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Nontrivial solutions for p-Laplacian systems

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Abstract

The paper deals with the existence and nonexistence of nontrivial nonnegative solutions for the sublinear quasilinear system

$$\begin{aligned}&\operatorname{div}\big(|\nabla u_i|^{p-2}\nabla u_i\big)+\lambda f_i(u_1,\dots,u_n)=0 \quad \text{in } \Omega,\\ &u_i=0 \quad \text{on } \partial\Omega,\ i=1,\dots,n,\end{aligned}$$

where $p>1$, Ω is a bounded domain in $\mathbb{R}^N$ $(N\geqslant 2)$ with smooth boundary, and f_i, $i=1,\dots,n$, are continuous, nonnegative functions. Let $\mathbf{u}=(u_1,\dots,u_n)$, $\|\mathbf{u}\|=\sum_{i=1}^n|u_i|$, we prove that the problem has a nontrivial nonnegative solution for small $\lambda>0$ if one of $\lim_{\|\mathbf{u}\|\to 0}\frac{f_i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}$ is infinity. If, in addition, all $\lim_{\|\mathbf{u}\|\to\infty}\frac{f_i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}$ is zero, we show that the problem has a nontrivial nonnegative solution for all $\lambda>0$. A nonexistence result is also obtained.

Keywords: Elliptic system; p-Laplacian; Schauder Fixed-Point Theorem

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1. Introduction

In this paper we consider the existence and nonexistence of nontrivial nonnegative solutions for the quasilinear elliptic system

$$\begin{cases} -\Delta_p u_i = \lambda f_i(u_1, \ldots, u_n) & \text{in } \Omega,\ i = 1, \ldots, n, \\ u_i = 0 & \text{on } \partial\Omega,\ i = 1, \ldots, n, \end{cases} \tag{1.1}$$

where $\Delta_p u_i = \operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i)$, $i = 1, \ldots, n$, $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with smooth boundary $\partial\Omega$, and $\lambda > 0$ is a parameter.

Problem (1.1) covers several important cases. When $p = 2$, (1.1) becomes the semilinear elliptic system

$$\begin{cases} -\Delta u_i = \lambda f_i(u_1, \ldots, u_n) & \text{in } \Omega,\ i = 1, \ldots, n, \\ u_i = 0 & \text{on } \partial\Omega,\ i = 1, \ldots, n, \end{cases} \tag{1.2}$$

and when $n = 1$, (1.1) becomes the p-Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In particular, when $n = 1$ and $p = 2$, (1.1) becomes the usual Laplacian problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Problem (1.4) has received extensive investigations in the past several decades, see, e.g., [1,2,7] and references therein. Lions [7] discussed, under various combinations of superlinearity or sublinearity of f at infinity, $f(0) = 0$ and $f(0) > 0$, the existence and nonexistence of positive solutions of (1.2). The results of [7] are also interpreted in terms of bifurcation diagrams. Recently, Hai and Shivaji [3–5] studied elliptic systems related to (1.1) and proved the existence of positive solutions to (1.1) in some sublinear cases. The results in [4,5] do not impose any sign conditions on the nonlinearities at zero. A necessary and sufficient condition for the existence of positive solutions for a class of sublinear quasilinear system was obtained in [3]. The main approach in [3] is based on the Schauder Fixed-Point Theorem and maximum principles. In several papers [9–11], Wang studied the number of nontrivial radial solutions of (1.1) on an annular domain and ball. For ODE case ($N = 1$) and annular domains, it was shown in [9] and other papers that the existence, multiplicity and nonexistence of positive solutions of (1.1) can be determined by appropriate combinations of superlinearity and sublinearity of $\mathbf{f}(u)$ at zero and infinity. When the domain is a ball, Wang [10,11] showed (1.1) has a nontrivial nonnegative solution for sublinear cases in a ball.

In this paper we shall study (1.1) in general domains. We shall show that (1.1) has at least one nontrivial nonnegative solution under sublinear assumptions. We also provide a nonexistence result. Our proofs make use of the Schauder Fixed-Point Theorem and weak comparison principles. Variational methods have been frequently used for Hamiltonian systems and gradient systems. However, there is apparently no possibility of using variational methods for the n-dimensional quasilinear elliptic system (1.1), and one has to use topological methods.

We now turn to the general assumptions for this paper. Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}^+$. Also, for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$, let $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$ and $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_n(\mathbf{u})) = (f_1(u_1, \ldots, u_n), \ldots, f_n(u_1, \ldots, u_n))$.

We make the following assumptions:

(H1) $f_i : \mathbb{R}_+^n \to \mathbb{R}_+$ is continuous, $i = 1, \dots, n$.
(H2) There exists $i_0 \in \{1, \dots, n\}$ such that

$$\lim_{\|\mathbf{u}\| \to 0} \frac{f_{i_0}(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} = \infty,$$

where $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$.
(H3) For all $i \in \{1, \dots, n\}$,

$$\lim_{\|\mathbf{u}\| \to \infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} = 0,$$

where $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$.

The main results of this paper are Theorems 1.1–1.3.

Theorem 1.1. *Assume* (H1) *and* (H2) *hold. Then there is* $\lambda_0 > 0$ *such that* (1.1) *has a nontrivial nonnegative solution for* $0 < \lambda < \lambda_0$.

Theorem 1.2. *Assume* (H1)–(H3) *hold and suppose that, for* i_0 *in* (H2),

$$f_{i_0}(\mathbf{u}) > 0 \quad \textit{for } 0 < \|\mathbf{u}\|, \ \mathbf{u} \in \mathbb{R}_+^n.$$

Then (1.1) *has a nontrivial nonnegative solution for all* $\lambda > 0$.

The following assumption will allow us to establish a nonexistence theorem:

(H4) For all $i \in \{1, \dots, n\}$,

$$\lim_{\|\mathbf{u}\| \to 0} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} < \infty, \qquad \lim_{\|\mathbf{u}\| \to \infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} < \infty,$$

where $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$.

Theorem 1.3. *Assume* (H1) *and* (H4) *hold. Then there is* $\lambda_0 > 0$ *such that* (1.1) *has no nontrivial solution for* $0 < \lambda < \lambda_0$.

We now give three examples to demonstrate these three theorems.

Example 1.

$$\begin{cases} \Delta_p u_1 + \lambda e^{(u_1 + \cdots + u_n)} = 0 & \text{in } \Omega, \\ \Delta_p u_i + \lambda f_i(u_1, \dots, u_n) & \text{in } \Omega, \ i = 2, \dots, n, \\ u_i = 0 & \text{on } \partial\Omega, \ i = 1, \dots, n, \end{cases} \tag{1.5}$$

where $p > 1$, f_i are any nonnegative continuous functions. Then (1.5) has a nontrivial solution for sufficient small $\lambda > 0$ according to Theorem 1.1.

Example 2.

$$\begin{cases} \Delta_p u_i + \lambda (u_1 + \cdots + u_n)^{p_i} = 0 & \text{in } \Omega, \ i = 1, \dots, n, \\ u_i = 0 & \text{on } \partial\Omega, \ i = 1, \dots, n, \end{cases} \tag{1.6}$$

where $p > 1$, $0 < p_1, p_2, \dots, p_n < p - 1$. Then (1.6) has a nontrivial solution for all $\lambda > 0$ according to Theorem 1.2.

Example 3.

$$\begin{cases} \Delta_p u_i + \lambda(u_1 + \cdots + u_n)^{p-1} e^{-(u_1+\cdots+u_n)} = 0 & \text{in } \Omega,\ i = 1,\dots,n, \\ u_i = 0 & \text{on } \partial\Omega,\ i = 1,\dots,n, \end{cases} \tag{1.7}$$

where $p > 1$. Then (1.7) has no nonnegative nontrivial solution for all sufficient small $\lambda > 0$ according to Theorem 1.3.

2. Preliminaries

We recall some basic results for the p-Laplacian. We refer to $\lambda_1 > 0$ as the first eigenvalue and ϕ_1 as the principal eigenfunction of the p-Laplacian on Ω, i.e.,

$$\begin{cases} -\Delta_p \phi_1 = \lambda_1 |\phi_1|^{p-2}\phi_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is known that ϕ_1 belongs to $C^{1+\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ and has one sign and we assume that $\phi_1 > 0$ in Ω.

Lemma 2.1. *Let $\phi \in C^1(\overline{\Omega})$ be the solution of*

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Then there exists a constant $c > 0$ such that $\phi \geqslant c\phi_1 > 0$ in Ω.

Proof. For constant $c > 0$, it is easy to see that $-\Delta_p(\frac{\phi}{c}) = \frac{1}{c^{p-1}}$. Now we choose $c > 0$ so that $\frac{1}{c^{p-1}} \geqslant \lambda_1\phi_1$ in Ω. Thus $-\Delta_p(\frac{\phi}{c}) \geqslant -\Delta_p(\phi_1)$ in Ω. It follows from the weak comparison principle [8] that $\phi \geqslant c\phi_1 > 0$ in Ω. $\square$

3. Proof of Theorem 1.1

Let E be the Banach space $\prod_{i=1}^n C(\overline{\Omega})$ with norm $\|\mathbf{v}\| = \sum_{i=1}^n \|v_i\|_\infty$ for $\mathbf{v} = (v_1,\dots,v_n) \in E$. For each $(v_1,\dots,v_n) \in E$, define $(u_1,\dots,u_n) = A_\lambda(v_1,\dots,v_n)$ by

$$\begin{cases} -\Delta_p u_i = \lambda f_i(v_1,\dots,v_n) & \text{in } \Omega,\ i = 1,\dots,n, \\ u_i = 0 & \text{on } \partial\Omega,\ i = 1,\dots,n. \end{cases} \tag{3.1}$$

Then $A_\lambda : E \to E$ is well defined, completely continuously, and fixed points of A_λ are solutions of (1.1) (see, e.g., [2,6]). Since we have

$$\lim_{\|\mathbf{u}\|\to 0} \frac{f_{i_0}(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} = \infty,$$

for $\mathbf{u} = (u_1,\dots,u_n) \in \mathbb{R}_+^n$, we can choose $\delta > 0$ so that

$$f_{i_0}(\mathbf{u}) > 0 \quad \text{for } 0 < \|\mathbf{u}\| \leqslant n\delta, \quad \mathbf{u} \in \mathbb{R}_+^n.$$

Let $\lambda_0 = \frac{\delta^{p-1}}{M\|\phi\|_\infty^{p-1}}$ and

$$M = \sup\bigl\{ f_j(\mathbf{u}):\ \|\mathbf{u}\| \leqslant n\delta,\ 1 \leqslant j \leqslant n,\ \mathbf{u} \in \mathbb{R}_+^n \bigr\} > 0.$$

We now only consider $0 < \lambda < \lambda_0$. Define a function $f_{i_0}^{\min} : [0, n\delta] \to [0,\infty)$ by

$$f_{i_0}^{\min}(t) = \min\bigl\{ f_{i_0}(\mathbf{u}):\ \mathbf{u} \in \mathbb{R}_+^n \text{ and } t \leqslant \|\mathbf{u}\| \leqslant n\delta \bigr\}.$$

In view of Lemma A.1 in Appendix A, condition (H2) implies

$$\lim_{t\to 0^+} \frac{f_{i_0}^{\min}(t)}{t^{p-1}} = \infty.$$

Therefore, for each $0 < \lambda < \lambda_0$, there exists a positive $\epsilon_1 < \delta$ such that

$$f_{i_0}^{\min}(\alpha) \geqslant \frac{\lambda_1}{\lambda}\alpha^{p-1}$$

if $0 < \alpha \leqslant \epsilon_1$. Now choose an $\epsilon > 0$ such that $\lambda\epsilon\|\phi_1\|_\infty < \epsilon_1$. We define a subset K of E by

$$K = \bigl\{(u_1, \dots, u_n) \in E\colon\ 0 \leqslant u_i \leqslant \delta \text{ for all } i \neq i_0,\ \lambda\epsilon\phi_1 \leqslant u_{i_0} \leqslant \delta \text{ in } \Omega\bigr\}$$

for each $0 < \lambda < \lambda_0$. Note that $\lambda\epsilon\phi_1 \leqslant \lambda\epsilon\|\phi_1\|_\infty < \epsilon_1 < \delta$ in Ω. It is easy to verify that K is a closed, bounded, convex subset of E. We claim that $A_\lambda : K \to K$. Let $(u_1, \dots, u_n) = A_\lambda(v_1, \dots, v_n)$ for $(v_1, \dots, v_n) \in K$. First, by the maximum principle [8], $u_i \geqslant 0$ in Ω, $i = 1, \dots, n$. On the other hand, since $\|v_i\|_\infty \leqslant \delta$, $i = 1, \dots, n$, we have

$$-\Delta_p u_i = \lambda f_i(v_1, \dots, v_n) \leqslant \lambda M \quad \text{in } \Omega,\ i = 1, \dots, n, \tag{3.2}$$

which implies, by the comparison principle [8], that

$$u_i \leqslant (\lambda M)^{\frac{1}{p-1}}\phi \leqslant (\lambda M)^{\frac{1}{p-1}}\|\phi\|_\infty \leqslant \delta, \quad i = 1, \dots, n.$$

Finally, in view of the definition of $f_{i_0}^{\min}$, we have

$$-\Delta_p u_{i_0} = \lambda f_{i_0}(v_1, \dots, v_n) \geqslant \lambda f_{i_0}^{\min}(\lambda\epsilon\phi_1).$$

By the choice of ϵ, for each $0 < \lambda < \lambda_0$, we have

$$-\Delta_p u_{i_0} \geqslant \lambda\frac{\lambda_1}{\lambda}(\lambda\epsilon\phi_1)^{p-1} = \lambda_1(\lambda\epsilon\phi_1)^{p-1}.$$

Again the comparison principle [8] implies that

$$u_{i_0} \geqslant \lambda\epsilon\phi_1 \quad \text{in } \Omega.$$

Hence, $(u_1, \dots, u_n) \in K$ and $A_\lambda : K \to K$. By the Schauder Fixed-Point Theorem, A_λ has a fixed point in K, which is the desired nontrivial solution of (1.1).

4. Proof of Theorem 1.2

Let E and A_λ be defined as in the proof of Theorem 1.1. For each $i = 1, \dots, n$, we define a function $f_i^{\max} : [0, \infty] \to [0, \infty)$ by

$$f_i^{\max}(t) = \max\bigl\{f(\mathbf{u})\colon\ \mathbf{u} \in \mathbb{R}_+^n \text{ and } \|\mathbf{u}\| \leqslant t\bigr\}.$$

In view of Lemma A.2 in Appendix A, the fact that

$$\lim_{\|\mathbf{u}\|\to\infty} \frac{f_i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} = 0, \quad i = 1, \dots, n,\ \mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n,$$

implies

$$\lim_{t\to\infty} \frac{f_i^{\max}(t)}{t^{p-1}} = 0, \quad i = 1, \dots, n.$$

We can choose a sufficient large $\delta > 0$ so that

$$\frac{f_i^{\max}(n\delta)}{(n\delta)^{p-1}} < \sigma, \quad i = 1, \dots, n,$$

where $\sigma > 0$ satisfying

$$n(\lambda\sigma)^{\frac{1}{p-1}} \|\phi\|_\infty \leqslant 1.$$

With this δ, we define a function $f_{i_0}^{\min} : [0, n\delta] \to [0, \infty)$ by

$$f_{i_0}^{\min}(t) = \min\bigl\{ f_{i_0}(\mathbf{u}) \colon \mathbf{u} \in \mathbb{R}_+^n \text{ and } t \leqslant \|\mathbf{u}\| \leqslant n\delta \bigr\}.$$

In view of Lemma A.1 in Appendix A, condition (H2) implies

$$\lim_{t \to 0^+} \frac{f_{i_0}^{\min}(t)}{t^{p-1}} = \infty.$$

It is easy to see that there exists a positive $\epsilon_1 < \delta$ such that

$$f_{i_0}^{\min}(\alpha) \geqslant \frac{\lambda_1}{\lambda} \alpha^{p-1}$$

if $0 < \alpha \leqslant \epsilon_1$. Now choose a positive ϵ such that $\lambda\epsilon\|\phi_1\|_\infty < \epsilon_1$.

We now define a subset K of E by

$$K = \bigl\{ (u_1, \dots, u_n) \in E \colon 0 \leqslant u_i \leqslant \delta \text{ for all } i \neq i_0,\ \lambda\epsilon\phi_1 \leqslant u_{i_0} \leqslant \delta \text{ in } \Omega \bigr\}.$$

Note that $\lambda\epsilon\phi_1 \leqslant \lambda\epsilon\|\phi_1\|_\infty < \epsilon_1 < \delta$ in Ω. Then K is a closed, bounded, convex subset of E. We claim that $A_\lambda : K \to K$. Let $(u_1, \dots, u_n) = A_\lambda(v_1, \dots, v_n)$ for $(v_1, \dots, v_n) \in K$. First, by the maximum principle [8], $u_i \geqslant 0$ in Ω, $i = 1, \dots, n$. On the other hand, since $\|v_i\|_\infty \leqslant \delta$, $i = 1, \dots, n$, we have, for $i = 1, \dots, n$,

$$-\Delta_p u_i = \lambda f_i(v_1, \dots, v_n) \leqslant \lambda f_i^{\max}(n\delta) \leqslant \lambda\sigma n^{p-1}\delta^{p-1} \quad \text{in } \Omega, \tag{4.1}$$

which implies, by the comparison principle [8], that

$$u_i \leqslant (\lambda\sigma)^{\frac{1}{p-1}} n\delta\phi \leqslant \delta, \quad i = 1, \dots, n.$$

Finally, in view of the definition of $f_{i_0}^{\min}$ and the choice of ϵ, ϵ_1, we have

$$-\Delta_p u_{i_0} \geqslant \lambda f_{i_0}(v_1, \dots, v_n) \geqslant \lambda f_{i_0}^{\min}(\lambda\epsilon\phi_1) \geqslant \lambda_1(\lambda\epsilon\phi_1)^{p-1}.$$

Again the comparison principle [8] implies that

$$u_{i_0} \geqslant \lambda\epsilon\phi_1 \quad \text{in } \Omega.$$

Hence, $(u_1, \dots, u_n) \in K$ and $A_\lambda : K \to K$. By the Schauder Fixed-Point Theorem, A_λ has a fixed point in K, which is the desired nontrivial solution of (1.1).

5. Proof of Theorem 1.3

It follows from (H1) and (H4) that there exists a constant $C > 0$ such that

$$f_i(\mathbf{u}) \leqslant C\left(\sum_{i=1}^n u_i\right)^{p-1} \quad \text{in } \Omega,\ i = 1, \dots, n,\ \mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n.$$

Choose $\lambda_0 > 0$ so that

$$(\lambda_0 C)^{\frac{1}{p-1}} \|\phi\|_\infty < \frac{1}{n}.$$

Now assume $\mathbf{v} = (v_1, \ldots, v_n) \in K$ is a nontrivial solution of (1.1). We will show that this leads to a contradiction if $0 < \lambda < \lambda_0$. Indeed, for $0 < \lambda < \lambda_0$ and $i = 1, \ldots, n$,

$$-\Delta_p v_i = \lambda f_i(v_1, \ldots, v_n) \leqslant \lambda C\left(\sum_{i=1}^n v_i\right)^{p-1} \leqslant \lambda C\left(\sum_{i=1}^n \|v_i\|_\infty\right)^{p-1}.$$

Hence, by the comparison principle, we have

$$v_i \leqslant (\lambda C)^{\frac{1}{p-1}} \sum_{i=1}^n \|v_i\|_\infty \phi < \alpha \sum_{i=1}^n \|v_i\|_\infty,$$

where $\alpha = (\lambda_0 C)^{\frac{1}{p-1}} \|\phi\|_\infty$. Thus

$$\sum_{i=1}^n \|v_i\|_\infty \leqslant n\alpha \sum_{i=1}^n \|v_i\|_\infty,$$

which is a contradiction since $n\alpha < 1$.

Appendix A

In this appendix we provide two lemmas, which simplify the proofs of our existence theorems. More importantly, they help to relax the monotonicity assumptions on the nonlinearities.

Let $\delta > 0$, $f : \mathbb{R}_+^n \to \mathbb{R}_+$ be continuous. We define two new functions: $f^{\min}(t) : [0, \delta] \to \mathbb{R}_+$ and $f^{\max}(t) : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f^{\min}(t) = \min\{f(\mathbf{u}):\ \mathbf{u} \in \mathbb{R}_+^n \text{ and } t \leqslant \|\mathbf{u}\| \leqslant \delta\}$$

and

$$f^{\max}(t) = \max\{f(\mathbf{u}):\ \mathbf{u} \in \mathbb{R}_+^n \text{ and } \|\mathbf{u}\| \leqslant t\}.$$

It is clear that both $f^{\min}$ and $f^{\max}$ are nondecreasing. Now we are able to prove the following two lemmas.

Lemma A.1. *If*

$$f(\mathbf{u}) > 0 \quad \text{for } 0 < \|\mathbf{u}\|,\ \mathbf{u} \in \mathbb{R}_+^n,$$

and

$$\lim_{\|\mathbf{u}\| \to 0} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} = \infty, \quad \mathbf{u} \in \mathbb{R}_+^n,$$

then

$$\lim_{t \to 0^+} \frac{f^{\min}(t)}{t^{p-1}} = \infty.$$

Proof. Let $M > 0$. Since

$$\lim_{\|\mathbf{u}\|\to 0} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} = \infty,$$

where $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$, there is $\delta_1 \in (0, \delta)$ such that

$$\frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} > M$$

for $0 < \|\mathbf{u}\| < \delta_1$ and $\mathbf{u} \in \mathbb{R}_+^n$. Now let

$$\bar{\delta} = \min\left\{\delta_1, \left(\frac{U}{M}\right)^{\frac{1}{p-1}}\right\} > 0,$$

where $U = \min\{f(\mathbf{u})\colon \delta_1 \leqslant \|\mathbf{u}\| \leqslant \delta,\ \mathbf{u} \in \mathbb{R}_+^n\}$.

We now claim that

$$\frac{f^{\min}(t)}{t^{p-1}} > M$$

for $0 < t < \bar{\delta}$. Indeed, for $t \in (0, \bar{\delta})$, there is a $\mathbf{u}_t \in \mathbb{R}_+^n$ and $t \leqslant \|\mathbf{u}_t\| \leqslant \delta$ such that $f^{\min}(t) = f(\mathbf{u}_t)$. If $\|\mathbf{u}_t\| < \delta_1$, we have

$$\frac{f^{\min}(t)}{t^{p-1}} = \frac{f(\mathbf{u_t})}{t^{p-1}} \geqslant \frac{f(\mathbf{u_t})}{\|\mathbf{u_t}\|^{p-1}} > M.$$

On the other hand, if $\|\mathbf{u}_t\| \geqslant \delta_1$, then

$$\frac{f^{\min}(t)}{t^{p-1}} = \frac{f(\mathbf{u_t})}{t^{p-1}} \geqslant \frac{U}{t^{p-1}} > \frac{U}{\bar{\delta}^{p-1}} > M.$$

This proves the claim and so does the lemma. □

A more general form of the following lemma was proved in Wang [9]. We give a proof here for completeness.

Lemma A.2. [9] *Let* $\mathbf{u} \in \mathbb{R}_+^n$ *and assume* $\lim_{\|\mathbf{u}\|\to 0} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}$ *and* $\lim_{\|\mathbf{u}\|\to 0} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}$ *exist* (*can be infinity*). *Then*

$$\lim_{t\to 0^+} \frac{f^{\max}(t)}{t^{p-1}} = \lim_{\|\mathbf{u}\|\to 0} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}$$

and

$$\lim_{t\to \infty} \frac{f^{\max}(t)}{t^{p-1}} = \lim_{\|\mathbf{u}\|\to \infty} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}.$$

Proof. It is easy to show that $\lim_{t\to 0^+} \frac{f^{\max}(t)}{t^{p-1}} = \lim_{\|\mathbf{u}\|\to 0} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}$. For the second part, we consider the two cases, (a) $f(\mathbf{u})$ is bounded and (b) $f(\mathbf{u})$ is unbounded. For case (a), it follows that $\lim_{t\to\infty} \frac{f^{\max}(t)}{t^{p-1}} = 0 = \lim_{\|\mathbf{u}\|\to\infty} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}$. For case (b), for any $\delta > 0$, let $M = f^{\max}(\delta)$ and

$$N_\delta = \inf\bigl\{\|\mathbf{u}\|\colon \mathbf{u} \in \mathbb{R}_+^n,\ \|\mathbf{u}\| \geqslant \delta,\ f(\mathbf{u}) \geqslant M\bigr\} \geqslant \delta,$$

then

$$\max\bigl\{f(\mathbf{u})\colon \|\mathbf{u}\| \leqslant N_\delta,\ \mathbf{u} \in \mathbb{R}_+^n\bigr\} = M = \max\bigl\{f(\mathbf{u})\colon \|\mathbf{u}\| = N_\delta,\ \mathbf{u} \in \mathbb{R}_+^n\bigr\}.$$

Thus, for any $\delta > 0$, there exists $N_\delta \geqslant \delta$ such that

$$f^{\max}(t) = \max\{f(\mathbf{u}):\ N_\delta \leqslant \|\mathbf{u}\| \leqslant t,\ \mathbf{u} \in \mathbb{R}^n_+\} \quad \text{for } t > N_\delta.$$

Now, suppose that $\lim_{\|\mathbf{u}\|\to\infty} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} = \alpha < \infty$. In other words, for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\alpha - \varepsilon < \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} < \alpha + \varepsilon \quad \text{for } \mathbf{u} \in \mathbb{R}^n_+,\ \|\mathbf{u}\| > \delta. \tag{A.1}$$

Thus, for $t > N_\delta$, there exist $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^n_+$ such that $\|\mathbf{u}_1\| = t$, $t \geqslant \|\mathbf{u}_2\| \geqslant N_\delta$ and $f(\mathbf{u}_2) = f^{\max}(t)$. Therefore,

$$\frac{f(\mathbf{u}_1)}{\|\mathbf{u}_1\|^{p-1}} \leqslant \frac{f^{\max}(t)}{t^{p-1}} = \frac{f(\mathbf{u}_2)}{t^{p-1}} \leqslant \frac{f(\mathbf{u}_2)}{\|\mathbf{u}_2\|^{p-1}}. \tag{A.2}$$

Now (A.1) and (A.2) yield that

$$\alpha - \varepsilon < \frac{f^{\max}(t)}{t^{p-1}} < \alpha + \varepsilon \quad \text{for } t > N_\delta. \tag{A.3}$$

Hence $\lim_{t\to\infty} \frac{f^{\max}(t)}{t^{p-1}} = \alpha$. Similarly, we can show

$$\lim_{t\to\infty} \frac{f^{\max}(t)}{t^{p-1}} = \lim_{\|\mathbf{u}\|\to\infty} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}}$$

if $\lim_{\|\mathbf{u}\|\to\infty} \frac{f(\mathbf{u})}{\|\mathbf{u}\|^{p-1}} = \infty$. $\square$

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