutions of the Dirichlet problems* $-\Delta u = u^p \pm u^q$*, in an annulus,* J. Differential Equations, 139 (1997), 194-217.

[19] S. Yadava, *Existence and exact multiplicity of positive radial solutions of semilinear elliptic problems in annuli,* Adv. Differential Equations, 6 (2001), 129-154.